An Empirical Comparison of the Performance of Classical Power Indices

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Power indices are general measures of the relative *a priori* voting power of individual members of a voting body. They are useful for both positive and normative analysis of voting bodies particularly those using weighted voting. This paper applies new algorithms for computing the rival Shapley-Shubik and Banzhaf indices for large voting bodies to shareholder voting power in a cross section of British companies. Each company is a separate voting body and there is much variation in ownership between them resulting in different power structures. Because the data are incomplete, both finite and 'oceanic' games of shareholder voting are analysed. The indices are appraised, using reasonable criteria, from the literature on corporate control. The results are unfavourable to the Shapley-Shubik index and suggest that the Banzhaf index much better reflects the variations in the power of shareholders between companies as the weights of shareholder blocs vary.

The Measurement of Voting Power

Many organizations have constitutions for their governance based upon systems of voting in which different members possess different numbers of votes. There are various reasons for designing voting bodies in this way. International organizations like the Council of the European Union have adopted weighted voting systems in order to combine the simplicity of majority voting with a need to ensure populations of different sizes in different countries are represented appropriately. International economic organizations such as the Bretton Woods institutions allocate votes to members on the basis of their financial contributions rather than their populations but the same principle of weighted majority voting is at the centre of their governance. Likewise a joint stock company allocates votes at company annual meetings and proxy votes to members in accordance with their financial commitment in share ownership.

In all these cases the rationale for the allocation of different voting weights to different members is the idea that this is also an allocation of voting power. It is the popular view that voting weight equates to voting power and indeed in many academic discussions of the voting systems in political organizations writers often refer to voting weights as power. For example the USA has about 18 percent of the voting weight in the International Monetary Fund and World Bank, and is often described as having 18 percent of the voting power.[1] In fact, in terms of its share of influence over the decisions taken by votes among the members, its power may be quite different from 18 percent. In general the number of votes a member casts bears little relation to the power that those votes represent. A member's power actually depends not only on its own voting strength but also on the complete configuration of votes allocated to all the other members as well. In order to

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measure power it is necessary to calculate a power index that takes account of all of them.

A power index measures each member's relative influence over decision making in the sense of its ability to use its vote to change a coalition of others from one which is losing to one which satisfies the majority requirement and wins. The number of times it can do this is expressed as a proportion of the number of voting outcomes that can occur, treating each outcome equally; this is usually expressed in probabilistic terms with voting outcomes assumed to be random and equally probable. Outcomes are defined in terms of dispositions of individuals on one side or the other in a general vote without reference to preferences. A power index therefore measures the power of individuals in an *a priori* sense within a particular voting system with given distribution of weights among members and majority requirement. It cannot predict the result of any particular vote but can be useful in helping understand or design a voting body in terms of the relative voting power of different members. (Good overviews of the field are Felsenthal and Machover, 1998; Straffin, 1994; Lucas, 1983).

This approach to the measurement of voting power is limited by the fact that different probability models for treating the outcomes of votes have been proposed. This has led to two different 'classical' indices being defined, the Shapley-Shubik index, and the Banzhaf index (Shapley and Shubik, 1954; Banzhaf, 1965). These indices give results that differ, sometimes substantially, when applied to the same data, and both sets are often presented side by side with the final choice being left to the reader. The resulting ambiguity has led to the power indices approach lacking credibility which has undoubtedly hindered its wider use for understanding and designing voting bodies such as those of the European Union.

The properties of the indices have been compared in theoretical terms against various criteria but the question of which is better is still an open one. Answering it is fraught with methodological problems, as is often the case when studying power. Ideally what would be required would be some independent evidence on the distribution of power in the particular voting body with which to compare the results for the indices. However this is very rarely available.

One area of application where there is a certain amount of independent evidence is shareholder power in companies. Case studies of actual firms and market experience have created a body of evidence on the relations between the structure of share ownership and the real power that particular shareholdings represent. It is therefore possible to identify firms in which a large shareholder is so powerful as effectively to be in control and whose power index one might reasonably expect to be very large. In others a large shareholding might be powerful if the other shares are widely held in many small holdings, but not powerful if they are held in larger blocks. Thus by looking to see how the distribution of share ownership in a cross section of companies varies and relating this to the corresponding power indices, it is possible to test the power index approach and also compare the different indices.

This paper computes power indices for shareholders in a large sample of British companies and tests them against suitable criteria in terms of how they are likely to behave as patterns of ownership vary across companies.[2] The results do not give any reason to reject the Banzhaf index but the results for the Shapley-Shubik index do not satisfy the criteria. The paper therefore leads to a definite conclusion: that the Shapley-Shubik index is not a suitable measure of power in this case.

The paper is organized as follows. First the power indices are defined in general terms and the comparative theoretical literature discussed. There is then a discussion of the methodological problems of testing them, a description of the methodology of this study, followed by a description of the criteria to be used. Next the data set is described before the power indices are defined precisely taking into account the nature of the data available. Because shareholding data is inevitably incomplete for each company – there are many thousands of shareholders and only the largest few are observed – this means the indices cannot be computed, even in principle, for the full voting body. This problem of incomplete data is dealt with by considering two cases that can be regarded as extremes. In the first case, the unobserved shareholdings are taken as being held in as few hands as possible; this defines a finite 'game' with hundreds of players where ownership is relatively concentrated. In the other case, the unobserved shareholdings are taken to be held by an infinity of players each holding an infinitesimal share; this defines an 'oceanic' game. For each case, both indices are calculated for the players whose shareholdings are actually observed and the results appraised in the light of the criteria that have been defined.

General Definition of Power Indices and Theoretical Comparisons

The model underlying the measurement of voting power is a game played by n players who co-operate by forming coalitions[3] by simply casting their votes; that is, they vote for or against a motion in a hypothetical meeting. Coalitions may be winning or losing according to the rules of decision making in the meeting and each member's power is then measured by his or her ability to influence the outcome by changing a coalition from losing to winning by voting with it rather than against it, an effect referred to henceforth as a swing. A power index is calculated for each player by considering each possible coalition of which he or she is not a member and evaluating the number of swings. The indices are not described in full detail here but in a later section after the discussion of methodology and the description of data because the form of the data available affects the precise nature of the games studied and their computational details. Before that I consider some theoretical aspects of the indices' comparative properties which have been discussed in the literature.

The classical power indices are derived from fundamentally different conceptions of the relationship between voting and power. The Shapley-Shubik index is derived as a special case of the Shapley value in co-operative game theory (Shapley, 1953), which assumes that the members of a winning coalition in a general n-person game divide up the spoils of victory among themselves by a process of bargaining, each player's share reflecting his or her contribution. The value is the expected value of the game to each player with respect to a model of random coalition formation and a characteristic function reflecting the payoff received by each coalition. The

Shapley-Shubik index is a special case of this in which the characteristic function specializes to having a value of either 1 or 0, according to whether the coalition wins or loses. Thus the assumption is that a winning coalition always has the same total value of 1 to be divided out among its members according to their respective power. On the other hand, the Banzhaf index has no such association with bargaining.

This led to the proposal that different *kinds* of power might actually be measured by the different indices (Felsenthal *et al*., 1998). The Shapley-Shubik index is associated with 'office-seeking' behaviour in which the process of constructing a winning coalition to attain power is accompanied by bargaining over how the spoils of office are to be distributed. The Banzhaf index has been associated with 'policy-seeking' behaviour where winning a vote means controlling the actions of the organization; the consequences of winning are in the nature of a public good, of which each member receives a fixed benefit, which cannot be redivided after bargaining among members of the winning coalition. This distinction becomes one between power as a prize, P-power, measured by the Shapley-Shubik index, and power as influence, I-power, appropriately measured by the Banzhaf index. In their book, however, Felsenthal and Machover express some scepticism about the concept of power as a prize (Felsenthal and Machover, 1998).

In his seminal paper, Coleman (1971) made a detailed critique of the realism of the behavioural assumptions behind the Shapley-Shubik index. He argued against an approach derived from a bargaining model on the grounds that policy-seeking behaviour is by far the more usual. He made two major specific criticisms of the realism of the model underlying the Shapley-Shubik index. First Coleman argued that there is little basis for the assumption that basic voting outcomes should be thought of as different *orderings* of players along a continuum. This assumption is the basis of the uniqueness property of the Shapley-Shubik index because it ensures that in any vote taken there is precisely one player whose contribution is pivotal. However it means that coalitions with different numbers of members are assigned different probabilities. A coalition with t members (in a game with n players) which player i can join to make it winning has a weight, in the definition of the index, of $t!(n-t-1)!/n!$. This expression is at its largest when t is very small or very large and at its smallest when t is in the middle of its range. This means that a coalition with a very few or a very large number of members dominates in the calculation of the index. It is not intuitively clear why this should be when the basic idea is to measure the ability of a player to change losing to winning and the size of coalition might seem to be irrelevant.

The second criticism Coleman made is of the idea that the Shapley value can be adapted to give a measure of power by assuming a special form for the characteristic function, 1 for a win and 0 otherwise. This is a quantitative representation of qualitative outcomes, the distinction between 0 and 1 representing the distinction between losing and winning, without any attempt at attaching any further value to those events. It is not appropriate therefore to treat these particular values assigned in this way as if they were *actual* quantities to be divided among members. Yet that is what this approach does. Moreover it assumes that the same amount, represented by 1, is available to the winning coalition each time there is a vote.

But for each vote the consequences are different and fundamentally incommensurable and this is not captured by this simple form of the characteristic function. As Coleman put it: 'The action is ordinarily one that carries its own consequences or distribution of utilities, and these cannot be varied at will, i.e. cannot be split up among those who constitute the winning coalition. Instead, the typical question is the determination of whether or not a given course of action will be taken, that is, the passage of a bill, a resolution, or a measure committing the collectivity to an action'. The Shapley-Shubik index is based on an unjustified step from first representing a qualitative event in quantitative form in the characteristic function then interpreting that as a pure quantity to be distributed in the definition of the value/index.

Other important theoretical comparisons of the indices are by Roth (1977), who showed that differences between them can be interpreted as reflecting different attitudes towards risk, and Straffin (1977), who characterized them in terms of probabilistic voting with different mechanisms for choosing the voting probability. Felsenthal *et al*. (Felsenthal *et al*., 1998) compare the indices in terms of a number of suggested theoretically desirable properties and conclude with some reservations about the Shapley-Shubik index.[4]

The Empirical Methodology

Most of the comparisons of power indices have been theoretical discussions of their foundations or properties. This section and the next describe the essentially empirical methodology used in this paper.

In order to test the performance of an index it is necessary to compare the value it assigns to a player's power with an independent measure of the actual power the player has. If, say, an index for a player were equal to 0.75, and if independent evidence suggested that this player might be able to swing about three quarters of votes, then we would be inclined to accept the index as empirically useful. On the other hand, if the player were known to have virtually no power, or to be so powerful as to be a virtual dictator, then we would be inclined to reject it. If, when assessing the comparative performance of two indices, one of them gave results different from the independent evidence and the other gave results which were not inconsistent with it, this would lead to rejection of the former and the non-rejection of the latter. If, in a comparison of the indices over a large number of games, the results obtained for one index were often or sometimes inconsistent with the independent evidence, while those obtained for the other index were never shown to be inconsistent with it, then this would suggest a clear preference for the latter and rejection of the former. The difficulty with this approach is that independent evidence of the type required is hard to gather because it is difficult to observe power empirically.

The general question of studying power is discussed at length by Morriss (1987). He argues, first, that power cannot be observed directly and that any evidence must be used in indirect ways; 'there is no easy mechanical way of establishing how much power someone has and the connection between the assertion that someone has power and the evidence for it is often complex and subtle'. Second, the interpretation of observed data depends on a theory of social processes. Third, he argues that power should best be studied using a variety of approaches within a regime of what he calls 'methodological tolerance'; specifically he maintains that research into power should not be confined to the use of 'hard' evidence.

Morriss suggests that there are five general approaches to gaining evidence about power that should be used in conjunction with each other:

(1) Experiments;
(2) Thought Experiments: considering the obvious;
(3) Natural Experiments: looking at naturally occurring situations in the real world;
(4) Consulting Experts: getting others to conduct experiments; consulting practitioners whose opinions might be informed by practically gained evidence;
(5) Resource-based Approaches: looking to the resources of players to give indications of power.

The approach adopted in this paper is a combination of (3) natural experiments, (4) the opinions of experts and (5) resource-based approaches. The research is not experimental in a direct sense. Thought experiments might be appropriate to situations where the conclusion is obvious, for example, where one of players has all the power and is a dictator, or where all players obviously have equal power despite different weights. But in these situations there is not a problem of choice of power index and so thought experiments have little to contribute to the main problem being considered.

Resource-based approaches involve examining the resources on which power is based, for example, using relative sizes of armed forces to compare military power of different countries. In the case of weighted voting the weights are the resources and power is monotone in a player's weight. But the numerical value of the power index depends also on the weights of all other players. Where player 2 (with the second-largest weight) has a large weight then player 1 (the largest player) is likely to have a smaller power than where player 2 has a smaller weight, although not in every case, depending on the full constellation of weights of all players; however player 1 is never less powerful than player 2.

Natural experiments are important in non-experimental social science where empirical research is largely based on passive observation of naturally occurring situations. The appraisal of the performance of power indices can be based on such data. One approach would look at a given voting body such as a legislature over a period of time and relate the variation of power indices for parties or voting blocks to independent indicators of the real disposition of power.[5] It would need to take account of other factors in the political process such as personalities of politicians and also of changing circumstances over time however which would be hard to control for.

Another approach studies a cross section of otherwise similar voting bodies with different structures in terms of weights and relates the indices to independent information on power. This is the approach adopted here using information from studies of company control by large shareholders through their voting weight as the basis of knowledge of actual power. This relies on the opinions of experts as well as case studies to indicate when there is control of the company by a particular identified shareholder and when the company is not so controlled. This evidence on company control is taken as indicating a power index that is very close to unity; how this varies in practice is the basis of the appraisal criteria.[6]

It is sometimes suggested that use be made of statistics of voting at company annual general meetings and in proxy voting to measure the power of shareholders. The argument is that the more powerful shareholders will be seen as determining the outcomes of more votes and observation of the frequency with which this occurs will provide a direct test of the power index. However power is not exercised in such an overt manner at company meetings; they are in any case often not well attended, at least partly because of this. Many fund managers with substantial holdings have tended to avoid voting as a matter of policy and either back incumbent management or use their influence to bring pressure behind the scenes. Their *a priori* voting power is nevertheless real and an important determining factor in the firm's governance. Power is important at the agenda-setting stage outside the company meeting and may not be observed directly or even at all. A powerful shareholder may be able to effectively control the company through informal contact with management on the basis of a common perception of the power represented by the shareholder's large voting weight. Proxy votes can provide evidence on power but they are too infrequent to be a source of simple statistical data. They have provided useful evidence however which informs the appraisal criteria described in the next section.

The question arises as to whether it is appropriate to model shareholder voting in terms of power indices. Power indices are about the measurement of *a priori* voting power whereas shareholders are usually regarded as solely interested in one issue: maximizing the value of their investment. If there were to be a vote on the simple question of whether an action should be taken which unambiguously increased the value of shares then all shareholders would be expected to vote the same way. It is therefore to be expected that such questions would never be put. Where votes are taken it is likely that the question is one of fundamental uncertainty where the consequences are unknown, such as between one proposed restructuring or merger and another, both of which are advocated by their proponents as being in the best long term interests of shareholders, or between two nominees for a director's position. Many such cases occur where the shareholders have the ultimate responsibility for making the decision.

Where the approach clearly *cannot* be applied is to a situation in which the question to be decided is the division of the profits among the shareholders, since that might lead to the absurd inference that a large minority shareholder, whose power is close to 1, is so powerful as to be able to expropriate the majority. The context therefore is a company whose shares are publicly traded in which powerful shareholders are prevented from being able to appropriate private benefits of control by high standards of corporate governance. This assumption means that the concept of power here is policy-seeking rather than office-seeking because the distribution of the benefits of winning a vote are fixed.

Appraisal Criteria in Terms of Shareholder Voting Power

An important study of power in real-world weighted voting games was the analysis of the ownership and control of large American corporations in the 1920s by Berle and Means (1932). They showed that the ownership of most corporations had become widely dispersed among a very large number of shareholders. The principal shareholdings of many large corporations were, in percentage terms, very small indeed, in many cases less than 1 per cent of the voting stock. From this Berle and Means inferred that in these cases no shareholder was sufficiently powerful to be able to exercise much influence over the company through voting at company meetings or in proxy votes and that there was a factual separation of ownership and control.

Not all companies had such dispersed ownership, however, some retaining a majority controlling shareholder, some having a large minority shareholder who was dominant and others were controlled by a legal device, such as a pyramidal structure or dual class shares, rather than through the sheer size of a shareholding voting block. Berle and Means defined a typology of control by which they attempted to classify every corporation. It is not necessary to describe this fully for present purposes, merely to confine attention to the voting power represented by a large holding of ordinary shares. The key distinction Berle and Means made was between a company with a substantial minority shareholding which was very powerful in voting terms and one without such a dominant shareholding. They deemed a minority shareholder to have 'working control' if it had 'sufficient stock interest to be in a position to dominate a corporation through their stock interest', and '... the ability to attract from scattered owners proxies sufficient when combined with their substantial minority interest to control a majority of the votes at the annual elections. Conversely, this means that no other stockholding is sufficiently large to act as a nucleus around which to gather a majority of the votes'. This definition corresponds to that of a voting weight with a very large power index.

Berle and Means used case studies to examine voting power in the former case by considering proxy voting. They used two sorts of evidence to examine if a minority shareholder had enough votes to be in effective control: whether a minority shareholder had been able to win a proxy fight with a minority of shares; and cases of stable ownership structures in which a minority shareholder was demonstrably in control without explicit voting taking place, based on information from press reports and elsewhere to determine control. In other cases they used information from others, in the form of what they called 'street knowledge' – that is Wall Street knowledge obtained from stockbrokers and others – which is a version of Morriss' approach of asking experts. Berle and Means were very careful in their research into voting power. They did not infer shareholder control unless they were confident they could observe it, albeit indirectly. They decided on the basis of their case studies that the dividing line between a shareholding being sufficiently large to have working control through voting and the complete separation of ownership and control was about 20 per cent of the voting capital, although this could vary, in some cases the figure being rather less. We would therefore require of a suitable index that in most cases where there is such a shareholding it assign a very high power to it, and the firm to be classifiable as minority controlled, although we might also expect there to be exceptions.[7]

Another, more recent, source of indirect information about the voting power of large shareholders is the opinion of practitioners in the world of corporate finance who may be using substantial amounts of their own money, or at least of their clients' money, to back their judgements (and whose reputations and careers certainly depend on them). This is another form of 'asking experts'. One manifestation of this is in the listing rules of the London Stock Exchange (the so called Yellow Book) – which are based on a body of opinion widely accepted by its members – which use, as a formal definition of a controlling shareholding, one which controls 30 percent or more of the votes at a company meeting.[8] Another authority which uses the 20 percent rule as the basis of ownership control is the recent extensive study by La Porta *et al*. (1999) who state simply, without giving any other authority: 'The idea behind using 20 percent of the votes is that this is usually enough to have effective control of the firm'.

This is the basis of the method used here. Minority control in the Berle and Means sense is identified with a very high value of the power index, close to 1, for the largest shareholder, player 1. Using the insights of Berle and Means it is possible to suggest some reasonable criteria which the indices should satisfy. These are as follows.

Appraisal Criteria

(1) The power index for player 1 should vary as voting weights vary.

(2) The power index for player 1 should vary as the weights for players 1 and 2 (w_1 and w_2) vary between companies. It should increase with w_1 and is likely to decrease as w_2 increases.

(3) The power index for player 1 should almost always be close to 1 whenever the weight for player 1 is above 30 per cent.

(4) The power index for player 1 should often be close to 1 whenever the weight for player 1 is between 20 per cent and 30 per cent.

(5) The power index for player 1 should sometimes be close to 1 whenever the weight for player 1 is between 15 and 20 per cent.

(6) The power index for player 1 is virtually never close to 1 when the weight is less than 15 per cent.

Notation

It is assumed that a company has n shareholders whose individual holdings (voting weights) are denoted $w_1, w_2, \dots, w_n$, where $0 < w_i < 0.5$ for all i and $\Sigma w_i = 1$. For convenience I assume the weights are ordered in decreasing order of size, so that: $w_i \geq w_{i+1}$ for all i. Votes are taken with a decision rule in terms of a quota q.[9] Sometimes it is necessary to refer to the collective size of a group of players with the largest weights. This is represented by s_j, where $s_j = \sum_{i \leq j} w_i$, the collective weight of the largest j players.

Voting outcomes are defined in terms of coalitions which are represented by subsets of the set of all players, $N = \{1, 2, \dots, n\}$. All members of a subset are assumed to cast all their votes in the same way. Let the total combined voting weight of all players in a subset T be $w(T)$; that is $w(T) = \sum_{i \in T} w_i$. If T is a winning coalition $w(T) \geq q$ and for a losing coalition $w(T) < q$.

The power indices are defined in terms of swings: losing coalitions that become winning when a particular player joins. Thus a swing for player i is a losing coalition, T_i, such that $q - w_i \leq w(T_i) < q$.

The Data Set: Share Ownership in Large British Companies

The data consists of ownership data on a cross section of 444 large British companies, mostly taken from the Times 1,000. For each company all shareholdings above 0.25 percent of the voting equity in 1985 or 1986 were collected.[10] The number of such large shareholdings observed varies in the sample between a minimum of 12 and a maximum of 56, with a median of 27. The equity shares represented by these observed shareholdings vary between 19 percent and 99 percent, the median being 66 percent. The dataset is therefore both detailed and fairly comprehensive in giving a picture of British firms.[11]

The data are summarized in Table 1. The table shows the distribution of the size of the largest shareholding, w_1, and also the joint distribution of w_1 with the second-largest holding, w_2, in order to indicate the variation in patterns of concentration of ownership in the sample. This variation makes it ideal for the study. Some 49 companies have relatively concentrated voting structures with w_1 greater than 30 percent, but in the great majority of cases w_1 is less than 30 percent. There is also a wide range of variation in the size of w_2 given w_1. For example in the group of 85 companies where w_1 is between 20 percent and 30 percent, w_2 is less than 10 percent in 38 cases, between 10 percent and 20 percent in a further 38 cases and greater than 20 percent in 9 cases; this is expected to give rise to a wide range of power distributions as given by the indices.

Table 1: The Data: The Largest Holding versus the Second Largest (Number of Companies)

Voting weight of second largest shareholder (w_2)	*Voting weight of largest shareholder (w_1)*						
	<5%	5–10%	10–20%	20–30%	30–40%	40–50%	Total
<5%	41	46	15	12	2	2	118
5–10%		98	73	26	10	9	216
10–20%			37	38	11	5	91
20–30%				9	4	2	15
30–40%					3	1	4
40–50%						0	0
All firms	**41**	**144**	**125**	**85**	**30**	**19**	**444**

The Problem of Incomplete Data

Information collected about company ownership is necessarily incomplete because of the very large number of shareholders there are in a typical large public company. Normally only the observations on a few of the largest shareholdings are easily available to researchers and in any case this is often all that is used in discussions of ownership and control.[12] It is, however, central to the approach adopted here that the power of the largest shareholder depends not only on its own size but also on the dispersion of the other, smaller holdings and it is necessary explicitly to deal with these in the way in which the indices are defined and computed. This incompleteness in the data therefore gives rise to important issues in deciding how to handle the missing observations.

The solution adopted here is to calculate two sets of indices, on two different assumptions about the unobserved weights – corresponding to two extremes of 'concentrated' and 'dispersed' ownership – both of which are arithmetically feasible given the observed data. The details are as follows. For any company the largest k shareholdings are observed and there is no information about the remaining $n - k$ holdings except that they are all smaller than w_k. Nor is n known, or needed; although the total number of shareholders could in principle be collected, it would add very little to the analysis. Two limiting cases are defined: the 'dispersed' case where w_i is assumed, for $i > k$, to be vanishingly small, and the number of shareholders to be infinite; and the 'concentrated' case where n is taken to be as small as possible consistent with the observed data. The former is referred to as limiting case D (Dispersed) and the latter as limiting case C (Concentrated).

For limiting case C it is necessary to adopt a value for n in the finite game. If w_k is the smallest weight observed in the data, then all the non-observed weights are no greater than w_k. The most concentrated pattern of ownership occurs when they are all equal[13] to w_k. Then the corresponding value of n, call it n', is defined as:

$$n' = g + k + 1$$

where g is the largest integer less than $(1 - s_k)/w_k$.

Let $w_i = w_k$ for all $i = k + 1, \dots, n' - 1$ and $w_{n'} = 1 - s_k - gw_k$.

Since the data in this study consist of shareholdings no smaller than 0.25 percent, $w_k = 0.0025$.

The Shapley-Shubik Index

The Shapley-Shubik index for a finite game is an n-vector γ. The index for player i, γ_i is defined as:

$$\gamma_i = \sum_{T_i} \frac{t!(n-t-1)!}{n!} \qquad i = 1, \dots, n \tag{1}$$

where the summation is over all swings for player i, T_i, t is the number of members of T_i, and n the number of members of N. It has a probabilistic interpretation as the probability of a swing for player i when the coalitions are formed by random orderings of the players, the term inside the summation being the probability of T_i occurring.

The direct evaluation of expression (1) is not feasible when n is large since it requires finding all subsets of N which are swings for each i: even for limiting case C the typical values of n are of the order of 300 or more and searching over all subsets of N would be prohibitively demanding of computer time. The values of γ_i are calculated in the two limiting cases using different approximation algorithms. For limiting case C I employ the method described in Leech (1998). This provides a very good approximation for this large finite game. For the limiting case D I assume an oceanic game and follow the approach of Shapley and Shapiro (1978).

The idea of an oceanic game seems to fit the current context very well: it is a game in which there are a finite number, m, of 'major' players with fixed voting weights, and a very large number (in the limit an ocean of 'non-atomic' players) with very small numbers of votes. Then as n goes to infinity the power index for player i converges on the value:

$$\gamma_i = \sum_{S \subseteq M_i} \int_a^b u^s (1-u)^{m-s-1} \mathrm{d}u \qquad i = 1, \dots, m \tag{2}$$

where $M = \{1, 2, \dots, m\}$, the set of major players, $M_i = M - \{i\}$, a = median $(0, (q - w(S))/(1 - w(M)), 1)$, b = median$(0, (q - w(S) - w_i)/(1 - w(M)), 1)$. In expression (2) the summation is taken over all subsets S of M_i and u is the dummy variable of integration. This expression is not difficult to evaluate requiring only a minor extension of the algorithm.[14]

The Shapley-Shubik index is found, for the finite game in case C, for every player with weights w_1 to w_k, $n' - k - 1$ players with weights w_k and player n' with weight $w_{n'} = 1 - s_k - (n' - k - 1)w_k$ and it sums to unity over all the n' players. For the oceanic game in case D, it is found for the m major players with weights w_1 to w_m, the indices summing to unity over all players. Here $m = 5$.[15]

The Banzhaf Index

The Banzhaf index is based on the idea of counting the number of swings in relation to all the possible voting outcomes, but the model of coalition formation underlying it is different from that associated with the Shapley-Shubik index, in that each coalition is given the same weight regardless of its size. That is, the way swings are counted in the index is different. The probability of any subset of N, say T_i, assuming random coalition formation, is now 2^{1-n} rather than $t!(n - t - 1)!/n!$. The probability of a swing for player i can then be written:

$$\beta_i' = 2^{1-n} \sum_{T_i} 1 \qquad i = 1, \dots, n \tag{3}$$

The summation is taken over all swings for player i. This is the Absolute (or Non-normalized) Banzhaf measure and it cannot be directly interpreted as giving a distribution of power among the players (as is conventionally a requirement for power indices) since in general it does not sum to unity. Introducing a normalization by defining

$$\beta_i = \beta_i' \Big/ \sum_i \beta_i' \qquad (4)$$

gives an index which does have this property but lacks the probability interpretation. This is the Normalized Banzhaf index.[16]

Computation of the Banzhaf index is easier than that of the Shapley-Shubik index. The algorithm for it described in Leech (1998) can be used for both limiting cases. Limiting case C is the finite game with n' players as described above. For limiting case D it is necessary to compute the values for the oceanic game. Banzhaf indices for oceanic games were studied by Dubey and Shapley (1979) who showed that under suitable conditions they can be obtained as the Banzhaf indices for the modified, finite game consisting only of the major players M with weights $w_1, w_2, \ldots, w_m$ and quota $q - (1 - w(M))/2$.[17] These indices are obtained by applying the same algorithm to this modified game. Here the set of major players M can be taken as all the observed shareholders, $M = \{1, 2, \ldots, k\}$. The results obtained are somewhat more sensitive than the Shapley-Shubik indices to which limiting case is assumed.

Results for Illustrative Companies

Tables 2 and 3 present power indices for some illustrative companies. The firms have been selected to span the range of variation in the first two shareholdings. Plessey has the most dispersed ownership with w_1 under 2 percent and Associated Newspapers is just short of being majority controlled. Two firms have been selected from each range of values for w_1:10–20 percent, 20–30 percent, 30–40 percent, 40–50 percent; these have relatively large and small values for w_2. Power indices are given for representative shareholders.

Table 2 shows the Shapley-Shubik indices. The index for case C assumes an extreme in which all the non-observed holdings are 0.25 percent, overstating concentration and understating the number of shareholders. For the most widely owned company, Plessey, for example, the voting game which is used as the basis of these indices is one with only 344 players. Case D assumes an oceanic game with 5 players with finite weights,[18] and the remaining votes widely dispersed among an ocean of infinitesimal holdings. For Plessey this consists of the five largest weights ranging from 1.94 percent to 1.05 percent, totalling 6.87 percent, and the remaining 93.13 percent distributed among the oceanic players.

The conclusions from Table 2 are, first, that despite the assumptions and methods of calculation used being so completely different for the two cases, the two sets of results are remarkably close. Secondly, the indices are relatively insensitive to the inequality in the data. Sun Life and Liberty, for example, both with w_1 around 22 percent, but very different values of w_2, 3.46 percent against 22.57 percent, have very similar results. The power of player 1 falls from 28 percent in Sun Life to just under 25 percent in the case of Liberty: it might be expected that the fact that the second-largest shareholding in Liberty was almost equal to the largest to have a profound effect. The third implication is the general insensitivity of the indices to the largest holding w_1. Although in every case power is more unequally distributed than ownership, the difference is never great. In terms of the appraisal

Table 2: Shapley-Shubik Indices for Illustrative UK Firms

Company	*Shareholder:*	*1*	*2*	*3*	*5*	*10*	*20*
Plessey	Weight	0.019	0.015	0.013	0.011	0.009	0.004
	Index (C)	0.020	0.015	0.013	0.011	0.009	0.004
	Index (D)	0.020	0.015	0.013	0.011		
Berisford	Weight	0.058	0.020	0.016	0.009	0.005	0.003
	Index (C)	0.061	0.020	0.016	0.009	0.005	0.003
	Index (D)	0.061	0.020	0.016	0.009		
United Spring	Weight	0.123	0.109	0.098	0.037	0.014	0.005
& Steel	Index (C)	0.134	0.117	0.103	0.036	0.014	0.005
	Index (D)	0.135	0.118	0.104	0.036		
Suter	Weight	0.128	0.065	0.053	0.031	0.107	0.009
	Index (C)	0.143	0.067	0.054	0.031	0.017	0.009
	Index (D)	0.144	0.067	0.055	0.031		
Sun Life	Weight	0.222	0.035	0.019	0.013	0.009	0.005
	Index (C)	0.283	0.033	0.017	0.012	0.008	0.004
	Index (D)	0.284	0.033	0.017	0.012		
Liberty	Weight	0.2263	0.2257	0.0894	0.0498	0.0181	
	Index (C)	0.2475	0.2465	0.0894	0.0460	0.0162	
	Index (D)	0.2486	0.2475	0.0922	0.0460		
Securicor	Weight	0.316	0.073	0.053	0.029	0.016	0.008
	Index (C)	0.448	0.059	0.043	0.023	0.013	0.006
	Index (D)	0.451	0.059	0.043	0.023		
Bulgin	Weight	0.310	0.222	0.045	0.028	0.009	0.003
	Index (C)	0.356	0.174	0.049	0.028	0.009	0.003
	Index (D)	0.355	0.172	0.049	0.029		
Ropner	Weight	0.410	0.060	0.050	0.020	0.012	0.003
	Index (C)	0.676	0.029	0.025	0.011	0.007	0.002
	Index (D)	0.680	0.028	0.025	0.011		
Steel Brothers	Weight	0.425	0.213	0.038	0.030	0.007	0.003
	Index (C)	0.616	0.055	0.035	0.028	0.006	0.002
	Index (D)	0.618	0.052	0.035	0.028		
Associated	Weight	0.4995	0.026	0.021	0.021	0.013	0.006
Newspapers	Index (C)	0.9839	0.0003	0.0003	0.0003	0.0003	0.0002
	Index (D)	0.9976	0.0000	0.0000	0.0000		

criteria, where a large power index for player 1, γ_1, would be expected, as in Securicor or Ropner, it remains very far below 1. This pattern is typical of the whole sample and indicates that the Shapley-Shubik index seriously understates the power of the largest shareholder in such cases. The conclusion is that the Shapley-Shubik index fails to satisfy criteria (1), (3) and (4).

Table 3 shows the corresponding Banzhaf indices for the same 11 illustrative companies.[19] These are always greater for limiting case D than for case C. Their values are sensitive to differences in ownership structure and vary considerably. Where

Table 3: Banzhaf Power Indices for Illustrative UK Firms

Company	Shareholder:	1	2	3	5	10	20
Plessey	weights	0.019	0.015	0.013	0.011	0.009	0.004
	Index (C)	0.020	0.015	0.013	0.011	0.009	0.004
	Index (D)	0.087	0.065	0.055	0.045	0.037	0.017
Berisford	weights	0.058	0.020	0.016	0.009	0.005	0.003
	Index (C)	0.080	0.019	0.016	0.009	0.005	0.003
	Index (D)	0.528	0.059	0.055	0.036	0.021	0.010
United Spring & Steel	weights	0.123	0.109	0.098	0.037	0.014	0.005
	Index (C)	0.143	0.124	0.112	0.033	0.013	0.005
	Index (D)	0.233	0.202	0.183	0.052	0.021	0.007
Suter	weights	0.128	0.065	0.053	0.031	0.017	0.009
	Index (C)	0.169	0.060	0.051	0.029	0.017	0.008
	Index (D)	0.270	0.093	0.080	0.046	0.026	0.013
Sun Life	weights	0.222	0.035	0.019	0.013	0.009	0.005
	Index (C)	0.981	0.0004	0.0003	0.0003	0.0002	0.0001
	Index (D)	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Liberty	weights	0.2263	0.2257	0.0894	0.0498	0.0181	na
	Index (C)	0.2025	0.2013	0.1121	0.0534	0.0189	na
	Index (D)	0.2348	0.2333	0.1312	0.0622	0.0220	na
Securicor	weights	0.316	0.073	0.053	0.029	0.016	0.008
	Index (C)	0.930	0.003	0.003	0.003	0.002	0.001
	Index (D)	0.971	0.002	0.002	0.002	0.001	0.001
Bulgin	weights	0.310	0.222	0.045	0.028	0.009	0.003
	Index (C)	0.372	0.059	0.053	0.034	0.011	0.003
	Index (D)	0.546	0.079	0.075	0.051	0.015	0.005
Ropner	weights	0.410	0.060	0.050	0.020	0.012	0.003
	Index (C)	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Index (D)	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Steel Brothers	weights	0.425	0.213	0.038	0.030	0.007	0.003
	Index (C)	0.9914	0.0004	0.0004	0.0004	0.0002	0.0001
	Index (D)	0.9994	0.0001	0.0001	0.0001	0.0000	0.0000
Associated Newspapers	weights	0.4995	0.0263	0.0213	0.0207	0.0128	0.0056
	Index (C)	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Index (D)	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000

ownership is dispersed as in the case of Plessey, power is almost equally dispersed. Where it is concentrated, as in Ropner, Steel Brothers or Associated Newspapers, with a shareholder with more than 40 percent having working control, the index reflects this. In other cases the Banzhaf index gives a richer variety of power distributions in response to differences in ownership structure.

A comparison of Sun Life and Liberty, for example, reveals a sensitivity of the index to the size of the second largest shareholding that the Shapley-Shubik index lacks. Shareholder 1 with a 22 percent weight in Sun Life has a Banzhaf power index of at least 98 percent, implying working control. In Liberty, however, both the largest two holdings are above 22 percent which must mean that player 1 is hardly more powerful than player 2. The Banzhaf index reflects this. A similar finding emerges for companies with a top shareholding of between 30 and 40 percent. A 31 percent shareholder in Securicor where there are no other large owners has over 93 percent of the voting power. On the other hand a similar-sized stake in Bulgin would have less than 55 percent of the voting power because of the presence of a large second shareholder with 22 percent. These results are entirely plausible in conforming with the appraisal criteria. The Banzhaf index generally is not in conflict with the appraisal criteria.

Power Indices for the Largest Shareholder in the Complete Sample

Table 4 gives the power of the largest shareholder according to both versions of each index for all the companies. Companies have been classified according to the appraisal criteria by the size of the largest shareholding. Because these are large companies, the great majority have relatively dispersed ownership and in 268 cases (more than half) the largest voting weight is below 15 percent. Nevertheless there is enough variation to compare the performance of the indices.

The Shapley-Shubik index does not meet the appraisal criteria. Only in 3 or 4 cases out of 19 where the largest holding is above 40 percent is it above 0.9, while the Banzhaf index exceeds 0.95 in all but one case.[20] A similar pattern is observed for the 30 companies where w_1 is between 30 and 40 percent: the Shapley-Shubik index is never above 0.8 and in half the cases it is below 0.5; the Banzhaf index varies considerably but in at least 11 cases it is greater than 0.95. For the group with w_1 between 20 and 30 percent, the Shapley-Shubik index is never above 0.5 while the Banzhaf index varies from 0.15 to 1. When the largest shareholding is less than 20 percent only the oceanic Banzhaf index is ever close to 1.

Figure 1 plots the respective power indices for the largest shareholding separately for each index.[21] Figure 1.1 shows the plot for the Shapley-Shubik index to be close to a simple functional relation. The scatter is in fact bounded above by the function $w_1/(1 - w_1)$ which is well known to be the value of the index for player 1 in an oceanic game with only one major player with weight w_1 (Shapley and Shapiro, 1978). Where the index is less than this value it is due to large weights for the other players. However the fact that power is almost always relatively low and that it exceeds 90 percent in only 3 or 4 companies means that this index fails to satisfy the criteria.

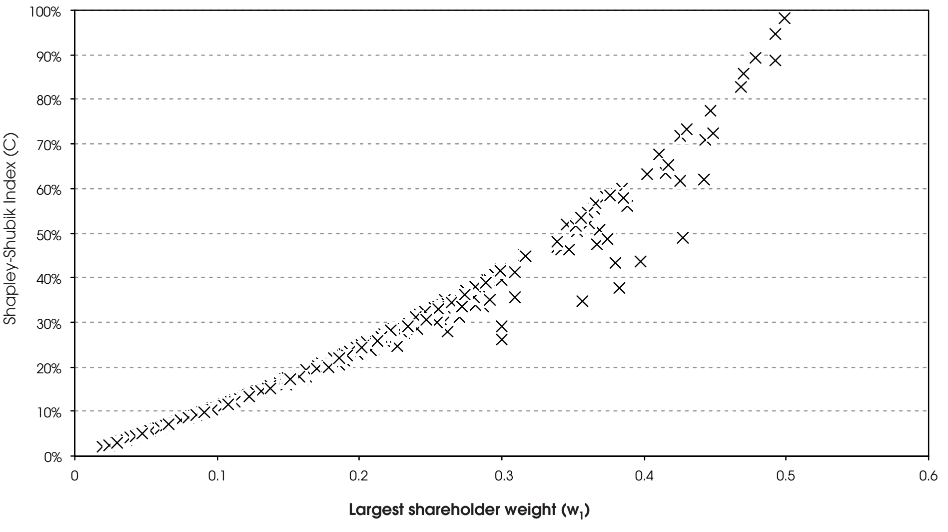
Figures 1.2 and 1.3 show the equivalent plots for the two limiting Banzhaf indices. Here there is much more variation consistent with the view that the index may be capturing well the effect of different ownership structures. There is very little effect up to about 15 percent, the index increasing with w_1, but after that power varies widely for a given value of w_1. These results suggest that shareholdings between 20 and 30 percent can be said to have voting control in many cases but equally not in many others. Most (but not all) holdings greater than 35 percent have an index almost equal to 1.

Table 4: Power Indices for Largest Shareholder, All Companies

	Shapley Shubik (C)	Shapley Shubik (D)	Banzhaf (C)	Banzhaf (D)
Largest shareholder (w_1) over 40%, (n = 19)				
Median	0.723	0.725	1	1
Minimum	0.490	0.490	0.575	0.671
Maximum	0.984	0.998	1	1
Index score is:		*Frequencies*		
<0.5	1	1	0	0
0.5–0.8	11	11	1	1
0.8–0.9	4	3	0	0
0.9–0.95	1	1	0	0
0.95–1	2	3	18	18
Largest shareholder (w_1) holds 30%–40%, (n = 30)				
Median	0.496	0.499	0.851	0.905
Minimum	0.348	0.348	0.262	0.318
Maximum	0.600	0.605	1	1
Index score is:		*Frequencies*		
<0.5	15	15	5	1
0.5–0.8	15	15	9	8
0.8–0.9	0	0	2	6
0.9–0.95	0	0	3	1
0.95–1	0	0	11	14
Largest shareholder (w_1) holds 20%–30%, (n = 85)				
Median	0.299	0.301	0.369	0.459
Minimum	0.225	0.226	0.157	0.224
Maximum	0.418	0.421	1.000	1
		Frequencies		
<0.5	85	85	55	46
0.5–0.8	0	0	19	19
0.8–0.9	0	0	4	6
0.9–0.95	0	0	3	4
0.95–1	0	0	4	10
Largest shareholder (w_1) holds 15%–20%, (n = 42)				
Median	0.206	0.207	0.219	0.296
Minimum	0.168	0.169	0.160	0.184
Maximum	0.243	0.245	0.652	0.998
Index score is:		*Frequencies*		
<0.5	42	42	39	32
0.5–0.8	0	0	3	6
0.8–0.9	0	0	0	1
0.9–0.95	0	0	0	0
0.95–1	0	0	0	3
Largest shareholder (w_1) holds under 15%, (n = 268)				
Median	0.088	0.089	0.092	0.153
Minimum	0.020	0.020	0.02	0.07
Maximum	0.174	0.174	0.331	0.745
Index score is:		*Frequencies*		
<0.5	268	268	268	264
0.5–0.8	0	0	0	4
0.8–0.9	0	0	0	0
0.9–0.95	0	0	0	0
0.95–1	0	0	0	0

Figure 1: Power Indices for Largest Shareholder versus Weight: All Companies

1.1 Shapley-Shubik Index: Limiting Case C



1.2 Banzhaf Index: Limiting Case C

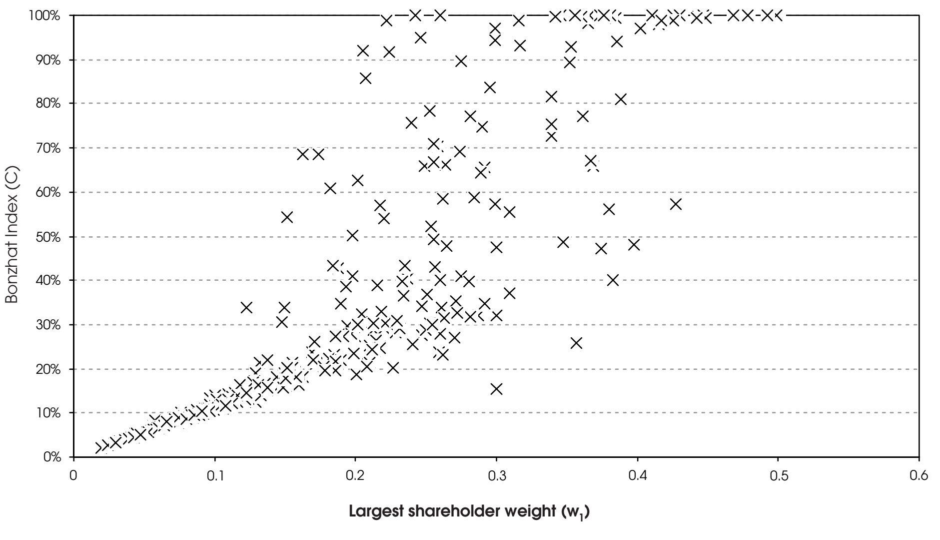
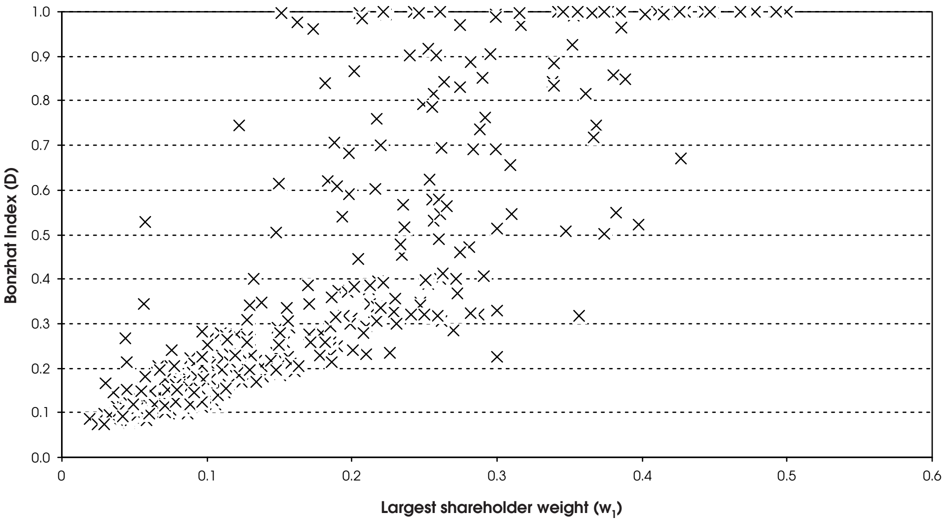


Figure 1: *Continued*

1.3 Banzhaf Index: Limiting Case D



Conclusions

This paper has reported on an exercise in the use of voting power indices to measure formal shareholder voting power in a large sample of companies, with a view to comparing the performance of the two 'classical' power indices. The criteria by which the indices have been appraised are based on independent analyses of shareholder voting power related to the separation of ownership and control. New accurate algorithms for computing power indices in large finite voting bodies and oceanic games have been applied.

The Shapley-Shubik index fails most of the appraisal criteria. The power index for the largest shareholder is almost never as high as independent evidence suggests it should be given the distribution of voting weights. Even when there is a dominant shareholder with at least 40 percent of the votes it is a lot less than 1. The index converges slowly to 1 as the largest shareholder's votes increase to a majority. The index is never very large in the important group of companies where the largest shareholder is above 20 percent, a case often regarded as having practical working control. On the other hand, the results obtained for the Banzhaf index did not fail to satisfy the appraisal criteria. The conclusion is that the Shapley-Shubik index should be rejected as an empirical measure and not to reject the Banzhaf index.

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Notes

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1 Though given that it requires an 85 percent majority for important decisions in the IMF, it actually has the power of veto. The inference is often drawn from this that it therefore has complete control but this is an over-simple inference since veto power is not control; the latter implies also the power to get ones own proposals agreed. Analysis of the voting system used by the IMF, based on the methods and results of the current paper, is in Leech (2002d). An analysis of the system of qualified majority voting in European Council of Ministers is in Leech (2002a).

2 It is of course possible to make the comparison using artificial data but it is felt that it gives much greater relevance by using real data containing real world complexity. Previous empirical applications of power indices to study shareholder voting power include Leech (1988), Pohjola (1988) and Rydqvist (1986); there are also several papers by Gambarelli (see Gambarelli, 1994).

3 The term coalition is used here to denote simply a group of players who vote in the same way on a particular ballot whether by prior arrangement or not. In politics the term is used for a more permanent arrangement where a group of members usually involving several parties make a prior agreement to vote together repeatedly.

4 See also Rapaport (1998) for a good account of the comparative theoretical properties of the indices.

5 Previous empirical tests of power indices include Riker (1959), Holler (1982) and Brams (1988). Riker studied the French National Assembly over 1953–54, looking at migrations of members from one party to another; his test was to use the Shapley-Shubik index to measure the power of the parties, assuming party discipline, and to see whether the deputies who migrated increased their power thereby. The results were negative. Holler tested the Banzhaf index using the Finnish parliament between 1948 and 1979 comparing the index with measures of real power such as participation in government. Brams found support for the Banzhaf over the Shapley-Shubik index as a measure of the relative power of the US House and Senate using data on the outcomes of legislative disagreements between them.

6 Voting by shareholders in the corporation has always been seen as an application of power indices. It was discussed at length in the seminal paper by Shapley and Shubik (1954), and developed in a subsequent paper by Shapley (1961), of which a condensed version was subsequently included as an appendix to Milnor and Shapley (1978).

7 Such would be where there are two or three very large holdings which would potentially control the company together but their rivalry, detected by the power index, prevents one of them having working control.

8 (London Stock Exchange, 1993). Presumably this should be interpreted as being either a single shareholder with 30 percent facing a large dispersed group of small shareholders, or two or more large shareholders and relatively fewer small shareholders. In the former case the power index would be close to 1 and in the latter case the number of player is so small that a controlling block can easily be formed.

9 It is conventional in the theoretical literature to require $q \geqslant 0.5$ to guarantee a unique decision, and that the voting game is 'proper'. (Strictly the number in this inequality should be slightly greater than 0.5.) In all the empirical work I take $q = 0.5$, which amounts to assuming that important decisions require a simple majority, which is the normal case with company meetings. There are exceptions to this, however, with some special decisions requiring a supermajority, but it is a broadly satisfactory assumption. In general ordinary decisions taken at company AGMs such as election of directors and passing of resolutions about the direction of the firm and which we might regard as bound up in the ordinary notion of control, are taken by simple majority. Games where $q > 0.5$ are not considered in this paper.

10 The source and method of construction of the data set are described in Leech and Leahy (1991). Companies with a majority shareholder were left out. The data can reasonably be regarded as representing beneficial shareholdings, since details of nominee holdings and names and addresses were used to identify ultimate owners and create blocks owned by linked or related individuals or insti-

tutions. There might remain a very slight underestimation of the true concentration of ownership to the extent this information was incomplete.

11 Analyses of the same firms, from the point of view of control, but using slightly different approaches to the measurement of voting power, are in Leech (2002b and c).

12 For example much international empirical work is based on ownership stakes greater than 5 percent and in Britain greater than the legally declarable level of 3 percent.

13 Strictly slightly smaller than w_k.

14 The algorithm combines the direct enumeration of the index with an approximation based on assuming the minor players vote probabilistically. It requires dividing players into an arbitrary number of major players and minor players, the latter being treated by an approximation. See Leech (1998). If it is assumed that the number of major players in the computation is the same as the number of major players in the oceanic game, expression (2) is easily found.

15 The value 5 for the number of finite players in this game was chosen for reasons of computational speed in calculating indices for all 444 firms but has little effect on the values obtained for the Shapley-Shubik indices. By changing the value of this number and re-calculating for the small sub-sample of companies reported here I have found the results to be practically invariant to the number of finite players.

16 Felsenthal and Machover (1998) reserve the use of the term 'power index' to one which is normalized. In this paper the same usage is employed and the comparison of properties is between the Normalized Banzhaf index and the Shapley-Shubik index.

17 This result depends on the quota q. For certain values of q the power indices are zero in the limit (the 'pitfall' points where the number of minor player swings become so numerous that the Banzhaf indices for major players go to zero). However in the cases studied here, with q always equal to $w(N)/2$, this problem does not arise.

18 See footnote 15.

19 The oceanic Banzhaf indices are found for all the observed weights, not just five.

20 This is the M and G Group in which $w_1 = 0.43$ and $w_2 = 0.32$, the top two shareholders between them holding 75 percent of the voting weight.

21 The oceanic Shapley-Shubik indices are practically identical to those for case C and are not shown.

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