SOCIAL CHOICE WITH FUZZY PREFERENCES

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1. INTRODUCTION

The most important concept of social choice theory is probably the concept of preference, be it individual preference or social preference. Typically, preferences are *crisp*, i.e., given by a binary relation over the set of options (social states, candidates, etc.). Then, for two options, either one is preferred to the other, or there is an indifference between them or there is no relation between them. In most cases, the possibility that there is no relation between them is excluded. The binary relation is then complete. Several authors, including Sen (1992), have considered that incompleteness was a way to deal with ambiguity. However, when the options are uncertain we must adopt preferences related in some way to probabilities and when they are complex, for instance when each option has manifold characteristics, we must try to take account of the vagueness this entails. To deal with vagueness, various authors have had recourse to, roughly speaking, three different techniques. The first is *probability theory.* Fishburn (1998) writes: 'Vague preferences and wavering judgments about better, best, or merely satisfactory alternatives lead naturally to theories based on probabilistic preference and probabilistic choice.' The second is *fuzzy set theory.* Each of these two theories has champions that apparently strongly disagree. There are, however, several signs indicating that the scientific war should eventually end (see Ross, Booker and Parkinson (2002) and the forewords in this book by Zadeh and by Suppes). The third is due to philosophers. Vagueness is an important topic within analytic philosophy (Keefe and Smith (1996), Burns (1991), Williamson (1994), Keefe (2000)). Philosophers have mainly been concerned with predicates such as tall, small, red, bald, heap, and solutions to the sorites paradox. (For instance, a nice example for the predicate 'small' is Wang's paradox discussed by Dummett (1975). Consider the inductive argument: 0 is small; If n is small, $n + 1$ is small: Therefore every number is small.) Although there is little agreement among them about the nature of vagueness, a theory has been designed to deal with it: supervaluation theory. In an interesting paper, Broome (1997) has jointly considered vagueness and incompleteness.

As social choice theorists, we are perhaps more interested in vagueness of relations than the kind of predicates mentioned above and most of the papers dealing with vagueness in choice theory have used fuzzy sets. The purpose of this chapter is to present some of the main results obtained on the aggregation of fuzzy preferences. Regarding the possible interpretations of preferences, it seems to us that we must confine ourselves to what Sen (1983, 2002) calls the *outcome evaluation.* Preference is about the fact that an option is judged to be a better state of affairs than another option. In Sen's typology, the two other interpretations are about choices, normative or descriptive. In particular, our analysis is not well adapted to voting, since the voters' ballot papers are not vague and the results of the election are not vague either, even if, before designing a crisp preference, each voter had rather fuzzy preferences.

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After introducing the concepts of fuzzy preference in Section 2, Section 3 will be devoted to Arrovian aggregation problems and Section 4 to other aspects, including a fuzzy treatment of Sen's impossibility of a Paretian liberal and the first results about fuzzy aggregation in economic environments.

2. FUZZY PREFERENCES

In (naive) set theory, given a set X, a subset A and an element $x \in X$, either $x \in A$ or $x \notin A$. Belonging to the subset can be defined by a function b from X to $\{0, 1\}$, where $x \in A$ is equivalent to $b(x) = 1$ and $x \notin A$ is equivalent to $b(x) = 0$. If the subset A refers to the description of some semantic concepts (events or phenomena or statements), there might be no clear-cut way to assert that an element is or is not in this subset. Classical examples are the set of tall men, the set of intelligent women or the set of beautiful spiders. The basic idea of replacing $\{0, 1\}$ by $[0, 1]$ as the set where the membership function takes its values is due to Zadeh. However, the origin of this is probably older, taking us back at least to Łukasiewicz who introduced many-valued logic in 1920. For excellent mathematical introductions, we recommend Dubois and Prade (1980) and Nguyen and Walker (2000).

For a fuzzy binary relation, the membership function associates a number in $[0, 1]$ to an ordered pair of options (x, y). The interpretation of a number $\alpha \in [0, 1]$ associated to (x, y) can be the degree of intensity with which x is preferred to y is α or the degree of truth that x is preferred to y is α. Though in some cases the first interpretation is possible and amounts to considering strength of preference, the second interpretation is always possible and is compulsory if, at some stage, fuzzy connectives (and, or ...) have been introduced.

The fact that some positive value α is associated to (x, y) does not entail, in general, that the value 0 must be assigned to (y, x). For instance, if we consider two versions of Beethoven's *Grosse Fuge*, say, by the Juilliard Quartet and the Berg Quartet, we may have mixed and conflicting feelings. We may prefer to some extent the Berg version because of its energy but have also some preference for the Juilliard because of its poetry. This will entail that some value $\alpha \in]0, 1]$ be given to the preference for the Berg version over the Juilliard, but also that some value $\beta \in]0, 1]$ be given to the preference for the Juilliard version over the Berg.

Moreover, on the basis of this example, it might seem strange to describe fuzziness by assigning a precise number. The logician Alasdair Urquhart (2001) has rightly observed :

> One immediate objection that presents itself to this line of approach is the extremely artificial nature of the attaching of precise numerical values to sentences like '73 is a large number' or 'Picasso's *Guernica* is beautiful'. In fact, it seems plausible to say that the nature of vague predicates precludes attaching precise numerical values just as much as it precludes attaching precise classical truth values.

One way to avoid this difficulty is to replace $[0, 1]$ ordered by $\geq$ by some set of qualitative elements ordered by a complete preorder. For instance, to give some intuition in the context of preference, the elements could be interpreted as being 'not at all', 'insignificantly', 'a little', 'mildly', 'much', 'very much', 'definitely', etc. We will consider both cases in the following subsections.

2.1. **Fuzzy preferences: numerical values.**

We will stick in this chapter to the notions and properties used in the fuzzy aggregation literature. There are many notions about transitivity and choice which will not be introduced (see, for instance, Barrett, Pattanaik and Salles (1990), Basu, Deb and Pattanaik (1992), Dasgupta and Deb ((1991), (1996), (2001)), Salles (1998)).

Let X be the set of alternatives with $\#X \geq 3$.

We will consider two types of fuzzy binary relations. Strict binary relations as described here were introduced in Barrett, Pattanaik and Salles (1986), hereafter denoted by BPS, and weak binary relations were introduced by Dutta (1987).

Definition 1. A *fuzzy binary relation* over X is a function $h : X \times X \to [0, 1]$.

Definition 2. A fuzzy binary relation p is a *BPS-fuzzy strict preference* if for all distinct $x, y, z \in X$,
(i) $p(x, x) = 0$
(ii) $p(x, y) = 1 \Rightarrow p(y, x) = 0$
(iii) $p(x, y) > 0$ and $p(y, z) > 0 \Rightarrow p(x, z) > 0$.

Definition 3. A BPS-fuzzy strict preference p is *BPS-complete* if for all distinct $x, y \in X$,
$p(x, y) > 0$ or $p(y, x) > 0$.

$p(x, y)$ can be interpreted as the degree of intensity with which x is preferred to y (or, see above, the degree of truth that x is preferred to y). (i) expresses that we consider strict preferences. (ii) means that when strict preferences are definite (in some sense non-fuzzy), they are asymmetric. (iii) is a rather mild transitivity property. It is now well known that many transitivity concepts are available for fuzzy binary relations (Dasgupta and Deb, 1996). In particular, the most widely used concept, max-min transitivity, which states that given any $x, y, z \in X$, $p(x, z) \geq min(p(x, y), p(y, z))$ is stronger than (iii).

Subramanian (1987) uses two different fuzzy strict preferences one of which is a variant of BPS-fuzzy strict preferences.

Definition 4. A fuzzy strict preference p^{S_1} is a *S_1-fuzzy strict preference* if it is a BPS-fuzzy strict preference for which for all distinct $x, y, z \in X$, $p^{S_1}(x, y) = p^{S_1}(y, x) = p^{S_1}(y, z) = p^{S_1}(z, y) = 0 \Rightarrow p^{S_1}(x, z) = p^{S_1}(z, x) = 0$.

A fuzzy strict preference p^{S_2} is a *S_2-fuzzy strict preference* if it satisfies properties (i) and (ii) of Definition 2, and if

for all distinct $x_1, x_2, ..., x_k \in X$, $p^{S_2}(x_1, x_2) > p^{S_2}(x_2, x_1)$ and $p^{S_2}(x_2, x_3) > p^{S_2}(x_3, x_2)$ and ... and $p^{S_2}(x_{k-1}, x_k) > p^{S_2}(x_k, x_{k-1}) \Rightarrow \neg(p^{S_2}(x_k, x_1) = 1$ and $p^{S_2}(x_1, x_k) = 0)$.

Definition 5. A fuzzy binary relation r is a *fuzzy weak preference* if it is reflexive, i.e., if for all $x \in X$, $r(x, x) = 1$.

The basic idea underlying the concept of fuzzy weak preference is that it will be possible to derive from it two components, viz. a symmetric component, the fuzzy indifference, i [1], and more importantly a strict component which is in some fuzzy sense asymmetric, p. All the numerical fuzzy weak preferences we are considering in this chapter are connected (or complete) with the following meaning.

Definition 6. A fuzzy weak preference r is *connected* if for all x, $y \in X$, $r(x, y) + r(y, x) \geq 1$.

We will present three decompositions of a fuzzy weak preference introduced respectively by Dutta (1987), Banerjee (1994), and Richardson (1998) and Dasgupta and Deb (1999). Theoretical results about these decompositions can be found in the cited papers and in

[1] Although the letter i is also used as the generic letter for individuals, there will clearly be no confusion possible.

Dasgupta and Deb (2001). We will not present these results here since they are not directly connected with the topic of the chapter.

Definition 7. A fuzzy weak preference r^D is a *D-fuzzy weak preference* if for all x, $y \in X$, $i^D(x,y) = min(r^D(x,y), r^D(y,x))$ and

$$p^D(x,y) = \begin{cases} r^D(x,y) & \text{if } r^D(x,y) > r^D(y,x) \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Banerjee objects to this decomposition on the basis that if $i^D(x,y) > 0$, $p^D(x,y)$ should be less than $r^D(x,y)$. It seems strange that when $r^D(x,y) = 1$ and $r^D(y,x) = 0.999$, and when $r^D(x,y) = 1$ and $r^D(y,x) = 0$, $p^D(x,y)$ has the same value (1). Richardson adds that the discontinuity of p^D seems also rather unreasonable. If $r^D(x,y) = 1$ and $r^D(y,x) = 0.999$, $p^D(x,y) = 1$, and if $r^D(x,y) = r^D(y,x) = 1$, $p^D(x,y) = 0$.

Definition 8. A fuzzy weak preference r^B is a *B-fuzzy weak preference* if for all x, $y \in X$, $i^B(x,y) = min(r^B(x,y), r^B(y,x))$ and $p^B(x,y) = 1 - r^B(y,x)$.

Richardson discusses this decomposition and the rôle of a property of strong connectedness. In particular, he notes that $p^B(x,y)$ has the same value when $r^B(x,y) = 1$ and $r^B(y,x) = 0.999$, and when $r^B(x,y) = 0.001$ and $r^B(y,x) = 0.999$.

Definition 9. A fuzzy weak preference r is *strongly connected* if for all x, $y \in X$, $max(r(x,y), r(y,x)) = 1$.

Definition 10. A fuzzy weak preference r^{RD^2} is a *RD^2-fuzzy weak preference* if for all x, $y \in X$, $i^{RD^2}(x,y) = min(r^{RD^2}(x,y), r^{RD^2}(y,x))$ and $p^{RD^2}(x,y) = max(r^{RD^2}(x,y) - r^{RD^2}(y,x), 0)$.

In these definitions D is for Dutta (1987), B is for Banerjee (1994) and RD^2 is for both Richardson (1998) and Dasgupta and Deb (1999).

We will now introduce several transitivity properties for fuzzy weak preferences.

Definition 11. A fuzzy binary relation r is
(i) *max-min transitive* if for all x, y, $z \in X$, $r(x,z) \geq min(r(x,y), r(y,z))$,
(ii) *max-δ transitive* if for all x, y, $z \in X$, $r(x,z) \geq r(x,y) + r(y,z) - 1$,
(iii) *exactly transitive* if for all x, y, $z \in X$, $r(x,y) = 1$ and $r(y,z) = 1 \Rightarrow r(x,z) = 1$,
(iv) *weakly max-min transitive* if for all x, y, $z \in X$, if $r(x,y) \geq r(y,x)$ and $r(y,z) \geq r(z,y)$, then $r(x,z) \geq min(r(x,y), r(y,z))$.

Max-min transitivity implies max-δ transitivity which implies exact transitivity. Also, it is obvious that max-min transitivity implies weak max-min transitivity (see Dasgupta and Deb (1996) for further results). Based on these four transitivities, we will define various social welfare functions in the next section. As mentioned above, there are many transitivity concepts for fuzzy binary relations (in fact, obviously an infinity). A basic requirement is that, when applied to values restricted to be 0 or 1 (i.e., to crisp relations), one must recover the standard notions of transitivity. For instance, if we consider (iii) of Definition 2, we have : $p(x,y) = 1$ and $p(y,z) = 1 \Rightarrow p(x,z) = 1$, which is the transitivity of the standard (strict) preference. But if we consider the following definition :

for all distinct x, y, $z \in X$, $p(x,y) > .05$ and $p(y,z) > .01 \Rightarrow p(x,z) = 1$

we have a transitivity notion for a fuzzy binary relation which is compatible with its crisp counterpart without being intuitively convincing. It is furthermore independent of (iii) of Definition 2. Of course, if, moreover, the transitivity properties are not independent, then, all other things being equal, for impossibility results, the weakest notion is the best, and for possibility results, the strongest notion the best.

2.2. **Fuzzy preferences: qualitative values.** We mentioned above that assigning precise numbers to elements to describe vagueness could seem paradoxical. Goguen (1967) and Basu, Deb and Pattanaik (1992) have proposed assigning some qualitative value, the set of these values being subject to some binary relation. We will follow here Barrett, Pattanaik and Salles (1992) and consider fuzzy strict preferences where fuzziness is given by elements in a finite set L completely preordered by a relation $\succeq$. This is, of course, very similar to Goguen's L-fuzzy sets, the only difference being that in Goguen $\succeq$ is a linear order, i.e., an anti-symmetric complete preorder. This means that, with a non-anti-symmetric complete preorder, there might be a non-unique way to define a degree of fuzziness (for instance 'a little' and 'mildly' can express the same fuzziness, though they are different elements of L).

Let L be a finite set and $\succeq$ a complete preorder on L with a unique $\succeq$-maximum, denoted $d^\star$, and a unique $\succeq$-minimum, denoted $d_\star$.

Definition 12. *An ordinally fuzzy binary relation* H is a function $H : X \times X \to L$.

Definition 13. An ordinally fuzzy binary relation P is a *BPS-ordinally fuzzy strict preference* if for all distinct x, y, $z \in X$,
(i) $P(x,x) = d_\star$
(ii) $P(x,y) = d^\star \Rightarrow P(y,x) = d_\star$
(iii) $P(x,y) = d^\star \Rightarrow P(x,z) \succeq P(y,z)$, and $P(y,z) = d^\star \Rightarrow P(x,z) \succeq P(x,y)$.

We will introduce a variety of transitivity conditions. As previously, it might be difficult to say which condition is the most appropriate. It can depend on the context. However we impose with (iii) a sort of transitivity condition which seems as compelling as (i) (exact irreflexivity) and (ii) (exact asymmetry). According to (iii) if x is definitely (or exactly) better than y, then x must 'fare as well' against z as y against z (in terms of preference in favour), and if y is definitely better than z, then x must 'fare as well' against z as against y. If one definitely prefers Bartok's concerto for orchestra to Gorecki's third symphony, the degree of his preference for Bartok's concerto over Dutilleux's first symphony (which may be 'mild') must be 'at least as strong as' the degree of his preference for Gorecki's symphony over Dutilleux's symphony (which may be null).

Definition 14. Let P be a BPS-ordinally fuzzy strict preference. P is
(i) *weakly max-min transitive* if for all x, y, $z \in X$, $P(x,y) \succeq P(y,x)$ and $P(y,z) \succeq P(z,y) \Rightarrow P(x,z) \succeq P(x,y)$ or $P(x,z) \succeq P(y,z)$,
(ii) *quasi-transitive* if for all x, y, $z \in X$, $P(x,y) \succ P(y,x)$ and $P(y,z) \succ P(z,y) \Rightarrow P(x,z) \succ P(z,x)$,
(iii) *acyclical* if there is no finite set $\{x_1, ..., x_k\} \subseteq X$ $(k > 1)$ such that $P(x_1,x_2) \succ P(x_2,x_1)$ and ... and $P(x_{k-1},x_k) \succ P(x_k,x_{k-1})$ and $P(x_k,x_1) \succ P(x_1,x_k)$,
(iv) *simply transitive* if for all x, y $z \in X$, $(P(x,y) \succ d_\star$ and $P(y,x) = d_\star)$ and $(P(y,z) \succ d_\star$ and $P(z,y) = d_\star) \Rightarrow P(x,z) \succ d_\star$ and $P(z,x) = d_\star$.

The following example justifies the choice of these transitivity properties as compared with others. Consider three options: a sum of money m, then $m+\delta$ $(\delta > 0)$ and x which is unspecified. Suppose $P(m+\delta,m) = d^\star$, $P(m,x) = d$, $P(x,m+\delta) = d'$, with $d^\star \succ d \succ d_\star$ and $d^\star \succ d' \succ d_\star$. If we consider the ordinal version of max-min transitivity, i.e., if for all x, y $z \in X$, $P(x,z) \succeq P(x,y)$ or $P(x,z) \succeq P(y,z)$, we should obtain $P(m,m+\delta) \succeq d$ or $P(m,m+\delta) \succeq d'$. Since $P(m+\delta,m) = d^\star$, by (ii) in Definition 13, $P(m,m+\delta) = d_\star$, a contradiction. The same is true if we consider the ordinal version of (iii) in Definition 2, i.e., for all x, y $z \in X$, $P(x,y) \succ d_\star$ and $P(y,z) \succ d_\star \Rightarrow P(x,z) \succ d_\star$. However, this example is compatible with our four transitivity properties.

3. AGGREGATION OF FUZZY PREFERENCES: ARROVIAN THEOREMS

We will introduce in all the cases defined above aggregation procedures and properties which are essentially the fuzzy replicates of Arrow's conditons (see Arrow (1963), Sen (1970)).

3.1. **The case of numerical values.** Let $N = \{1, ..., n\}$ be a finite set of individuals ($n \geq 2$).

Definition 15. A *fuzzy aggregation function* is a function that associates a social fuzzy binary relation over X, denoted h_S, to an n-list of individual fuzzy binary relations over X, denoted $(h_1, ..., h_i, ..., h_n)$.

$h_i(x, y)$ can be interpreted as the degree of intensity with which individual i prefers (weakly or strictly) x to y (or the degree of confidence we have that i prefers (weakly or strictly) x to y).

Definition 16. Let f be a fuzzy aggregation function, h_i, h_S, etc., be fuzzy binary relations, p_i, p_S, etc., be BPS-fuzzy strict preferences, or strict components of r_i, r_S of any type (i.e. D, B or RD^2), etc. f satisfies

FI (fuzzy independence of irrelevant alternatives) if for all n-lists $(h_1, ..., h_n), (h'_1, ..., h'_n)$ and all distinct $x, y \in X$, $h_i(x, y) = h'_i(x, y)$ and $h_i(y, x) = h'_i(y, x)$ for every $i \in N$ implies $h_S(x, y) = h'_S(x, y)$ and $h_S(y, x) = h'_S(y, x)$, where $h_S = f(h_1, ..., h_n)$ and $h'_S = f(h'_1, ..., h'_n)$;

FPC (fuzzy Pareto criterion) if for all $(h_1, ..., h_n)$, all distinct $x, y \in X$, $p_S(x, y) \geq min_i p_i(x, y)$, where $h_S = f(h_1, ..., h_n)$;[2] and

FPR (fuzzy positive responsiveness) if for all $(r_1, ..., r_n), (r'_1, ..., r'_n)$ and all distinct $x, y \in X$, $r_i = r'_i$ for all $i \neq j$, $r_S(x, y) = r_S(y, x)$, and ($p_j(x, y) = 0$ and $p'_j(x, y) > 0$) or ($p_j(y, x) > 0$ and $p'_j(y, x) = 0$) $\Rightarrow p'_S(x, y) > 0$.

FI is the natural counterpart of Arrow's independence of irrelevant alternatives and FPC means that if every individual prefers x to y with at least degree t , then the society must reflect this unanimity.

Let $\mathbb{A}_{BPS}$ be the set of BPS-fuzzy strict preferences and $\mathbb{A}'_{BPS} \subseteq \mathbb{A}_{BPS}$, $\mathbb{A}'_{BPS} \neq \emptyset$.

Definition 17. A *BPS-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{A}'^n_{BPS} \to \mathbb{A}_{BPS}$.

Definition 18. Let $f : \mathbb{A}'^n_{BPS} \to \mathbb{A}_{BPS}$ and $J_f = \{(t_1, t_2) \in [0, 1] \times [0, 1]$: for some $p \in \mathbb{A}'_{BPS}$ and some distinct $a, b \in X$, $p(a, b) = t_1$ and $p(b, a) = t_2\}$.

f is said to have a *non-narrow domain* for distinct $x, y, z \in X$ if

for all $(t_1, t_2) \in J_f$, there exists $p \in \mathbb{A}'_{BPS}$ such that $p(x, y) = p(x, z) = 1$ and $p(y, z) = t_1$ and $p(z, y) = t_2$, and also there exists $p' \in \mathbb{A}'_{BPS}$ such that $p'(y, x) = p'(z, x) = 1$ and $p'(y, z) = t_1$ and $p'(z, y) = t_2$, and

for all $(t_1, t_2) \in [0, 1] \times [0, 1]$ for which $(t_1, 0)$, $(t_2, 0) \in J_f$ and $t_2 \geq t_1$, there exists $p \in \mathbb{A}'_{BPS}$ such that $p(x, y) = t_1$, $p(y, z) = 1$, $p(x, z) = t_2$ and $p(y, x) = p(z, y) = p(z, x) = 0$, and also there exists $p' \in \mathbb{A}'_{BPS}$ such that $p'(x, y) = 1$, $p'(y, z) = t_1$, $p'(x, z) = t_2$ and $p'(y, x) = p'(z, y) = p'(z, x) = 0$.

Of course, if $\mathbb{A}'_{BPS} = \mathbb{A}_{BPS}$, it can be easily seen that the condition defining a non-narrow domain is satisfied for all distinct $x, y, z \in X$. This condition is weaker than a

[2] $(h_1, ..., h_n)$ is either $(p_1, ..., p_n)$ in the BPS framework or $(r_1, ..., r_n)$, with h_S being respectively p_S or r_S.

universality condition requiring that $\mathbb{A}'_{BPS} = \mathbb{A}_{BPS}$ and is sufficient to obtain the following theorems. A coalition is a non-empty subset of N.

Theorem 1. *Let $f : \mathbb{A}'^n_{BPS} \to \mathbb{A}_{BPS}$ be a BPS-fuzzy social welfare function satisfying FI, FPC and having a non-narrow domain for all distinct $x, y, z \in X$. Then there exists a unique coalition C such that*

for all distinct $x, y \in X$ and all $(p_1, ..., p_n) \in \mathbb{A}'^n_{BPS}$, if $p_i(x, y) > 0$ and $p_i(y, x) = 0$ for every $i \in C$, then $p_S(x, y) > 0$, where $p_S = f(p_1, ..., p_n)$; and

for all distinct $x, y \in X$ and all $(p_1, ..., p_n) \in \mathbb{A}'^n_{BPS}$, if for some $j \in C$, $p_j(x, y) > 0$ and $p_j(y, x) = 0$, then $p_S(y, x) = 0$, where $p_S = f(p_1, ..., p_n)$.

This theorem is reminiscent of Gibbard's oligarchy theorem (Gibbard (1969)). If the individuals in coalition C share some agreement in their preferences, they can exert some positive (fuzzy) power. Furthermore, each individual in the coalition has some fuzzy veto power.

One can verify that the fuzzy aggregation function given by $p_S(x, y) = min_i p_i(x, y)$ is a BPS-fuzzy social welfare function. This does not contradict Theorem 1. In this case the unique coalition C is the entire N.

Theorem 2. *Consider the further requirement that if $p_S \in f(\mathbb{A}'^n_{BPS})$, then p_S is BPS-complete. Then $\#C = 1$, i.e., the coalition C shrinks:*

there exists an individual $i \in N$ such that for all distinct $x, y \in X$ and all $(p_1, ...p_n) \in \mathbb{A}'^n_{BPS}$, if $p_i(x, y) > 0$ and $p_i(y, x) = 0$, then $p_S(x, y) > 0$ and $p_S(y, x) = 0$, where $p_S = f(p_1, ..., p_n)$.

This theorem is a fuzzy analog of Arrow's theorem (Arrow (1963)) and the individual of Theorem 2 could be considered a *BPS-fuzzy dictator.*

We will now consider fuzzy weak preferences.

Let $\mathbb{A}_{D_1}$ be the set of D-fuzzy weak preferences which are max-min transitive.

Definition 19. A *D_1-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{A}^n_{D_1} \to \mathbb{A}_{D_1}$.

For such functions, one obtains a result similar to Theorem 1.

Theorem 3. *Let $f : \mathbb{A}^n_{D_1} \to \mathbb{A}_{D_1}$ be a D_1-fuzzy social welfare function satisfying FI and FPC. Then there exists a unique coalition C such that*

for all distinct $x, y \in X$ and all $(r^D_1, ..., r^D_n) \in \mathbb{A}^n_{D_1}$, if $p^D_i(x, y) > 0$ for every $i \in C$, then $p^D_S(x, y) > 0$, where $r^D_S = f(r^D_1, ..., r^D_n)$; and

for all distinct $x, y \in X$ and all $(r^D_1, ..., r^D_n) \in \mathbb{A}^n_{D_1}$, if for some $j \in C$, $p^D_j(x, y) > 0$, then $p^D_S(y, x) = 0$, where $r^D_S = f(r^D_1, ..., r^D_n)$.

Although Dutta (1987) provides an example showing that Theorem 2 cannot be directly extended (take $r_S(x, x) = 1$, and for $x \neq y$ $r_S(x, y) = 1$ if for all $i \in N$ $r_i(x, y) > r_i(y, x)$, and $r_S(x, y) = \alpha \in]1/2, 1[$ otherwise), a result similar to Theorem 2 is possible if one further assumes that f satisfies the positive responsiveness conditon defined above (Definition 16).

Theorem 4. *Let $n \geq 3$ and $f : \mathbb{A}^n_{D_1} \to \mathbb{A}_{D_1}$ be a D_1-fuzzy social welfare function satisfying FI, FPC and FPR. Then there exists an individual $i \in N$ such that for all distinct $x, y \in X$ and all $(r^D_1, ...r^D_n) \in \mathbb{A}^n_{D_1}$, if $p^D_i(x, y) > 0$, then $p^D_S(x, y) > 0$, where*

$r_S^D = f(r_1^D, ..., r_n^D)$.

The result is reminiscent of a result of Mas-Colell and Sonnenschein (1972). The individual shown to exist is a sort of fuzzy dictator. However, if max-min transitivity is replaced by max-δ transitivity, the kind of impossibility of Theorem 4 vanishes.

Let $\mathbb{A}_{D_2}$ be the set of D-fuzzy weak preferences which are max-δ transitive.

Definition 20. A *D_2-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{A}_{D_2}^n \to \mathbb{A}_{D_2}$.

Theorem 5. *For all $(r_1^D, ..., r_n^D) \in \mathbb{A}_{D_2}^n$ and all $x, y \in X$, let $r_S(x, y) = (1/n)\Sigma_i r_i^D(x, y)$.*[3] *Then this function is a D_2-fuzzy social welfare function satisfying FI, FPC and FPR for which there is no individual $i \in N$ such that for all distinct $x, y \in X$ and all $(r_1^D, ..., r_n^D) \in \mathbb{A}_{D_2}^n$, if $p_i^D(x, y) > 0$, then $p_S^D(x, y) > 0$, where $r_S^D = f(r_1^D, ..., r_n^D)$.*

In fact this rule is also obviously anonymous, with anonymity defined in the usual way as symmetry over individuals (incidentally, it is also symmetric over options).

Banerjee (1994) shows that on substituting fuzzy weak preferences r^B, Theorem 5 is no longer true and a theorem similar to Theorem 4 is obtained. However, as Richardson (1998) observes, Banerjee uses the hidden fact that two of the sufficient conditions to obtain his decomposition imply that the fuzzy weak preferences r^B are strongly connected. One of these two conditions is in fact imposed by all the mentioned contributors. It is the sort of fuzzy asymmetry mentioned above. It says that the strict component p must satisfy $p(x, y) > 0 \Rightarrow p(y, x) = 0$. If it is obviously the case for r^D and r^{RD^2}, it is not for r^B. However, by adding strong connectedness, it also becomes true for r^B.

Let $\mathbb{A}_B$ be the set of B-fuzzy weak preferences which are strongly connected and max-δ transitive.

Definition 21. A *B-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{A}_B^n \to \mathbb{A}_B$.

Theorem 6. *Let $f : \mathbb{A}_B^n \to \mathbb{A}_B$ be a B-fuzzy social welfare function satisfying FI and FPC. Then there exists an individual $i \in N$ such that for all distinct $x, y \in X$ and all $(r_1^B, ..., r_n^B) \in \mathbb{A}_B^n$, if $p_i^B(x, y) > 0$, then $p_S^B(x, y) > 0$, where $r_S^B = f(r_1^B, ..., r_n^B)$.*

We will now consider the third decomposition, i.e., fuzzy weak preferences r^{RD^2}.

Let $\mathbb{A}_{RD^2}$ be the set of RD^2-fuzzy weak preferences which are strongly connected and exactly transitive.

Definition 22. A *RD^2-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{A}_{RD^2}^n \to \mathbb{A}_{RD^2}$.

We may note that exact transitivity is a very weak condition, so the following theorem due to Richardson is quite interesting even though strong connectedness may appear as rather constraining.

Theorem 7. *Let $f : \mathbb{A}_{RD^2}^n \to \mathbb{A}_{RD^2}$ be a RD^2-fuzzy social welfare function satisfying FI and FPC. Then there exists an individual $i \in N$ such that for all distinct*

[3] This function is sometimes called the mean rule; for extended studies see García-Lapresta and Llamazares (2000) and Ovchinnikov (1991).

$x, y \in X$ and all $(r_1^{RD^2}, ..., r_n^{RD^2}) \in \mathbb{A}^n_{RD^2}$, if $p_i^{RD^2}(x, y) > 0$, then $p_S^{RD^2}(x, y) > 0$, where $r_S^{RD^2} = f(r_1^{RD^2}, ..., r_n^{RD^2})$.

However, if we assume that the RD^2-fuzzy weak preferences are only max-δ transitive, we get again a kind of possibility result.

Let $\mathbb{B}_{RD^2}$ be the set of RD^2-fuzzy weak preferences which are max-δ transitive.

Definition 23. A *RD^2_δ-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{B}^n_{RD^2} \to \mathbb{B}_{RD^2}$.

Theorem 8. *For all $(r_1^{RD^2}, ..., r_n^{RD^2}) \in \mathbb{B}^n_{RD^2}$ and all $x, y \in X$, let $r_S(x, y) = (1/n)\Sigma_i r_i^{RD^2}(x, y)$. Then this function is a RD^2_δ-fuzzy social welfare function satisfying FI, FPC and FPR for which there is no individual $i \in N$ such that for all distinct $x, y \in X$ and all $(r_1^{RD^2}, ..., r_n^{RD^2}) \in \mathbb{B}^n_{RD^2}$, if $p_i^{RD^2}(x, y) > 0$, then $p_S^{RD^2}(x, y) > 0$, where $r_S^{RD^2} = f(r_1^{RD^2}, ..., r_n^{RD^2})$.*

Weak max-min transitivity is found by Dasgupta and Deb ((1996), (2001)) to perform well in permitting non-trivial fuzzy preferences and in preventing cycles of strict preferences. By using RD^2 decomposition they obtain again a rather negative result, showing the existence of an individual having disproportionate power.

Let $\mathbb{C}_{RD^2}$ be the set of RD^2-fuzzy weak preferences which are weakly max-min transitive.

Definition 24. A *RD^2_w-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{C}^n_{RD^2} \to \mathbb{C}_{RD^2}$.

Theorem 9. *Let $f : \mathbb{C}^n_{RD^2} \to \mathbb{C}_{RD^2}$ be a RD^2_w-fuzzy social welfare function satisfying FI and FPC. Then there exists an individual $i \in N$ such that for all distinct $x, y \in X$ and all $(r_1^{RD^2}, ..., r_n^{RD^2}) \in \mathbb{C}^n_{RD^2}$, if $p_i^{RD^2}(x, y) = 1$, then $p_S^{RD^2}(x, y) > 0$, where $r_S^{RD^2} = f(r_1^{RD^2}, ..., r_n^{RD^2})$.*

The kind of fuzzy dictator we have here, called a weak dictator by Dasgupta and Deb (1999), exerts power only in case he or she exactly prefers one option to another.

3.2. **The case of qualitative values.** This subsection will be entirely based on Barrett, Pattanaik and Salles (1992). Fuzzy aggregation in an ordinal framework has rarely been explored. The only other work we know is Barrett and Pattanaik (1990) where Barrett, Pattanaik and Salles (1992) is extended in characterizing rank-based aggregation rules such as the median rule.

Definition 25. An *ordinally fuzzy aggregation function* is a function that associates a social ordinally fuzzy binary relation over X, denoted H_S, to an n-list of individual ordinally fuzzy binary relations over X, denoted $(H_1, ..., H_i, ..., H_n)$.

Definition 26. Let f be an ordinally fuzzy aggregation function, H_i, H_S, etc., be ordinally fuzzy binary relations, P_i, P_S, etc., be BPS-ordinally fuzzy strict preferences. f satisfies

OFI (*ordinally fuzzy independence of irrelevant alternatives*) if for all n-lists $(H_1, ..., H_n)$, $(H'_1, ..., H'_n)$ and all $x, y \in X$, $H_i(x, y) \sim H'_i(x, y)$ and $H_i(y, x) \sim H'_i(y, x)$ for every $i \in N$ implies $H_S(x, y) \sim H'_S(x, y)$ and $H_S(y, x) \sim H'_S(y, x)$; and

OFU (*ordinally fuzzy unanimity*) if for all n-lists $(P_1, ..., P_n)$ and all $x, y \in X$, there exists $i \in N$ such that $P_i(x, y) \succeq P_S(x, y)$ and there exists $j \in N$ such that $P_S(x, y) \succeq$

$P_j(x, y)$.

Condition OFU is a sort of Pareto criterion. Consider $(P_1, ..., P_n)$ and $x, y \in X$. The set $\{max_\ell P_\ell(x, y)\}$ $(\ell = 1, ...n)$ contains some element α. [4] Similarly, $\{min_\ell P_\ell(x, y)\}$ contains some element β. The condition means that the ordinally fuzzy strict social preference $P_S(x, y)$ must be 'between' α and β or 'at the same level' as α or β, i.e., $\alpha \succeq P_S(x, y) \succeq \beta$.

Let $\mathbb{O}$ be the set of BPS-ordinally fuzzy strict preferences and $\mathbb{O}_w$, $\mathbb{O}_q$, $\mathbb{O}_a$, and $\mathbb{O}_s$ be respectively the set of BPS-ordinally fuzzy strict preferences which are weakly max-min transitive, quasi-transitive, acyclical, and simply transitive.

Theorem 10. $\mathbb{O}_w \subseteq \mathbb{O}_q \subseteq \mathbb{O}_a$.

We will now introduce several ordinally fuzzy aggregation functions distinguished according to the set in which these functions take their values. The domains of these functions are identical, viz., the Cartesian product of $\mathbb{O}_w$ (or, of course, the Cartesian product of any superset of $\mathbb{O}_w$, though it will not be indicated).

Definition 27. A w-*(respectively* q-*,* a-*,* s-*) ordinally fuzzy social welfare function* is an ordinally fuzzy social welfare function $f : \mathbb{O}_w^n \to \mathbb{O}_w$ (respectively $\mathbb{O}_w^n \to \mathbb{O}_q$, $\mathbb{O}_w^n \to \mathbb{O}_a$, $\mathbb{O}_w^n \to \mathbb{O}_s$).

According to the different transitivity properties and other conditions imposed, we obtain the five following results.

Theorem 11. *Let $\#X \geq n$ and let $f : \mathbb{O}_w^n \to \mathbb{O}_a$ be an a-ordinally fuzzy social welfare function satisfying OFI and OFU. Let $d \in L$, $d \succ d_\star$. Then there exists an individual j such that for all $x, y \in X$ and for all $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_j(x, y) \succeq d \succ P_j(y, x)$ and $d \succeq P_i(y, x)$ for all $i \in N - \{j\} \Rightarrow P_S(x, y) \succeq P_S(y, x)$, where $P_S = f(P_1, ..., P_n)$.*

This is clearly an expression of veto power. This sort of veto power can even be strengthened if the number of options is increased.

Theorem 12. *Let $\#X \geq 2n$ and let $f : \mathbb{O}_w^n \to \mathbb{O}_a$ be an a-ordinally fuzzy social welfare function satisfying OFI and OFU. Let $d \in L$, $d \succ d_\star$. Then there exists an individual j such that for all $x, y \in X$ and for all $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_j(x, y) \succeq d \succ P_j(y, x) \Rightarrow P_S(x, y) \succeq P_S(y, x)$, where $P_S = f(P_1, ..., P_n)$.*

It is possible to extend these two theorems when no restriction is imposed on the number of elements in X. Regarding Theorem 11 if $\lceil n/\#X \rceil$ is the smallest integer $\geq n/\#X$, N can be partitioned into at most $\#X$ coalitions of size $\leq \lceil n/\#X \rceil$. Then, the kind of veto power assigned to individual j, will be assigned to some coalition belonging to the partition. For Theorem 12, it is sufficient to replace $\lceil n/\#X \rceil$ by $\lceil 2n/\#X \rceil$.

We will consider now the case of quasi-transitivity.

Theorem 13. *Let $f : \mathbb{O}_w^n \to \mathbb{O}_q$ be a q-ordinally fuzzy social welfare function satisfying OFI and OFU. Let $d \in L$, $d \succ d_\star$. Then there exists a coalition C such that:*

for all $x, y \in X$ and for all $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_i(x, y) \succeq d \succ P_i(y, x)$ for all $i \in C \Rightarrow P_S(x, y) \succ P_S(y, x)$, where $P_S = f(P_1, ..., P_n)$; and

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[4]Since $\succeq$ is a complete preorder, we have not excluded the possibility that two different elements essentially represent the same 'degree of preference' (for instance 'a little' and 'mildly'). Then $\{max_\ell P_\ell(x, y)\}$ may contain more than one element.

for all $i \in C$, *for all* $x, y \in X$ *and for all* $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_i(x,y) \succeq d \succ P_i(y,x) \Rightarrow P_S(x,y) \succeq P_S(y,x)$, *where* $P_S = f(P_1, ..., P_n)$.

Replacing $\mathbb{O}_q$ by $\mathbb{O}_w$, we obtain the following theorem.

Theorem 14. *Let* $f : \mathbb{O}_w^n \to \mathbb{O}_w$ *be a* w*-ordinally fuzzy social welfare function satisfying* OFI *and* OFU. *Let* $d \in L$, $d \succ d_\star$. *Then there exists a coalition* C *such that:*
for all $x, y \in X$ *and for all* $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_i(x,y) \succeq d \succ P_i(y,x)$ *for all* $i \in C \Rightarrow P_S(x,y) \succeq d \succ P_S(y,x)$, *where* $P_S = f(P_1, ..., P_n)$*; and*
for all $i \in C$, *for all* $x, y \in X$ *and for all* $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_i(x,y) \succeq d \succ P_i(y,x) \Rightarrow P_S(x,y) \succeq d$ *or* $d \succ P_S(y,x)$, *where* $P_S = f(P_1, ..., P_n)$.

Finally, we consider the case of simple transitivity.

Theorem 15. *Let* $f : \mathbb{O}_w^n \to \mathbb{O}_s$ *be an* s*-ordinally fuzzy social welfare function satisfying* OFI *and* OFU. *Then there exists a coalition* C *such that:*
for all $x, y \in X$ *and for all* $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_i(x,y) \succ d_\star$ *and* $P_i(y,x) = d_\star$ *for all* $i \in C \Rightarrow P_S(x,y) \succ d_\star$ *and* $P_S(y,x) = d_\star$, *where* $P_S = f(P_1, ..., P_n)$*; and*
for all $i \in C$, *for all* $x, y \in X$ *and for all* $(P_1, ..., P_n) \in \mathbb{O}_w^n$, $P_i(x,y) \succ d_\star$ *and* $P_i(y,x) = d_\star \Rightarrow P_S(x,y) \succ d_\star$ *or* $P_S(y,x) = d_\star$, *where* $P_S = f(P_1, ..., P_n)$.

By Theorem 10, one can replace appropriately the transitivity properties in Theorems 11, 12 and 13. Theorem 15 can be compared to Theorem 1. The result of Theorem 15 is slightly weaker regarding the kind of veto power, since we have $P_S(x,y) \succ d_\star$ or $P_S(y,x) = d_\star$ rather than $p_S(y,x) = 0$.

4. OTHER ASPECTS

In this section, we will present two aspects which are heretofore less developed than what can be called Arrovian aspects, viz., fuzzy versions of Sen's impossibility of a Paretian liberal and considerations of some economic types of restriction on fuzzy preferences.

4.1. Aggregation of fuzzy preferences and Sen's impossibility theorem.

This subsection is essentially based upon Subramanian (1987).

Sen's impossibility theorem demonstrates that for some class of aggregation functions (generally called social decision functions) that includes Arrovian social welfare functions, given a sufficiently large domain, there is an inconsistency between unanimity—or the weak Pareto principle—(whenever every individual prefers alternative a to alternative b, so does the society) and a condition called minimal liberalism (there are at least two individuals i and j and for each of them two alternatives, a_i, b_i for i (resp. a_j, b_j for j), such that the (strict) preference of i (resp. j) over his alternatives is reflected by the social (strict) preference over these alternatives. The intuition is that the alternatives a_i, b_i for i (resp. a_j, b_j for j) belong to i's (resp. j's) personal sphere, or, more precisely, differ on characteristics concerning i (resp. j) only. This condition may appear as being very strong and giving too much power to these two individuals. Rather than describing individual liberty, it can be interpreted as some kind of local dictatorship (Salles (2000)). In Subramanian's fuzzy version, the condition is weakened. Whenever i (resp. j) exactly prefers one of his two alternatives to the other, say a_i to b_i, then the degree of the fuzzy social preference of a_i over b_i must be greater than (or at least as great as) the degree of the fuzzy social preference of b_i over a_i.

Let $\mathbb{S}_1$ be the set of S_1-fuzzy strict preferences, $\mathbb{S}_2$ the set of S_2-fuzzy strict preferences, $\mathbb{S}_{1e}$ the subset of S_1 made up of all exact S_1-fuzzy strict preferences (those for which the only possible values are 0 or 1), $\mathbb{S}_{2e}$ the subset of S_2 made up of all exact S_2-fuzzy strict preferences. (See Definition 4.)

Definition 28. A S_1*-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{S}_1^n \to \mathbb{S}_2$. A S_{1e}*-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{S}_{1e}^n \to \mathbb{S}_2$. A S'_{1e}*-fuzzy social welfare function* is a fuzzy aggregation function $f : \mathbb{S}_{1e}^n \to \mathbb{S}_1$.

In the following definition, f will be any of the fuzzy aggregation functions defined in Definition 28, and we will use generically $(p_1, ..., p_n)$ for any n-list in the domains of these functions.

Definition 29. Let f be any of the fuzzy aggregation functions of Definition 28. f satisfies

$SFPC$ *(S-fuzzy Pareto criterion)* if for all $(p_1, ..., p_n)$, all distinct $x, y \in X$, $p_i(x, y) = 1$ and $p_i(y, x) = 0$ for all $i \in N \Rightarrow p_S(x, y) = 1$ and $p_S(y, x) = 0$, where $p_S = f(p_1, ..., p_n)$; and

FML_1 *(fuzzy minimal liberalism-1)* if there exist two individuals $i, j \in N$ and for each of them two options, a_i and b_i for i and a_j and b_j for j, such that

for all $(p_1, ..., p_n)$, $p_i(a_i, b_i) = 1$ and $p_i(b_i, a_i) = 0 \Rightarrow p_S(a_i, b_i) > p_S(b_i, a_i)$, and for all $(p_1, ..., p_n)$, $p_i(b_i, a_i) = 1$ and $p_i(a_i, b_i) = 0 \Rightarrow p_S(b_i, a_i) > p_S(a_i, b_i)$; and

for all $(p_1, ..., p_n)$, $p_j(a_j, b_j) = 1$ and $p_j(b_j, a_j) = 0 \Rightarrow p_S(a_j, b_j) > p_S(b_j, a_j)$, and for all $(p_1, ..., p_n)$, $p_j(b_j, a_j) = 1$ and $p_j(a_j, b_j) = 0 \Rightarrow p_S(b_j, a_j) > p_S(a_j, b_j)$; and

FML_2 *(fuzzy minimal liberalism-2)* if there exist two individuals $i, j \in N$ and for each of them two options, a_i and b_i for i and a_j and b_j for j, such that

for all $(p_1, ..., p_n)$, $p_i(a_i, b_i) = 1$ and $p_i(b_i, a_i) = 0 \Rightarrow p_S(a_i, b_i) \geq p_S(b_i, a_i)$, and for all $(p_1, ..., p_n)$, $p_i(b_i, a_i) = 1$ and $p_i(a_i, b_i) = 0 \Rightarrow p_S(b_i, a_i) \geq p_S(a_i, b_i)$; and

for all $(p_1, ..., p_n)$, $p_j(a_j, b_j) = 1$ and $p_j(b_j, a_j) = 0 \Rightarrow p_S(a_j, b_j) \geq p_S(b_j, a_j)$, and for all $(p_1, ..., p_n)$, $p_j(b_j, a_j) = 1$ and $p_j(a_j, b_j) = 0 \Rightarrow p_S(b_j, a_j) \geq p_S(a_j, b_j)$.

It should be noted that both liberalism conditions are based on individual exact preferences. The first one is interpreted by Subramanian as the fuzzy counterpart of the condition due to Sen ((1970), (1970a)) and the second of the version due to Karni (1978). Also the Pareto criterion (unanimity) is based on individual and social exact preferences.

Theorem 16. *There does not exist a function* $f : \mathbb{S}_{1e}^n \to \mathbb{S}_2$ *satisfying conditions* $SFPC$ *and* FML_1*.*

Theorem 17. *There does not exist a function* $f : \mathbb{S}_{1e}^n \to \mathbb{S}_1$ *satisfying conditions* $SFPC$ *and* FML_2*.*

Theorem 18. *The fuzzy aggregation function* f *defined by*
for all $x, y \in X$*, all* $(p_1, ..., p_n) \in \mathbb{S}_1^n$*,* $p_S(x, y) = min_i p_i(x, y)$*, where* $p_S = f(p_1, ..., p_n)$*,*
is a S_1*-fuzzy social welfare function satisfying* $SFPC$ *and* FML_2*.*

The crucial rôle of exact preferences in these theorems must be underlined. Exact preferences are the only preferences in the domains for Theorems 16 and 17, and furthermore, individual preferences are exact in the definitions of the Pareto condition ($SFPC$) and the fuzzy versions of minimal liberalism conditions.

Dimitrov (2004) uses intuitionistic fuzzy sets introduced by Atanassov (1999). Rather than associating a (unique) number to an ordered pair (x, y), he associates two numbers, the first one expressing the degree to which x is preferred to y and the second the degree to which x is not preferred to y, requiring that the sum of these numbers be ≤ 1. He considers fuzzy weak preferences with a decomposition à la Dutta and obtains a possibility result.

4.2. **Aggregation of fuzzy preferences and economic environments.** When social choice theory started its modern development in the 1940's, Black (1958) introduced a condition (in a kind of geometric way), called single-peakedness, restricting the individual preferences. In the exact case, if the individual preferences given by complete preorders are single-peaked, majority rule is a social welfare function (given some mild condition on the number of individuals having these preferences). A number of developments took place from the 1960's. They are excellently surveyed in Gaertner (2001, 2002). Among the restrictive conditions on individual preferences, those used in standard microeconomic theory are particularly important. For instance, consider exchange economies. Since equilibrium redistributions are Pareto-optimal, it seems crucial to be able to rank these redistributions on the basis of an aggregation function that satisfies some properties related to ethical and social justice considerations. A major difficulty arises about these considerations because individuals have preferences over their consumption sets but not on redistributions. This can be resolved by assuming selfishness and by identifying an individual's preferences over the redistributions with this individual's preferences over his or her individual bundles. In the case of (pure) public goods, this difficulty disappears as individuals and society have their preferences on the same set of alternatives, which is generally taken to be the positive orthant of a Euclidean space. Then an individual's preference is generally given by a complete preorder which is, furthermore, monotonic, continuous and (strictly) convex. In a fundamental paper that started the literature on aggregation in an economic environment, Kalai, Muller and Satterthwaite (1979) dealt with the case of public goods. The excellent overview of this topic by Le Breton and Weymark (2004) is highly recommended. Geslin, Salles and Ziad (2003) consider the public good case when individual and social preferences are fuzzy. They test the robustness of the results of Barrett, Pattanaik and Salles (1986) when the set of alternatives is the positive orthant of a Euclidean space (that is a pure public good economy where the social and individual preferences are defined over the same set), and when the individuals' fuzzy strict preferences satisfy some monotonicity properties.

Geslin, Salles and Ziad (GSZ) consider the case where $X = \mathbb{R}^{\ell}_{+}$, the positive orthant of ℓ-dimensional Euclidean space. They introduce two monotonicity properties on individual BPS-fuzzy strict preferences. Although the first is probably specific to their paper, the second is an adaptation of the monotonicity of preferences of microeconomics texts.

We will use the standard notation regarding inequalities between vectors in $\mathbb{R}^{\ell}$, i.e., given $x = (x_1, ..., x_\ell)$ and $y = (y_1, ..., y_\ell)$, $x \geq y$ if $x_h \geq y_h$ for $h = 1, ...\ell$; $x > y$ if $x \geq y$ and $x \neq y$; and $x \gg y$ if $x_h > y_h$ for $h = 1, ..., \ell$. We will consider BPS-fuzzy strict preferences (see Definition 2).

Definition 30. A BPS-fuzzy strict preference p satisfies $F-monotonicity$ if for any y, x, $x' \in \mathbb{R}^{\ell}_{+}$,
(1) if $x \leq y$, $p(x, y) = 0$, and
(2) otherwise, $x > x' \Rightarrow p(x, y) > p(x', y)$ if $p(x', y) \neq 1$ and $p(x, y) = 1$ if $p(x', y) = 1$. [5]

This definition is intuitively appealing. It means that the degree to which x is preferred to y is greater (when it is possible) than the degree to which x' is preferred to y, when x is greater (in the vector sense) than x'.

Let $\mathbb{M}_1$ be the set of F-monotonic BPS-fuzzy strict preferences.

[5]This corrects the definition in GSZ which is insufficient to prove Theorem 19.

Definition 31. A *F-monotonic BPS-social welfare function* is a fuzzy aggregation function $f : \mathbb{M}_1^n \to \mathbb{A}_{BPS}$.

Theorem 19. *Let $X = \mathbb{R}_+^\ell$. The fuzzy aggregation function f defined by, for all $x, y \in \mathbb{R}_+^\ell$ and all $(p_1, ..., p_n) \in \mathbb{M}_1^n$, $p_S(x, y) = (1/n)\Sigma_i p_i(x, y)$ is a F-monotonic BPS-fuzzy social welfare function satisfying FI and FPC.*

Of course, as previously, one can remark that this fuzzy social welfare function satisfies other properties (in particular, properties of symmetry). In BPS (1986), it is indicated that the mean rule is not a BPS-fuzzy social welfare function because (iii) of Definition 3 is not satisfied by the social preference. The preceding result indicates that F-monotonicity is a sufficient condition to obtain (iii). GSZ introduce a second monotonicity property which they call E-monotonicity.

Definition 32. A BPS-fuzzy strict preference p satisfies *E-monotonicity* if for all distinct $x, y \in \mathbb{R}_+^\ell$, $x > y \Rightarrow p(x, y) = 1$ and $p(y, x) = 0$.

This is, of course, similar to the standard property of microeconomics. Let $\mathbb{M}_2$ be the set of E-monotonic BPS-fuzzy strict preferences.

Definition 33. An *E-monotonic BPS-social welfare function* is a fuzzy aggregation function $f : \mathbb{M}_2^n \to \mathbb{A}_{BPS}$.

GSZ show that Theorems 1 and 2 are essentially preserved when individual fuzzy preferences are E-monotonic BPS-fuzzy strict preferences.

Theorem 20. *Let $X = \mathbb{R}_+^\ell$ and $f : \mathbb{M}_2'^n \subseteq \mathbb{M}_2^n \to \mathbb{A}_{BPS}$ be an E-monotonic BPS-fuzzy social welfare function satisfying FI and FPC and having a non-narrow domain for all distinct $x, y, z \in \mathbb{R}_+^\ell$ such that there is no $>$-relation between any two of them. Then there exists a unique coalition C such that*

for all distinct $x, y \in \mathbb{R}_+^\ell$ and all $(p_1, ..., p_n) \in \mathbb{M}_2'^n$, if $p_i(x, y) = 1$ and $p_i(y, x) = 0$ for every $i \in C$, then $p_S(x, y) > 0$, where $p_S = f(p_1, ..., p_n)$; and

for all distinct $x, y \in X$ and all $(p_1, ..., p_n) \in \mathbb{M}_2'^n$, if for some $j \in C$, $p_j(x, y) = 1$ and $p_j(y, x) = 0$, then $p_S(y, x) = 0$, where $p_S = f(p_1, ..., p_n)$.

The following theorem is a simple corollary (in the same way as Theorem 2 is a corollary of Theorem 1).

Theorem 21. *Let $X = \mathbb{R}_+^\ell$, $f : \mathbb{M}_2'^n \subseteq \mathbb{M}_2^n \to \mathbb{A}_{BPS}$ be an E-monotonic BPS-fuzzy social welfare function satisfying FI and FPC and having a non-narrow domain for all distinct $x, y, z \in \mathbb{R}_+^\ell$ such that there is no $>$-relation between any two of them, and $p_S \in f(\mathbb{M}_2'^n)$ be BPS-complete. Then $\#C = 1$: there exists an individual $i \in N$ such that for all distinct $x, y \in \mathbb{R}_+^\ell$ and all $(p_1, ..., p_n) \in \mathbb{M}_2'^n$, if $p_i(x, y) = 1$ and $p_i(y, x) = 0$, then $p_S(x, y) > 0$ and $p_S(y, x) = 0$, where $p_S = f(p_1, ..., p_,)$.*

These two theorems can be stated and proved in an essentially similar manner for another monotonicity assumption: *$F^\star$-monotonicity.* p is said to be *$F^\star$-monotonic* if for all distinct $x, y \in \mathbb{R}_+^\ell$, $x > y \Rightarrow p(x, y) > 0$ and $p(y, x) = 0$.

GSZ consider another subclass of E-monotonic BPS-fuzzy strict preferences, which appears as very restrictive and rather arbitrary. Their purpose, here, is to show that they obtain similar results even in this very restrictive case, which exemplifies the robustness of their quasi-negative results.

Definition 34. Let $a \in \mathbb{R}^\ell_+$, $a \gg 0$, and $\alpha \in]0,1[$. The strictly positive vector a is said to *α-parametrize* a fuzzy binary relation p^{a_α} over $\mathbb{R}^\ell_+$ if for all $x, y \in \mathbb{R}^\ell_+$,

$$p^{a_\alpha}(x,y) = \begin{cases} 1 & \text{if } ax > ay \\ \alpha & \text{if } ax = ay \text{ and } x \neq y \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where ax and ay are dot products.

Let $\mathbb{P}$ be the set of fuzzy binary relations over $\mathbb{R}^\ell_+$ that are α-parametrized by some strictly positive vector for some $\alpha \in]0,1[$.

Definition 35. A *$P - BPS$-social welfare function* is a fuzzy aggregation function $f : \mathbb{P}^n \to \mathbb{A}_{BPS}$.

Theorem 22. If $p \in \mathbb{P}$, then $p \in \mathbb{M}_2$.

We now show again that Theorems 1 and 2 are essentially preserved when $\mathbb{A}'_{BPS} = \mathbb{P}$.

Theorem 23. *Let $X = \mathbb{R}^\ell_+$ and $f : \mathbb{P}^n \to \mathbb{A}_{BPS}$ be a $P - BPS$-fuzzy social welfare function satisfying FI and FPC. Then there exists a unique coalition C such that*

for all distinct $x, y \in \mathbb{R}^\ell_+$ and all $(p_1, ..., p_n) \in \mathbb{P}^n$, if $p_i(x,y) = 1$ and $p_i(y,x) = 0$ for every $i \in C$, then $p_S(x,y) > 0$, where $p_S = f(p_1, ..., p_n)$,

and

for all distinct $x, y \in X$ and all $(p_1, ..., p_n) \in \mathbb{P}^n$, if for some $j \in C$, $p_j(x,y) = 1$ and $p_j(y,x) = 0$, then $p_S(y,x) = 0$, where $p_S = f(p_1, ..., p_n)$.

The following theorem is a simple corollary (in the same way as Theorem 2 is a corollary of Theorem 1).

Theorem 24. *Let $X = \mathbb{R}^\ell_+$, $f : \mathbb{P}^n \to \mathbb{A}_{BPS}$ be a P-BPS-fuzzy social welfare function satisfying FI and FPC and $p_S \in f(\mathbb{P}^n)$ be BPS-complete. Then $\#C = 1$:*

there exists an individual $i \in N$ such that for all distinct $x, y \in \mathbb{R}^\ell_+$ and all $(p_1, ..., p_n) \in \mathbb{P}^n$, if $p_i(x,y) = 1$ and $p_i(y,x) = 0$, then $p_S(x,y) > 0$ and $p_S(y,x) = 0$, where $p_S = f(p_1, ...p_n)$.

5. CONCLUDING REMARKS

It is clear from this chapter (which we hope is as complete as possible) that there are many routes that have not yet been explored within fuzzy set theory. The most striking feature of this approach is its flexibility. Exploring various assumptions each with intuitive appeal, one can obtain divergent results. This is true when we consider different ways to decompose fuzzy weak preferences. It is also true, in economic environments, when we consider different monotonicity conditions.

Finally, let us mention that fuzzy sets are only one way to deal with imprecision and vagueness. Other approaches are possible, even though largely unexplored in economic theory. This is the case with *rough sets* theory which provides a nice way to describe similarities (indifferences). It is also the case with *supervaluation theory* , particularly in vogue among philosophers, as mentioned in the introduction, and with the theory of *interval orders* (Fisburn (1985)) and other aspects of measurement theory.

First, let us consider *rough sets.* Given a set X and a partition of X (geometrically, one can imagine some kind of grid), any subset S of X will possibly contain elements of the partition (that is, entire subsets belonging to the partition). The union of these elements will be considered as an inner approximation of S (it will be included in S). Also, if we consider elements of the partition that have a non-empty intersection with S, one can take the union of these elements. This union will contain S and will be the outer approximation of S. This construction due to Pawlak (1982) (see also Polkowski (2002)) has been used by Bavetta and del Seta (2001) to describe problems in the freedom of choice literature, in particular to deal with the difficulty raised by indistinguishable alternatives.

In *supervaluation theory* (a theory pertaining to philosophical logic), a proposition containing a vague term is true if it is true in all sharpenings of the term. A 'sharpening' is an 'admissible' way in which a vague term can be made precise. Consider, for instance, the term 'old.' 'Old' can be interpreted as 'being over 55 years of age.' Alternatively, it can be interpreted as 'being over 60 years of age,' and so on. None of these 'sharpenings' is the actual meaning of 'old,' which is vague, but they are ways 'old' might be sharpened. On the other hand, 'being over 20 years of age' cannot be a 'sharpening' of 'old' since it is clearly not an 'admissible' way of making 'old' precise. So the notion of 'being admissible' is crucial. For supervaluation theory and the (philosophical) study of vagueness Williamson (1994), Keefe (2000) and Piggins (1999) are recommended. We have drawn our presentation from Piggins (1999) which also included applications to welfare economics.

Finally, in *measurement theory* (Fisburn (1970), (1988), Krantz, Luce, Suppes and Tversky (1971)), a binary relation $\succ_{ST}$ on $X \times X$, where X is the set of alternatives, is interpreted as a comparison of the strengths of preference between ordered pairs, $(x, y) \succ_{ST} (z, w)$ meaning that the strength of preference of x over y exceeds the strength of preference of z over w. An underlying preference $\succ$ over X is defined by $x \succ y$ if $(x, y) \succ_{ST} (y, y)$. Given a set of axioms, a utility representation for comparable differences, i.e., a real-valued function u such that $(x, y) \succ_{ST} (z, w) \Leftrightarrow u(x) - u(y) > u(z) - u(w)$ can be derived. This, of course, is reminiscent of the von Neumann-Morgenstern representation of a complete preorder over lotteries, but despite this the formalization has been, to the best of our knowledge, largely ignored by fuzzy set theorists. (A problem is that it is also rather difficult to interpret the notion of strength of preference of y over y. We would need for this to have a notion of (unique) minimum for the strength of preference relation.)

References

Arrow, K.J. (1963) *Social choice and individual values, Second Edition*. New York: Wiley.

Atanassov, K.T. (1999) *Intuitionistic fuzzy sets: theory and applications*. Heidelberg: Physica-Verlag.

Banerjee, A. (1994) 'Fuzzy preferences and Arrow-type problems in social choice', *Social Choice and Welfare*, 11: 121-130.

Barrett, C.R. and P.K. Pattanaik (1990) 'Aggregation of fuzzy preferences', in: J. Kacprzyk and M. Fedrizzi, eds., *Multiperson decision making using fuzzy sets and possibility theory*. Dordrecht: Kluwer.

Barrett, C.R., P.K. Pattanaik and M. Salles (1986) 'On the structure of fuzzy social welfare functions', *Fuzzy Sets and Systems*, 19: 1-10.

Barrett, C.R., P.K. Pattanaik and M. Salles (1990) 'On choosing rationally when preferences are fuzzy', *Fuzzy Sets and Systems*, 34: 197-212.

Barrett, C.R., P.K. Pattanaik and M. Salles (1992) 'Rationality and aggregation of preferences in an ordinally fuzzy framework', *Fuzzy Sets and Systems*, 49: 9-13.

Basu, K., R. Deb and P.K. Pattanaik (1992) 'Soft sets: an ordinal formulation of vagueness with some applications to the theory of choice', *Fuzzy Sets and Systems*, 45: 45-58.

Bavetta S. and M. Del Seta (2001) 'Constraints and the measurement of freedom of choice', *Theory and Decision*, 50: 213-238.

Black, D. (1958) *The theory of committees and elections.* Cambridge: Cambridge University Press.

Broome, J. (1997) 'Is incommensurability vagueness?', in: R. Chang, ed., *Incommensurability, incomparability, and practical reason.* Cambridge (Mass.): Harvard University Press.

Burns, L.C. (1991) *Vagueness.* Dordrecht: Kluwer.

Dasgupta, M. and R. Deb (1991) 'Fuzzy choice functions', *Social Choice and Welfare*, 8: 171-182.

Dasgupta, M. and R. Deb (1996) 'Transitivity and fuzzy preference', *Social Choice and Welfare*, 13: 305-318.

Dasgupta, M. and R. Deb (1999) 'An impossibility theorem with fuzzy preferences', in: H. de Swart, ed., *Logic, game theory and social choice.* Tilburg: Tilburg University Press.

Dasgupta, M. and R. Deb (2001) 'Factoring fuzzy preferences', *Fuzzy Sets and Systems*, 118: 489-502.

Dimitrov, D. (2004) 'The Paretian liberal with intuitionistic fuzzy preferences: a result', *Social Choice and Welfare*, 21.

Dubois, D. and H. Prade (1980) *Fuzzy sets and systems: theory and applications.* New York: Academic Press.

Dummett M. (1975) 'Wang's paradox', *Synthese*, 30: 325-365.

Dutta, B. (1987) 'Fuzzy preferences and social choice', *Mathematical Social Sciences*, 13: 215-229.

Fishburn, P.C. (1970) *Utility theory for decision making.* New York: Wiley.

Fishburn, P.C. (1985) *Interval orders and interval graphs: a study of partially ordered sets.* New York: Wiley.

Fishburn, P.C. (1988) *Nonlinear preference and utility theory.* Wheatsheaf: Brighton.

Fishburn, P.C. (1998) 'Stochastic utility', in: S. Barbera, P.J. Hammond and C. Seidl, eds., *Handbook of utility theory, Volume 1.* Dordrecht: Kluwer.

Gaertner, W. (2001) *Domains Conditions in Social Choice Theory.* Cambridge: Cambridge University Press.

Gaertner, W. (2002) 'Domain restrictions', in: K.J. Arrow, A.K. Sen and K. Suzumura, eds., *Handbook of social choice and welfare, Volume 1.* Amsterdam: North-Holland.

García-Lapresta, J.L. and B. Llamazares (2000) 'Aggregation of fuzzy preferences: some rules of the mean', *Social Choice and Welfare*, 17: 673-690

Geslin, S., M. Salles and A. Ziad (2003) 'Fuzzy aggregation in economic environments I: quantitative fuzziness, public goods and monotonicity assumptions', *Mathematical Social Sciences*, 45: 155-166.

Gibbard, A. (1969) 'Intransitive social indifference and the Arrow dilemma', unpublished.

Goguen, J.A. (1967) 'L-fuzzy sets', *Journal of Mathematical Analysis and Applications*, 18: 145-174.

Kalai, E., E. Muller and M. Satterthwaite (1979) 'Social welfare functions when preferences are convex, strictly monotonic, and continuous', *Public Choice*, 34: 87-97.

Karni, E. (1978) 'Collective rationality, unanimity and liberal ethics', *Review of Economic Studies*, 45: 571-574.

Keefe, S. (2000) *Theories of Vagueness.* Cambridge: Cambridge University Press.

Keefe, S. and P. Smith, eds. (1996) *Vagueness: A Reader.* Cambridge (Mass.):MIT Press.

Krantz, D.H., R.D. Luce, P. Suppes and A. Tversky (1971). *Foundations of measurement, Volume 1.* New York: Academic Press.

Le Breton, M. and J.A. Weymark (2004) 'Arrovian social choice theory on economic domains', in: K.J. Arrow, A.K. Sen and K. Suzumura, eds., *Handbook of social choice and welfare, Volume 2.* Amsterdam: North-Holland.

Mas-Colell, A. and H.F. Sonnenschein (1972), 'General possibility theorems for group decisions', *Review of Economic Studies*, 39: 185-192.

Nguyen, H.T. and E.A. Walker (2000) *A first course in fuzzy logic, Second edition.* Boca Raton: Chapman & Hall.

Ovchinnikov, S.V.(1991) 'Social choice and Łukasiewicz logic', *Fuzzy Sets and Systems*, 43: 275-289.

Pawlak, Z. (1982) 'Rough sets', *International Journal of Information and Computer Sciences*, 11: 341-356.

Piggins, A. (1999) 'Vagueness, quasi-ordering and supervaluation theory', mimeographed, University of Caen.

Polkowski, L. (2002) *Rough sets, mathematical foundations.* Heidelberg: Physica-Verlag.

Richardson, G. (1998) 'The structure of fuzzy preferences: social choice implications', *Social Choice and Welfare*, 15: 359-369.

Ross, T.J., J.M. Booker and W.J. Parkinson, eds. (2002) *Fuzzy logic and probability applications: bridging the gap.* Alexandria: American Statistical Association and Philapelphia: Society for Industrial and Applied Mathematics.

Salles, M. (1998) 'Fuzzy utility', in: S. Barbera, P.J. Hammond and C. Seidl, eds., *Handbook of utility theory, Volume 1.* Dordrecht: Kluwer.

Salles, M. (2000) 'Amartya Sen: droits et choix social', *Revue Economique*, 51: 445-457.

Sen, A.K. (1970) *Collective choice and social welfare.* San Francisco: Holden-Day.

Sen, A.K. (1970a) 'The impossibility of a Paretian liberal', *Journal of Political Economy*, 78: 152-157.

Sen, A.K. (1983) 'Liberty and social choice', *Journal of Philosophy*, 80: 5-28.

Sen, A.K. (1992) *Inequality reexamined.* New York: Russell Sage Foundation and Oxford: Clarendon Press.

Sen, A.K. (2002) *Rationality and freedom.* Cambridge (Mass.): Harvard University Press.

Subramanian, S. (1987) 'The liberal paradox with fuzzy preferences', *Social Choice and Welfare*, 4: 213-218.

Urquhart, A. (2001) 'Basic many-valued logic', in: D.M. Gabbay and F. Guenthner, eds., *Handbook of philosophical logic, Second edition, Volume 2.* Dordrecht: Kluwer.

Williamson, T. (1994) *Vagueness.* London: Routledge.

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