Uncertainty quantification in computer experiments with polynomial chaos

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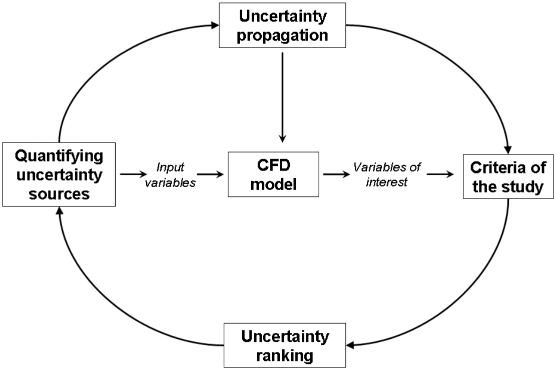
Uncertainty quantification (UQ) in computer experiments

- Context: Deterministic and complex numerical simulator are used to model real dynamic systems and they can be computationally expensive to run
- We are interested to study the effect of epistemic (lack of knowledge) and aleatoric (inherent to system) uncertainties on the model outputs
- Sources include initial condition, boundary condition & model parameters
- Example: drug clearance in circulation as an exponential decay response $\frac{d\theta}{dt} = -C\theta$ with C as a r.v. that represents the population response
- Conventional approaches such as MC are not practical in studying these expensive simulators
- Goal: PC construct a metamodel that mimics the complex model's behaviour and conduct UQ, SA, quantile estimation, optimization, calibration, *etc*.

Probabilistic framework

The UQ of a computer experiment follows the following iterative steps:

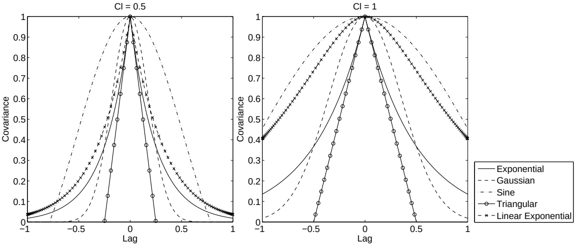
1. representation of input uncertainties - random variable or process
2. uncertainty propagation - MC, GP or gPC
3. quantification of solution uncertainty - mean, variance, pdf or sensitivity



De Rocquigny (2006)

Stochastic input representation: stochastic process

Any second order random process $\kappa(x, \omega)$, with continuous and bounded covariance kernel $C(x_1, x_2) = \mathbb{E}(\kappa(x_1, \omega) \otimes \kappa(x_2, \omega))$, can be represented as an infinite sum of random variables. It is real, symmetric and positive–definite.



- Karhunen-Loève (KL) expansion represents the random process with an orthogonal set of deterministic functions with random coefficients as

$$\kappa(x, \omega) = \mu_\kappa(x) + \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_n(x) \xi_n(\omega).$$

- For a continuous kernel, the convergence of the KL expansion is uniform as $N \to \infty$. Karhunen (1948) & Loève (1977)
- $\psi_n(x)$ and λ_n solved from Fredholm integral equation of 2nd kind with $C(x_1, x_2)$.

Stochastic input representation: random variables

- Represent the random variable, $\kappa(\omega)$, with orthogonal functions of the stochastic variable with deterministic coefficients

$$\kappa(\omega) = \sum_{m=0}^{\infty} \kappa_m \phi_m(\boldsymbol{\xi}(\omega)).$$

- Wiener-Chaos: representation of a Gaussian random variable using Hermite polynomials with L^2 convergence as $M \to \infty$. Wiener (1938), Ghanem & Spanos (1991) and Cameron & Martin (1947)
- generalized Polynomial Chaos: generalized representation to non-Gaussian random variables with polynomials from the Wiener–Askey scheme. Xiu & Karniadakis (2002)
- if $\kappa(\omega)$ follows a normal distribution, it can be represented exactly as $\kappa(\omega) = \mu_\kappa + \sigma_\kappa \boldsymbol{\xi}$ where $\boldsymbol{\xi}$ is the linear term in Hermite

Selection of orthogonal basis

- In the propagation step, we need to evaluate the inner product w.r.t. the probability space measure, $\rho(\boldsymbol{\xi})d\boldsymbol{\xi}$ as

$$\langle \phi_i(\boldsymbol{\xi}), \phi_j(\boldsymbol{\xi}) \rangle = \int_\Gamma \phi_i(\boldsymbol{\xi})\phi_j(\boldsymbol{\xi})\rho(\boldsymbol{\xi})d\boldsymbol{\xi}.$$

- Correspondence between the *pdf* of ξ, $\rho(\xi)$, and the weighting function of classical orthogonal polynomials, $w(\xi)$, determines the polynomial basis

Distribution	Random variable, ξ	Wiener-Askey PC, $\phi(\xi)$	Support, Γ
Continuous	Gaussian	Hermite-chaos	$(-\infty, \infty)$
	gamma	Laguerre-chaos	$[0, \infty)$
	beta	Jacobi-chaos	$[a, b]$
	uniform	Legendre-chaos	$[a, b]$
Discrete	Poisson	Charlier-chaos	$\{0, 1, 2, \dots\}$
	binomial	Krawtchouk-chaos	$\{0, 1, \dots, N\}$
	negative binomial	Meixner-chaos	$\{0, 1, 2, \dots\}$
	hypergeometric	Hahn-chaos	$\{0, 1, \dots, N\}$
Periodic	uniform	Fourier-chaos*	$[-\pi, \pi)$

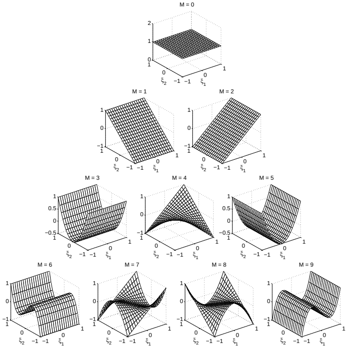
Multivariate basis

Multivariate basis is the tensor products of 1D polynomials

$$
\begin{aligned}
\phi_m(\boldsymbol{\xi}) &= \phi^{\alpha_{m,n=1}}(\xi_1) \otimes \phi^{\alpha_{m,n=2}}(\xi_2) \otimes \cdots \otimes \phi^{\alpha_{m,n=N}}(\xi_N), \quad \text{for } m = 0, \cdots, M, \\
&= \phi^{\boldsymbol{\alpha}_m}(\boldsymbol{\xi}), \quad \text{for } m = 0, \cdots, M.
\end{aligned}
$$

Truncation depends on input dimension, N, and output nonlinearity, P

m	P	$\prod$ Notation	Legendre Polynomials
0	0	$P^0(\xi_1)P^0(\xi_2)$	1
1	1	$P^1(\xi_1)P^0(\xi_2)$	ξ_1
2		$P^0(\xi_1)P^1(\xi_2)$	ξ_2
3	2	$P^2(\xi_1)P^0(\xi_2)$	$3/2\xi_1^2 - 1/2$
4		$P^1(\xi_1)P^1(\xi_2)$	$\xi_1\xi_2$
5		$P^0(\xi_1)P^2(\xi_2)$	$3/2\xi_2^2 - 1/2$
6	3	$P^3(\xi_1)P^0(\xi_2)$	$5/2\xi_1^3 - 3/2\xi_1$
7		$P^2(\xi_1)P^1(\xi_2)$	$3/2\xi_2\xi_1^2 - 1/2\xi_2$
8		$P^1(\xi_1)P^2(\xi_2)$	$3/2\xi_1\xi_2^2 - 1/2\xi_1$
9		$P^0(\xi_1)P^3(\xi_2)$	$5/2\xi_2^3 - 3/2\xi_2$



Stochastic Galerkin method: intrusive approach

PC represent the stochastic solution $u(\mathbf{x}, \boldsymbol{\xi})$ with the same orthogonal basis as the input, *i.e.* $u(\mathbf{x}, \boldsymbol{\xi}) = \sum u_m(\mathbf{x})\phi_m(\boldsymbol{\xi})$
Substitute the expansions into the system of equations, $\mathcal{L}(\mathbf{x}, \boldsymbol{\xi}; u) = f(\mathbf{x}, \boldsymbol{\xi})$.
Take the Galerkin projection, *i.e.*

$$\langle \mathcal{L}\left(\mathbf{x}, \boldsymbol{\xi}; \sum u_m(\mathbf{x})\phi_m(\boldsymbol{\xi})\right), \phi_m(\boldsymbol{\xi})\rangle \quad = \quad \langle f\left(\mathbf{x}, \boldsymbol{\xi}\right), \phi_m(\boldsymbol{\xi})\rangle, \quad \text{for } m = 0, \ldots, M.$$

- $u_m(\mathbf{x})$ are solved from the system of $(M+1)$ coupled equations.
- The system is deterministic and can be solved using a standard discretization technique.
- Extensive modification on the simulator is needed.

Stochastic Galerkin method: intrusive approach

Example

First-order linear ODE: $\dot{\Theta}(t, \boldsymbol{\xi}) = -C(\boldsymbol{\xi})\Theta(t, \boldsymbol{\xi})$ with rate of decay as a normal r.v., *i.e.* $C(\boldsymbol{\xi}) = \sum_{i=0}^{M} C_i \phi_i(\boldsymbol{\xi})$. The gPC expansions of $C(\boldsymbol{\xi})$ and $\Theta(t, \boldsymbol{\xi})$ are substituted into the ODE to give

$$\sum_{k=0}^{M_\theta} \dot{\Theta}_k(t)\phi_k(\boldsymbol{\xi}) \quad = \quad -\sum_{i=0}^{M_C} \sum_{j=0}^{M_\theta} C_i \Theta_j(t)\phi_i(\boldsymbol{\xi})\phi_j(\boldsymbol{\xi}).$$

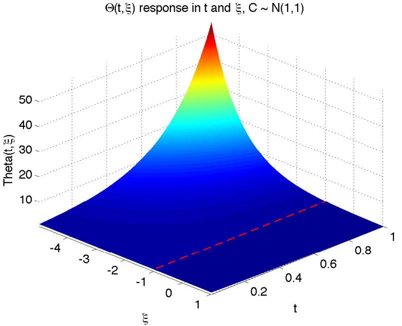
The Galerkin projection of the expanded ODE with orthogonal polynomial:

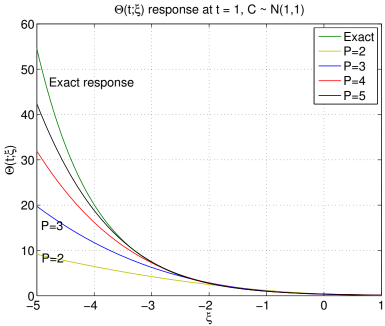
$$\dot{\Theta}_k(t) = -\sum_{i=0}^{M_C} \sum_{j=0}^{M_\theta} \frac{\langle \phi_i \phi_j \phi_k \rangle}{\langle \phi_k^2 \rangle} C_i \Theta_j(t), \quad \text{for k} = 0, \ldots, M_\theta.$$

This coupled deterministic system of equations is solved with an initial condition $\Theta(t=0) = \sum \Theta_m(t=0)\phi_m(\boldsymbol{\xi})$. With increasing t, the modal coefficients are propagated from the lower Θ_m to higher Θ_m, *i.e.* propagation of uncertainty as increasing non–linear response in the random space.

Surface response of the linear ODE

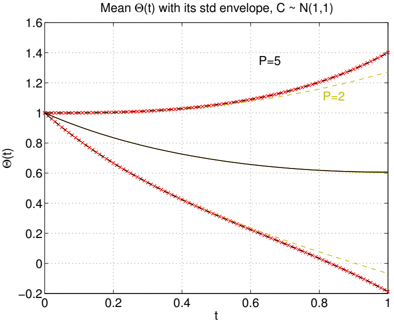
- $\dot{\Theta}(t, \boldsymbol{\xi}) = -C(\boldsymbol{\xi})\Theta(t, \boldsymbol{\xi})$
- $\Theta(t, \xi)$ response is exponential in t with $\Theta(t = 0) = 1$.
- Treating the coefficient of decay as a random variable, $C(\xi) \sim \mathcal{N}(1, 1)$
- We represent the univariate stochastic output $\Theta(t; \xi)$ as a linear combination of Hermite polynomials $\Theta(t; \boldsymbol{\xi}) = \sum \Theta_m(t)\phi_m(\boldsymbol{\xi})$.
- Uncertainty propagation visualized as solution response surface evolution in random space, ξ

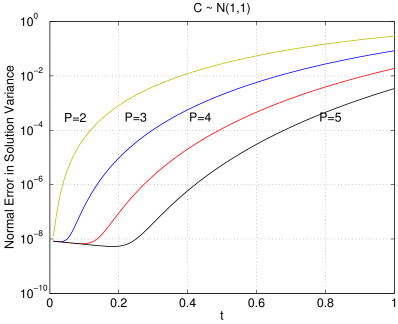




The choice of polynomial chaos truncation

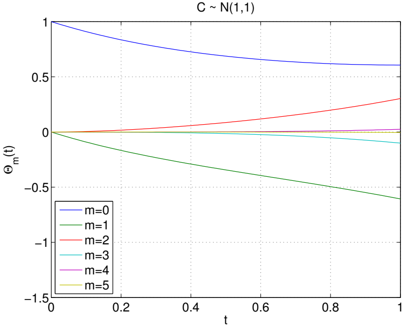
- As response in ξ becomes more non–linear with t, the higher order P in $\phi_m(\xi)$ are needed in gPC expansion
- Estimation of higher order statistics also require higher P
- Premature truncation leads to large error in the response surface and the solution statistics

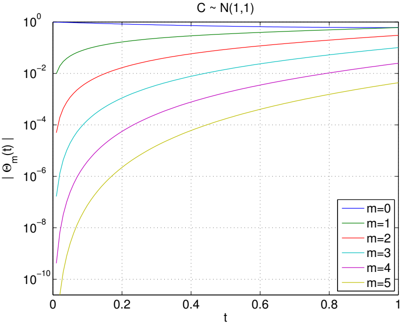




Evolution of the PC coefficients

- Increasing t propagates the initial uncertainty from lower order coefficients to higher order coefficients





- The task now is to determine the coefficients of expansion, $\Theta_m(t)$ in the representation.
- This simple system of equation easily solved with the intrusive approach
- Complex numerical solvers can benefit from a non–intrusive approach

Probabilistic collocation method (PCM)

Projecting directly the stochastic solution, $u(\mathbf{x}, \boldsymbol{\xi}) = \sum u_m(\mathbf{x})\phi_m(\boldsymbol{\xi})$, onto the orthogonal basis, $\phi_m(\boldsymbol{\xi})$, we obtain the following $(M+1)$ decoupled equations:

$$u_m(\mathbf{x}) = \frac{\langle u(\mathbf{x}, \boldsymbol{\xi}), \phi_m(\boldsymbol{\xi})\rangle}{\langle \phi_m^2(\boldsymbol{\xi})\rangle}, \quad \text{for } m = 0, ..., M.$$

The inner–product can be evaluated using Monte Carlo or related methods. We investigate a numerical quadrature approach to approximate the inner product where the numerical solver is treated as a black box from which samples are repeated taken.

One–dimensional quadrature rules

Integrals are approximated as the weighted sum of function evaluations on deterministic quadrature points, *i.e.*

$$\begin{aligned}\langle u(\mathbf{x},\boldsymbol{\xi}),\phi_m(\boldsymbol{\xi})\rangle &= \int_{\boldsymbol{\Gamma}} u(\mathbf{x},\boldsymbol{\xi})\phi_m(\boldsymbol{\xi})\rho(\boldsymbol{\xi})d\boldsymbol{\xi},\\ &\approx \sum_{j=0}^{N_q} w_j u(\mathbf{x},\mathbf{z}_j)\phi_m(\mathbf{z}_j).\end{aligned}$$

The accuracy of the method depends on the selection of the quadrature approach, *i.e.* constructions of w_j and $\mathbf{z}_j$.

	Γ	P	N_q	Nestedness
Gauss-Legendre	(-1,1)	$2L-1$	L	No
Clenshaw-Curtis	[-1,1]	$L-1$	$2^{L-1}+1$	Yes
Gauss-Laguerre	$[0,\infty)$	$2L-1$	L	No
Gauss-Hermite	$(-\infty,\infty)$	$2L-1$	L	No
Hermite Kronrod-Patterson	$(-\infty,\infty)$	$2m+n-1^{*}$	1-3-9-19-35 or 1-4-18-30	Yes

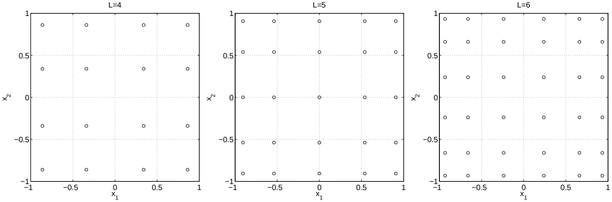
Multi–dimensional quadrature rules are constructed from 1D quadrature rules.

Full–tensor quadrature

Multi–dimensional full–tensor quadrature relies on tensor product of 1D quadrature rules, *e.g.* N–dimensional quadrature points are

$$\mathcal{Q}_L^N(f) = (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_N})(f).$$

Example: Two–dimensional Gauss–Legendre quadrature:



Accuracy: Theoretical polynomial exactness $P = 2L - 1$ in each dimension where L is the number of quadrature points in each dimension
Cost: Number of quadrature points grows as $\mathcal{O}(L^N)$ and error converges as $\epsilon(Z) = \mathcal{O}(Z^{-r/N})$. – *"curse of dimensionality"*

Sparse quadrature: the Smolyak approach

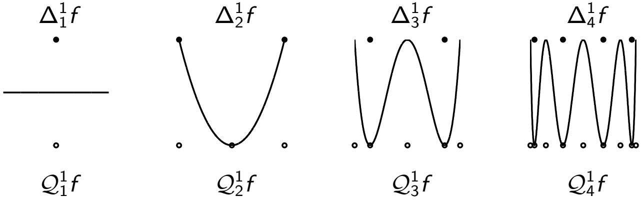
"Curse of dimensionality" could be 'broken' with the sparse grid. Its construction is based on the following three steps: Gerstner & Griebel (1998)

1. Constructed from 1D difference grid
2. Tensor product of 1D difference grids: cost reduction
3. Linear combination of the tensor products: embeddedness → refinement cost reduction

Accuracy: Theoretical polynomial exactness at least $P \leq 2L - 1$ where L is the quadrature level. Smolyak (1963), Novak & Ritter (1996)
Cost: Error converges as $\epsilon(Z) = \mathcal{O}(Z^{-r}(\log(Z)^{(N-1)(r+1)}))$. Novak & Ritter (1996)

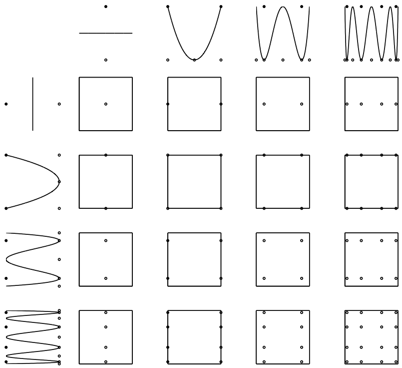
Sparse quadrature: with nested Clenshaw-Curtis quadrature rule

1D difference grid: $\triangle_k^1 f := \left(\mathcal{Q}_k^1 - \mathcal{Q}_{k-1}^1\right) f$



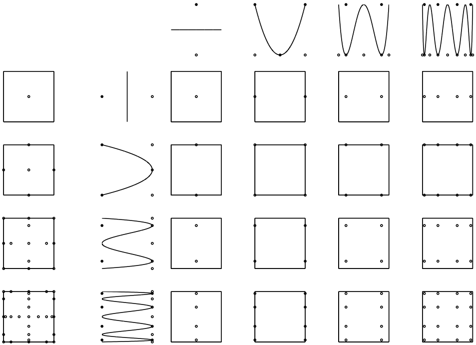
Sparse quadrature: with nested Clenshaw-Curtis quadrature rule

1D difference grid: $\triangle_k^1 f := \left(\mathcal{Q}_k^1 - \mathcal{Q}_{k-1}^1\right) f$
Tensor product: $\left(\triangle_{k_1}^1 \otimes \cdots \otimes \triangle_{k_N}^1\right) f$



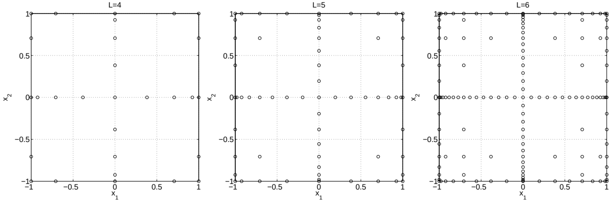
Sparse quadrature: with nested Clenshaw-Curtis quadrature rule

1D difference grid: $\triangle_k^1 f := \left(\mathcal{Q}_k^1 - \mathcal{Q}_{k-1}^1\right) f$
Tensor product: $\left(\triangle_{k_1}^1 \otimes \cdots \otimes \triangle_{k_N}^1\right) f$
Linear combination: $\mathcal{Q}_L^N[f] := \sum \left(\triangle_{k_1}^1 \otimes \cdots \otimes \triangle_{k_N}^1\right) f$

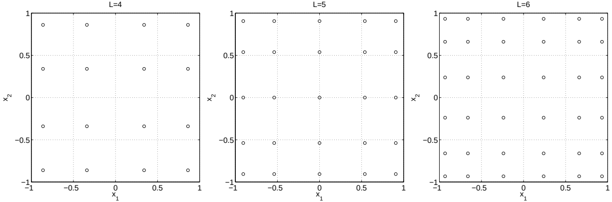


Sparse quadrature: comparison with full–tensor quadratures

Sparse Clenshaw-Curtis Chebyshev: P=7, P=9 & P=11



Full Gauss–Legendre Quadrature: P=7, P=9 & P=11



Canonical, maximum and anisotropic expansions

M is determined by the accuracy of the quadrature approach. If the quadrature has a polynomial accuracy of P or $\mathbf{P}$, there are the following expansions for

$$f_r(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^N} f_{\boldsymbol{\alpha}} \phi_{\boldsymbol{\alpha}}(\mathbf{x})$$

- Canonical: *total* degrees not greater than P, *i.e.* $\{\phi_{\boldsymbol{\alpha}} \ / \ |\boldsymbol{\alpha}| \leq P\}$
- Maximum: degree in *each* n not greater than P, *i.e.* $\{\phi_{\boldsymbol{\alpha}} \ / \ \boldsymbol{\alpha} \leq P\}$.
- Anisotropic: degree in *each* n not greater than P_n, *i.e.* $\{\phi_{\boldsymbol{\alpha}} \ / \ \boldsymbol{\alpha} \leq \mathbf{P}\}$.

M	P	Legendre polynomial	Canonical, $P = 2$	Maximum, $P = 2$	Anisotropic $\mathbf{P} = [3, 1]$
0	0	1	$P^0(x_1)P^0(x_2)$	$P^0(x_1)P^0(x_2)$	$P^0(x_1)P^0(x_2)$
1	1	x_1	$P^1(x_1)P^0(x_2)$	$P^1(x_1)P^0(x_2)$	$P^1(x_1)P^0(x_2)$
2		x_2	$P^0(x_1)P^1(x_2)$	$P^0(x_1)P^1(x_2)$	$P^0(x_1)P^1(x_2)$
3	2	$3/2x_1^2 - 1/2$	$P^2(x_1)P^0(x_2)$	$P^2(x_1)P^0(x_2)$	$P^2(x_1)P^0(x_2)$
4		x_1x_2	$P^1(x_1)P^1(x_2)$	$P^1(x_1)P^1(x_2)$	$P^1(x_1)P^1(x_2)$
5		$3/2x_2^2 - 1/2$	$P^0(x_1)P^2(x_2)$	$P^0(x_1)P^2(x_2)$	$P^0(x_1)P^2(x_2)$
6	3	$5/2x_1^3 - 3/2x_1$	$P^3(x_1)P^0(x_2)$	$P^3(x_1)P^0(x_2)$	$P^3(x_1)P^0(x_2)$
7		$(3/2x_1^2 - 1/2)x_2$	$P^2(x_1)P^1(x_2)$	$P^2(x_1)P^1(x_2)$	$P^2(x_1)P^1(x_2)$
8		$x_1(3/2x_2^2 - 1/2)$	$P^1(x_1)P^2(x_2)$	$P^1(x_1)P^2(x_2)$	$P^1(x_1)P^2(x_2)$
9		$5/2x_2^3 - 3/2x_2$	$P^0(x_1)P^3(x_2)$	$P^0(x_1)P^3(x_2)$	$P^0(x_1)P^3(x_2)$
10	4	$35/8x_1^4 - 15/4x_1^2 + 3/8$	$P^4(x_1)P^0(x_2)$	$P^4(x_1)P^0(x_2)$	$P^4(x_1)P^0(x_2)$
11		$(5/2x_1^3 - 3/2x_1)x_2$	$P^3(x_1)P^1(x_2)$	$P^3(x_1)P^1(x_2)$	$P^3(x_1)P^1(x_2)$
12		$(3/2x_1^2 - 1/2)(3/2x_2^2 - 1/2)$	$P^2(x_1)P^2(x_2)$	$P^2(x_1)P^2(x_2)$	$P^2(x_1)P^2(x_2)$
13		$x_1(5/2x_2^3 - 3/2x_2)$	$P^1(x_1)P^3(x_2)$	$P^1(x_1)P^3(x_2)$	$P^1(x_1)P^3(x_2)$
14		$35/8x_2^4 - 15/4x_2^2 + 3/8$	$P^0(x_1)P^4(x_2)$	$P^0(x_1)P^4(x_2)$	$P^0(x_1)P^4(x_2)$

gPC as a Uncertainty Quantification (UQ) & Sensitivity Analysis (SA) tool

Statistical moments:

$$\begin{aligned}\mu_u(\mathbf{x}) &= \int_\Gamma u_r(\mathbf{x};\omega)\phi_0(\boldsymbol{\xi})\rho(\boldsymbol{\xi})d\boldsymbol{\xi} = u_0(\mathbf{x}),\\ \sigma^2_{u,gPC}(\mathbf{x}) &= \int_\Gamma \left[\sum_{m=0}^{M} u_m(\mathbf{x})\phi_m(\boldsymbol{\xi}) - u_0(\mathbf{x})\right]^2 \rho(\boldsymbol{\xi})d\boldsymbol{\xi} = \sum_{m=1}^{M} u_m^2(\mathbf{x})\langle\phi_m^2(\boldsymbol{\xi})\rangle.\end{aligned}$$

Solution sensitivity: Partial differentiation wrt ξ_n Agarwal (2008)

$$\mathbf{S}_{\xi_n}(\mathbf{x}) = \frac{\partial u_r(\mathbf{x};\boldsymbol{\xi})}{\partial \xi_n}.$$

Sensitivity analysis: partial variances Sobol' (1993)

$$\sigma_u^2(\mathbf{x}) = \sum_{i_1=1}^{N} D_{i_1}(\mathbf{x}) + \sum_{i_1=1}^{N}\sum_{i_2=1}^{i_1} D_{i_1 i_2}(\mathbf{x}) + \sum_{i_1=1}^{N}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} D_{i_1 i_2 i_3}(\mathbf{x}) \cdots + D_{i_1 i_2 \ldots i_N}(\mathbf{x}).$$

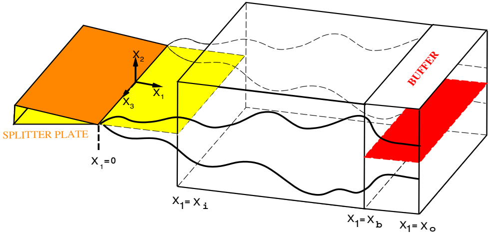
Probability density function (PDF): numerical computation from the histogram of a large MC sample of $u_r(\mathbf{x},\boldsymbol{\xi})$ based on the distribution of $\boldsymbol{\xi}$

Application of gPC to some examples

Examples	Tasks	N	R.V / Representations
Mixing layer magnitude	UQ & SA	2 & 3	Uniform/Legendre
Mixing layer phase	UQ & SA	1 & 2	Periodic/Fourier
Toy models	QE	1 to 10	Gauss.&Uni./Herm.&Leg.
Global circulation model	SA & CAL.	5	Log-uni.&Uni./Leg.

Sensitivity of spatially developing mixing layer

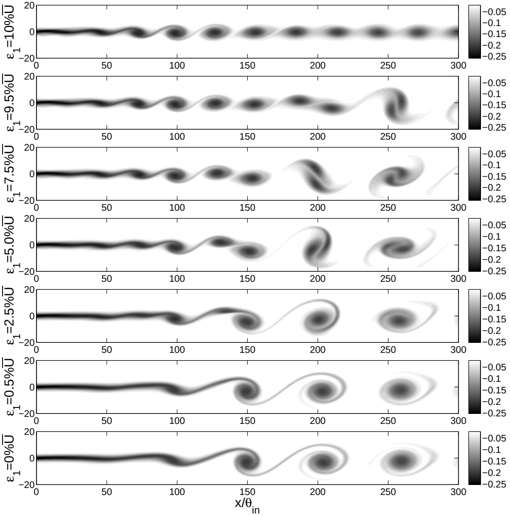
- Coherent vortical structures triggered by inflow forcing Brown & Roshko (1974)
- Shear layer at the inflow approximated as $\overline{U}_{in}(y) = 1 + \lambda \tanh(y/2)$
- Downstream shear layer growth is very sensitive to forcing definition
- Forcing with LST fundamental mode, *i.e.* most unstable, and its subharmonic modes: $u_p(y,t) = \sum \epsilon_n f_n(y) \exp(i(\omega_n t + \gamma_n))$
- 3D flow structure is largely 2D $\rightarrow$ 2D DNS Delville et al. (1999)
- Goal: To generalize the approach to design discrete forcing with random magnitude or phasing



De Brun (2001)

Sensitivity to forcing: magnitude ϵ_n

- Instantaneous vorticity contours with bimodal perturbation
- Vortical structure variation as the relative frequency content in inflow forcing changes



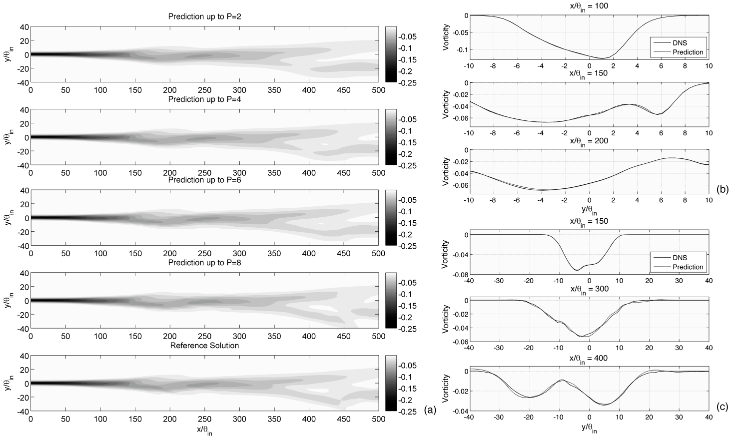
Stochastic mixing layer with random magnitudes, ϵ_n

Treat ϵ_n and γ_n as random variables to determine the most general way to control mixing layer growth with inflow forcing.

- Bimodal forcing and trimodal forcing examined
- Stochastic forcing magnitudes ϵ_n as uniform variables in $[0, 10\%\overline{U}]$
- Legendre-Chaos expansion of stochastic fields
- Mixing layer solutions with 2D spectral/hp DNS solver
- $Re = 100$, $\lambda = 0.5$
- Non–intrusive Probabilistic Collocation Method with full–tensor Gauss–quadrature
- 81 full–tensor quadrature points for bimodal forcing (N=2, L=9, P=8) & 1000 for trimodal (N=3, L=10, P=9)
- Examine time-averaged mixing layer thickness, *e.g.* momentum thickness θ

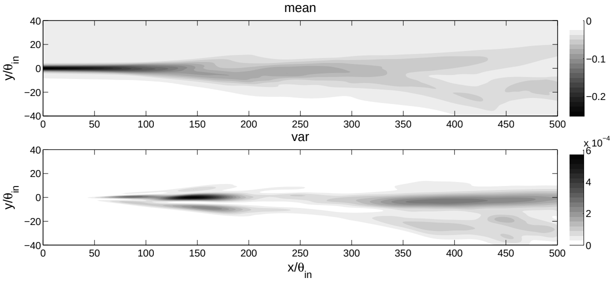
Accuracy of the gPC expansion: solution prediction

With $u(\mathbf{x}, \boldsymbol{\xi}) = u_m(\mathbf{x})\phi_m(\boldsymbol{\xi})$, we can predict the solution at an arbitrary point within $\boldsymbol{\Gamma}$. Accuracy of the prediction increases with increasing M or P.

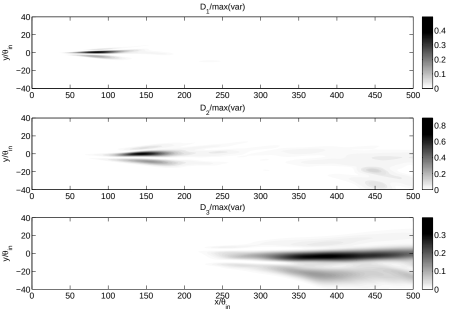


Response variability in trimodal perturbation case

- Initial response up to $x/\theta_{in} = 250$ similar to the bimodal case
- Large local variance at the location associated with the onset of deterministic subharmonic vortex merging

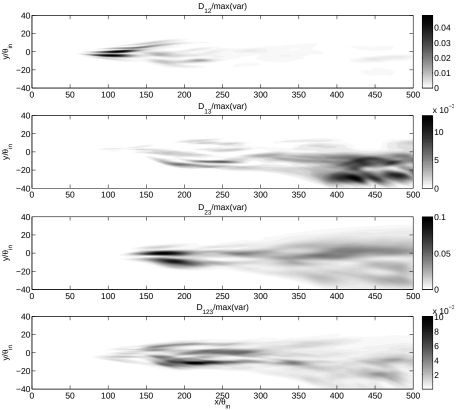


Partial variance contour in trimodal vorticity



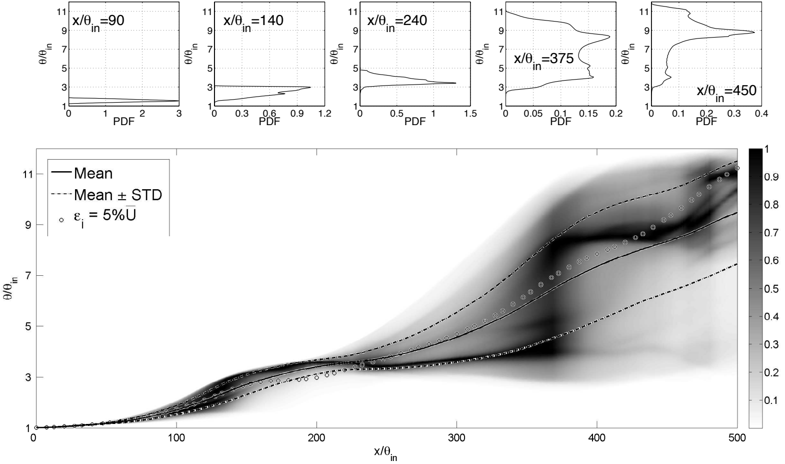
- D_n: sensitivities of the solution to ϵ_n
- Contours of each sensitivity index correspond closely to the deterministic vortex-roll up of each mode

Partial variance in trimodal vorticity contour



- D_{ij}: sensitivities of the solution to interaction between ϵ_i and ϵ_j
- Large D_{12} and D_{23} $\longrightarrow$ interactions between successive modes are dominant Kelly (1967)
- D_{123}: sensitivities of the solution to the mutual interaction amongst all modes

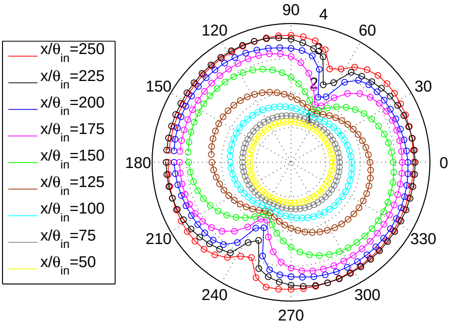
θ PDF in trimodal perturbation case



Stochastic mixing layer with random phase γ_n

- Bimodal forcing and trimodal forcing examined
- Stochastic phase shifts γ_n as uniform random variables in $[0, 2\pi)$
- Forcing magnitudes maintained at $\sum \epsilon_n = 10\%\overline{U}$
- SCM with Newton-Cotes quadrature
- Fourier-Chaos expansion of stochastic fields
- Discrete Fourier transformation (DFT) speeds up coefficient computations
- 72 equidistant quadrature samples are used (nested points)
- Examine time-averaged mixing layer thickness, *e.g.* momentum thickness θ

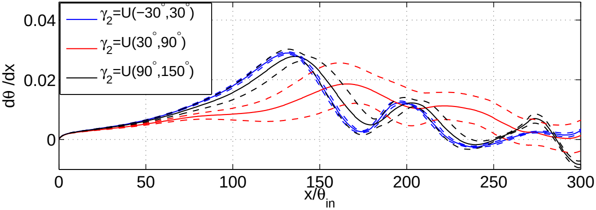
Response of momentum thickness



- Symmetry observed as $\gamma_2 \in [0, 2\pi]$ includes two periods of fund. forcing
- Mixing layer growth strongly delayed over small γ_2 range near 70° Inoue (1995)
- Delayed growth reported for $\gamma_2 = 0$ at merging locations Stanley & Sarkar (1997)
- 45° difference between inflow forcing formulations
- Phase shift at inflow does not correspond to phase shift at merging locations

Mixing layer growth rate statistics

- $\partial\theta/\partial x$ examined for:
  Normal growth: $\gamma_2 = U(-30^\circ, 30^\circ)$ & $\gamma_2 = U(90^\circ, 150^\circ)$
  Delayed growth: $\gamma_2 = U(30^\circ, 90^\circ)$



- 'Normal growth': Fast growth near inflow followed by sharp drop in $\partial\theta/\partial x$. Drop or contraction of the mixing layer Oster & Wygnansk (1982)
- 'Delayed growth': Slower growth with less $\partial\theta/\partial x$ fluctuation. Large variance due to solution sensitivity in $\gamma_2 \in [45^\circ, 80^\circ]$. Range of sensitivity is small Stanley & Sarkar (1997)

PC as a quantile estimation tool

Empirical quantile: estimated from $\hat{Y}_\alpha = \inf\{y; \hat{F}(y) \geq \alpha\}$ which gives

$$\hat{Y}_\alpha = Y_{(\lceil \alpha Z \rceil)}, \tag{1}$$

where $\{Y_{(i)}\}_{i=1}^{Z}$ are the ordered set of the Z MC samples.
The metamodel accurately determines the statistical moments but fails in extreme quantile estimations, *i.e.* α near 0 or 1.
We propose a multi–element refinement approach: global gFC metamodel is complimented by local metamodel constructed around design points $\boldsymbol{\xi}_\alpha$.
Design point: *most likely* random input that corresponds to $u_r(\mathbf{x}, \boldsymbol{\xi}) = u_\alpha(\mathbf{x})$.
This gives a constraint nonlinear minimization problem , *i.e.*

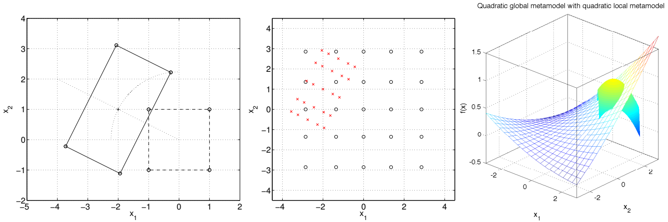
$$\min \|\boldsymbol{\xi}\|, \quad \text{s.t.} \quad \sum_{m=0}^{M} u_m(\mathbf{x})\phi_m(\boldsymbol{\xi}) - \hat{Y}_\alpha = 0.$$

The above problem is solved by the method of **Lagrangian multipliers**.

Multi–Element Monte Carlo simulation

Local gPC metamodels are created around the design points. The multielement solution is used as the metamodel, *i.e.*

$$D_{\mathrm{ME}} = \begin{cases} D_{\mathrm{global}} = D \setminus D_{\mathrm{local}}, & \text{domain of global gPC,} \\ D_{\mathrm{local}} = \cup D_{\beta_i}, & \text{domains of refinement about } \hat{\boldsymbol{\xi}}_{\alpha_i}, \text{ for } i = 1, \ldots, N_\beta. \end{cases}$$



The final multi–element gPC (MEgPC) metamodel is

$$f_{\mathrm{ME}}(\mathbf{x}) = \begin{cases} \sum_{m=0}^{M} f_m \phi_m(\mathbf{x}), & \text{if } \mathbf{x} \in D_{\mathrm{global}}, \\ \sum_{m=0}^{M_i^*} f_{m,i}^* \psi_{m,i}(\mathbf{T}_i^{-1}(\mathbf{x})), & \text{if } \mathbf{x} \in D_{\beta_i}. \end{cases}$$

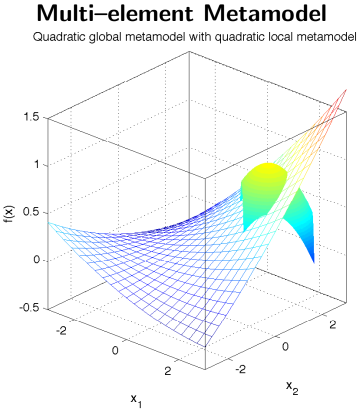
where $\mathbf{T}_i$ a transformation operator that maps a point in the uniform bounded support $\mathbf{x}^* \in [-1, 1]^N$ to the local domain $\mathbf{x} \in D_{\beta_i}$.

Example: Gaussian–like response

We examine the quantile of the output of a Gaussian-like function:

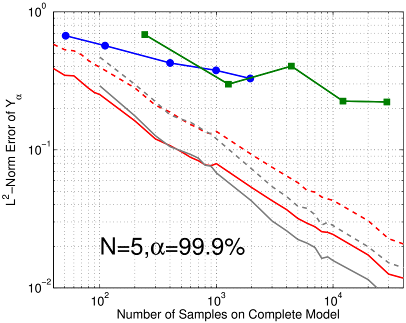
$$f(\mathbf{x}) = \sum_{i=1}^{N_\alpha} \prod_{n=1}^{N} \exp\left(\frac{-(x_n - \mu_{n,i})^2}{2\sigma_{n,i}^2} \right), \qquad (2)$$

where $\|\boldsymbol{\mu}\| = 2$, $\boldsymbol{\sigma} = 1$, $\mathbf{x}$ are i.i.d. random variables and $x_n \in \mathcal{N}(0,1)$.



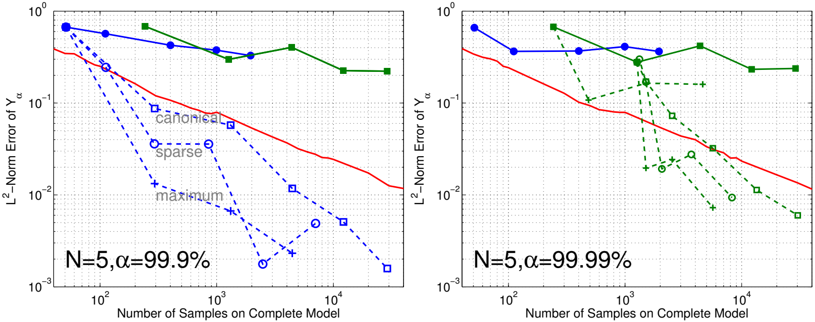
α–quantile estimator convergence for MC, IS and global gPC

- Monte Carlo $\hat{Y}_\alpha$ converges as $1/\sqrt{Z}$
- Importance sampling $\hat{Y}_\alpha$ computed at selected Z: Z/2 MC samples for first estimate of $\hat{Y}_\alpha$, at most $Z/4$ for GPM and the rest for IS
- Global **full** and **sparse** gPC estimations of $\hat{Y}_{\alpha,r}$ (from $L = 3$ to 7) are poor



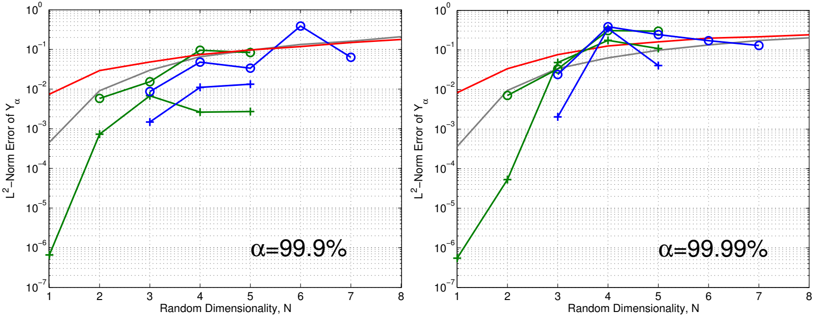
Effects of different local refinements

- Local **full** (canonical & maximum) and **sparse** gPC metamodel refinements
- Maximum expansion improves the accuracy of $\hat{Y}_{\alpha,\mathrm{ME}}$ given the same Z
- Seek best $\hat{\boldsymbol{\xi}}_{\alpha,r}$ estimation by maximizing Z in global gPC metamodel



Target cost study

- An arbitrary target cost that increases linearly with N: $Z_{total} = 100N$
- Monte Carlo and importance sampling $\hat{Y}_\alpha$ with entire sampling budget
- Global **full** and **sparse** + local full maximum (+) and sparse (∘) supplemental metamodels
- Maximize global metamodel cost while not exceeding the entire budget



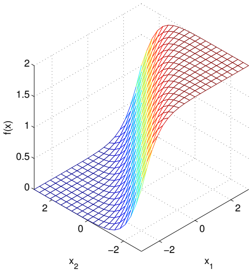
Example: Hypertangent response

We examine the quantile of the output of a hypertangent function:

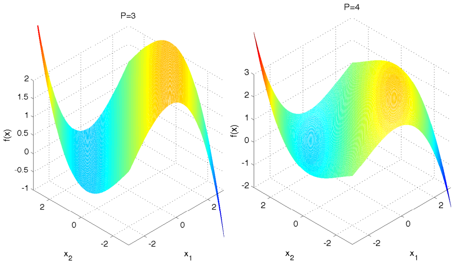
$$Y(\mathbf{x}) = 1 + \tanh\left(\sum_{n=1}^{N} \sigma_n (x_n - \mu_n)\right).$$

where the N–dimensional input are i.i.d. random variables $\mathbf{x} \in \mathcal{N}(0,1)$.

Double Peak Response

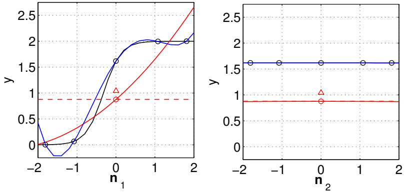


Global gPC Metamodel

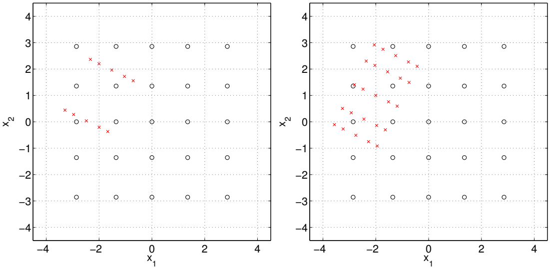


Anisotropic grid

- The dominance of some random variables can be revealed by examining the partial variance of the global gPC metamodel
- One–dimensional metamodels about $\hat{\boldsymbol{\xi}}_{\alpha,r}$ can identify dominant directions

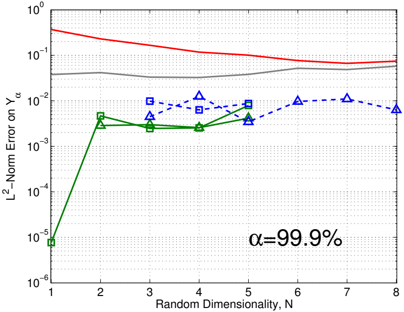


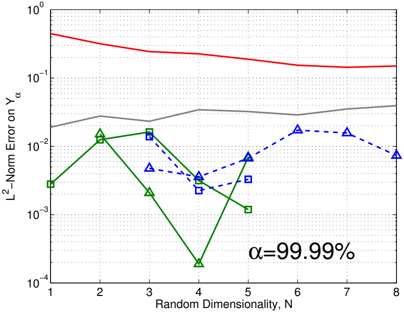
- Anisotropic grids, P in $\hat{\boldsymbol{\xi}}'_{\alpha,r}$ and linear in transverse directions, reduce cost



Target cost study

- An arbitrary target cost that increases linearly with N: $Z_{total} = 100N$
- Monte Carlo and importance sampling $\hat{Y}_\alpha$ with entire sampling budget
- Global **full** and **sparse** + local full canonical (□) and anisotropic (△) supplemental metamodels
- Maximize global metamodel cost while not exceeding the entire budget



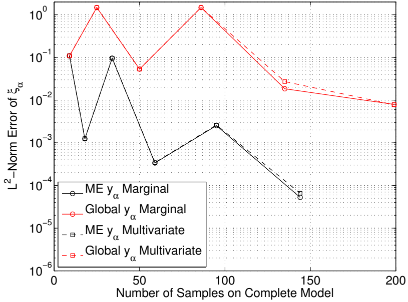


Quantile of multivariate output

We assume that all components of the random output $\mathbf{Y}$ are extreme and define the multivariate α–quantile as the point $\mathbf{y}_\alpha$ where the multivariate and marginal *cdf*'s satisfy the following conditions

$$F(\mathbf{y}_\alpha) = \alpha \quad \text{and} \quad F_1(y_{\alpha,1}) = F_2(y_{\alpha,2}) = \cdots = F_K(y_{\alpha,K}) \tag{3}$$

where K is the number of outputs. Results with N=2 and $\alpha = 99\%$ case for multiple Gaussian peaks:



Calibration and sensitivity analysis GCM

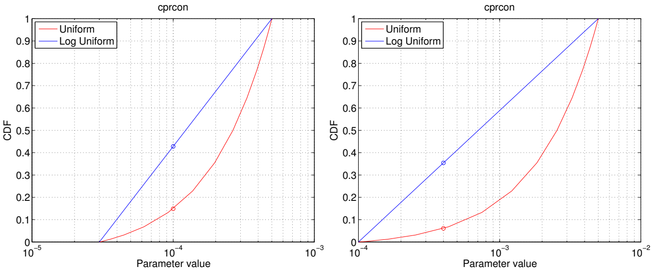
- Examine the AGCM ECHAM6 with uncertain parameters in cloud modeling
- 1977 climatological distributions of sea ice and surface temperature used as initial condition
- Five R.V. in the expert range transformed to the Gaussian space
- Ensemble of model output created for a *single* year run
- Full–tensor quadrature with squadratic accuracy, *i.e.* 243 points

Selection of the input random variables

Table: Expert parameter range and their default values

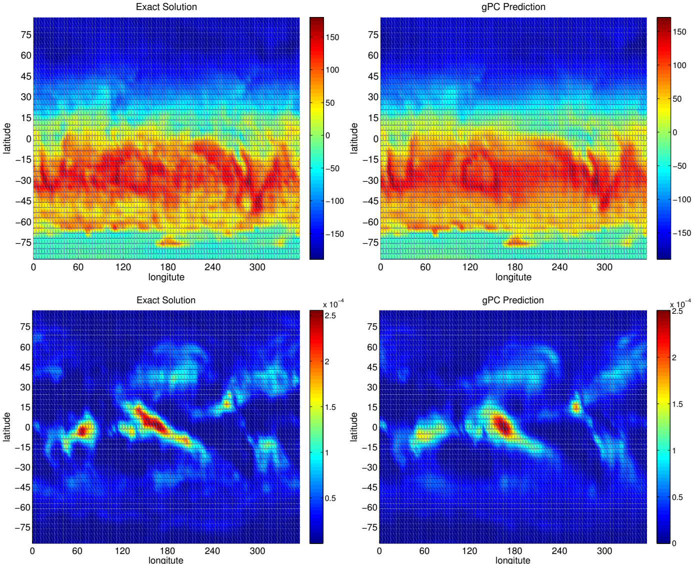
Parameter	Range	Default value
entrainment rate for shallow convection (entrscv)	0.0003-0.001	0.0003
entrainment rate for penetrative convection (entrpen)	0.00003-0.0005	0.0001
inhomogeneities of ice clouds (zinhomi)	0.65-1.0	0.7
inhomogeneities of liquid clouds (zinhoml)	0.65-1.0	0.7
conversion rate of cloud water to rain (cprcon)	0.0001-0.005	0.0004

- zinhomi & zinhoml are treated as uniform r.v.
- entrscv, entrpen & cprcon are treated as uniform r.v or log uniform r.v.
- A dependent parameter, $cmfctop = entrscv \times \frac{1000}{3}$, is included
- A uniform distribution under–weights the entire lower range



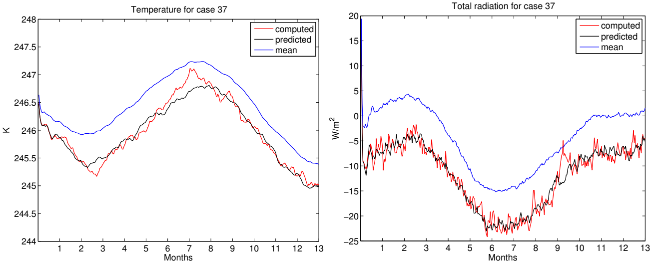
Validation: Comparison of computed global contours and gPC predictions

- Comparison at an arbitrary point within the support
- Exact solution vs gPC prediction for global radiation and precipitation
- For December 1970, large–scale patterns resolved in time-averaged results



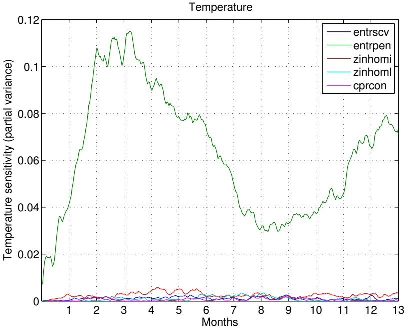
Validation: Comparison of computed global means and gPC predictions

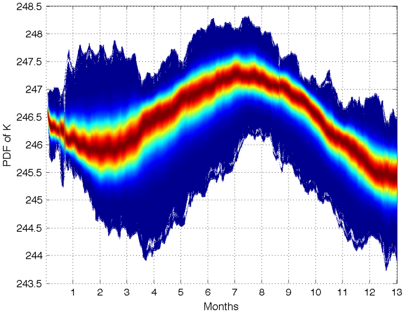
- Global mean should be consider to avoid small eccentric scales



Sensitivity analysis

- Partial variances reveal strong effects from 'entrpen'.
- Couple terms in the partial variance is much smaller
- Temperature PDF generated from the gPC metamodels with 10^5 Monte Carlo samples.





Code calibration

For optimization problem with K objective functions, we seek all the $\boldsymbol{\xi}$ that satisfy the following minimization problem, *e.g.*

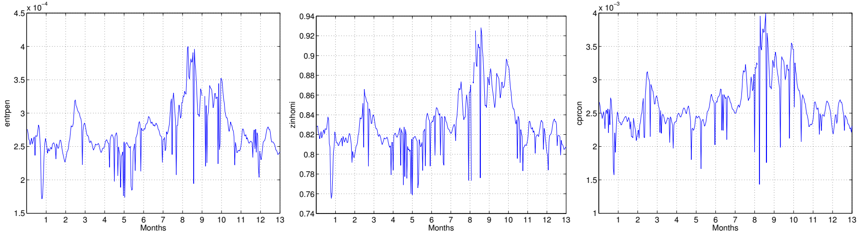
$$\boldsymbol{\xi}^* = \underset{\boldsymbol{\xi}}{\operatorname{argmin}} \sum_{k=1}^{K} \omega_k \left(\sum_{m=0}^{M} u_{m,k}(t)\phi_m(\boldsymbol{\xi}) - u_{\mathrm{obs},k}(t) \right)^2 \quad \text{for t} = 1,\ldots,364$$

The choice of weight vector $\boldsymbol{\omega}$ is arbitrary. Many optimization algorithms exist. So far K=1

- Lagrange multiplier algorithm used to solve the constraint nonlinear minimization problem for global averaged temperature
- u_{obs} are the daily global averaged temperature in 1970 from ECMWF
- the following figures show the daily 'optimal' value for each parameter
- with additional objective functions, there is likely to be non-dominant sets, *i.e.* one cannot make one objective better without worsening the other objectives Neelin (2001)

Calibration results

Parameter	Range	Default value
entrainment rate for shallow convection (entrscv)	0.0003-0.001	0.0003
entrainment rate for penetrative convection (entrpen)	0.00003-0.0005	0.0001
inhomogeneities of ice clouds (zinhomi)	0.65-1.0	0.7
inhomogeneities of liquid clouds (zinhoml)	0.65-1.0	0.7
conversion rate of cloud water to rain (cprcon)	0.0001-0.005	0.0004



Some concluding remarks

- PC and gPC constructs metamodels that accurately mimics the behaviours of complete simulators about the mean of the stochastic inputs
- Initial used as a UQ ans SA tool in engineering problems
- It has potential as a multi–objective optimization tool
- There is no free lunch – it suffers from the "curse of dimensionality"
- Adaptive techniques (multi–element, anisotropic quadrature) can reduce cost
- To investigate anisotropic spare quadrature & sparse gPC representation
- Reduce input dimension via non–dimensional analysis or identification of dominant inputs
- Orphan points (difference between sample budget and quadrature cost) - can we use them in a sequential design – with Hugo?
- Including data assimilation and Bayesian analysis in gPC/PC framework
- Practical issues: need better random input measurement