# MICHAEL DUMMETT ON SOCIAL CHOICE AND VOTING

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Abstract. Michael Dummett worked on the theoretical aspects of aggregation of individual preferences and on the strategic aspects of voting theory. He also extended Black's analysis of single-peaked preferences for majority rule to the case of voting games (majority games), offering a greater flexibility for the expression of voters' preferences. He is also with Donald Saari one of the major advocates of the use of Borda's rule in actual voting. In two books and a paper, he proposed many examples showing the advantages and defects of many voting rules used in the world

At the beginning of the 1970's, there has been an upsurge of the publications in social choice and voting theory with, for instance, the books of Murakami (1968), Sen (1970), Pattanaik (1971) and Fishburn (1973). On the contrary, the publications in this area were quite rare in the 1950's and the beginning 1960's after the foundational works of Arrow (1951, 1963) and Black (1958). In 1970, I just started my research for my doctoral dissertation. I was particularly interested in the restrictions of individual preferences that could guarantee the transitivity of the social preference and/or the existence of a 'best' element generated by some aggregation procedure. I was intrigued by a paper by Michael Dummett and Robin Farquharson published in *Econometrica* in 1961. I was intrigued for three reasons. Firstly, I had never heard of the authors. Secondly, in the first pages of the paper, what would become, more than ten years later, one of the most famous results of the social choice literature, viz. the Gibbard-Satterthwaite Theorem, was conjectured in passing. Thirdly, the used proof technique was 'different.'[1] Doing researches in social choice theory, I rapidly heard of Farquharson's book (1969) and was very surprised to discover in this book bibliography that Farquharson published a paper in French on his equilibrium concept in the *Comptes Rendus de l'Académie des Sciences* as early as in 1955. [2] Incidentally, a recent paper by Michael Dummett (2002) is devoted to Farquharson. It took me some time (probably a few years, after I started to develop an amateurish interest in philosophy, particularly in analytic philosophy—surely a major disease for most French 'intellectuels'—) before I came to know that Michael Dummett was one of the greatest living philosophers.

In 1984, a book by Dummett entitled *Voting Procedures* appeared. I think that this book is one of the main works in voting theory and in the analysis of actual voting procedures—I mean procedures used or usable in real life, political or else. A declared objective of this book was to remedy a 'deplorable situation,' viz. that the theory of voting 'was known only to a small circle of people,' excluding 'politicians, national and local' as well as, most probably, 'experts in political institutions and political theory.' *Voting Procedures* was not an easy book. I had the impression from discussions with Michael Dummett that he thought the aforementioned objective had not been reached. On the other hand, it is already a classic even if, precisely because its author is a philosopher, it has not been so far

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[1]The originality of the proof technique is reminiscent of the proof used to show, given an acyclic binary relation over a finite set of elements, the existence of maximal elements in this finite set. To the best of my knowledge this was first proved in von Neumann and Morgenstern (1944, 1953) and rediscovered by Sen (1970) and Pattanaik (1971).

[2]it was not exceptional at that time that important papers by English-speaking natives (and, for this matter, also by Germans and Japaneses), particularly in mathematics, be published in French. These days are, alas for me, over.

widely read by the so-called 'small circle of people.' *Principles of Electoral Reform* (1997) was surely written to reach the initial objective, and it is, I believe, totally successful. Here is a book easy to read, even for laymen, and full of substance and wit. I am not sure, however that in these times when electoral reforms are in order in many countries or when groups of countries have to devise procedures (as in the European Union), it has been read by politicians and journalists.

One may wonder why philosophers have been interested at all in social choice and voting theory. On the one hand, social choice theory and welfare economics always had strong links with moral philosophy and it is not a surprise that many philosophers contributed to this area. It has also been the case that some great authors were at the same time philosophers and economists. I can mention without trying to be exhaustive Adam Smith, John Stuart Mill and Henry Sidgwick. In our time, it is rather difficult to ascertain that Amartya Sen or John Broome are basically economists rather than philosophers. Sen has published many of his recent papers in philosophical journals and some of his books (1987, 2003) are obviously related to moral philosophy. Moreover, important philosophers (professionally they are philosophers because they are or have been academics within departments of philosophy in universities) made major contributions in social choice, from different perspectives. Let me mention Gibbard (1973), Davidson (1986), Gauthier (1986), Griffin (1986), Jeffrey (1992), Suppes (1996), Nozick (1997). On the other hand, Michael Dummett's work is essentially devoted to social choice theory as a paradigm of voting, in its practical and theoretical guise. In this sense, he is a real heir of another Oxford logician: C. L. Dodgson (Lewis Carroll). It is a fact that today there are several logicians working in this area (such as Moshé Machover, Sven Ove Hansson or Harrie de Swart) and that meetings are specifically organized to bring logicians together with game theorists and social choice theorists.

In this essay, I will outline first Dummett's theoretical contribution. This includes an original approach to the aggregation of preferences problem and an analysis of strategic voting. It also includes what was the core of his 1961 paper co-authored with Robin Farquharson, where a sufficient condition on individual preferences for the existence of a solution to voting games is given. This condition is related to, but more general than, Duncan Black's single-peaked preferences. Then, I will describe some of Dummett's work on practical voting methods.

## 1. THE IMPOSSIBILITY OF SOCIAL WELFARE FUNCTIONS

I will first recall the canonical version of Arrow's impossibility theorem. I will use what are now rather standard definitions, notations etc.

Let $X$ be a set of alternatives (options, social states, candidates, allocations of standard microeconomic theory etc.). $\#X$ is the cardinal of $X$ (the number of its elements when $X$ is finite). A binary relation $\succeq$ over $X$ is a set of ordered pairs $(x, y)$ with $x$ and $y$ in $X$, i.e., $\succeq$ is a subset of the Cartesian product $X \times X$. I use the notation $x \succeq y$ rather than $(x, y) \in \succeq$. Intuitively $x \succeq y$ means in our context '$x$ is as good as $y$.' The asymmetric component of $\succeq$, denoted by $\succ$, meaning 'is better than,' is defined by $x \succ y$ if $x \succeq y$ and $\neg y \succeq x$ ($\neg$ is the negation symbol). The symmetric component, denoted by $\sim$, meaning 'there is an indifference between,' is defined by $x \sim y$ if $x \succeq y$ and $y \succeq x$. The binary relation $\succeq$ is reflexive if for all $x \in X$, $x \succeq x$. It is complete if for all $x$, $y \in X$, $x \succeq y$ or $y \succeq x$ (note that if $\succeq$ is complete, it is reflexive and that, in this case, $x \succ y \Leftrightarrow \neg y \succeq x$). It is transitive if for all $x$, $y$, $z \in X$, $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$. A binary relation which is complete and transitive is a complete preorder. With $X$ finite, a complete preorder ranked the alternatives from a most preferred to a least preferred with possible ties.

Let $N$ be a finite set of individuals (economic agents, voters) of cardinal $n$, i.e., $N = \{1, ..., n\}$. Each individual $i \in N$ has a preference over $X$ given by a complete preorder $\succeq_i$. The aggregation question is about the construction of a social (collective, synthetic) preference $\succeq_S$ (or in some cases a social choice) from a list of individual preferences (one preference per individual).

Let $Ord(X)$ be the set of complete preorders over $X$, $Ord(X)'$ be a subset of $Ord(X)$, and $Bin(X)$ be the set of complete binary relations over $X$.

**Definition 1.** An *aggregation function* is a function from $Ord(X)'^n$ into $Bin(X)$, where $Ord(X)'^n$ is the Cartesian product of $Ord(X)'$ $n$ times.

This means that the aggregation function $f$ associates a complete (social) binary relation $\succeq_S$ to a $n$-list $(\succeq_1, ..., \succeq_n)$ of (individual) complete preorders: $f : (\succeq_1, ..., \succeq_n) \mapsto \succeq_S$.

**Definition 2.** A *social welfare function* is an aggregation function for which $Bin(X) = Ord(X)$.

In this case, the sort of rationality required for the social preference is absolutely identical with the rationality we assumed for individuals. If we consider the simple case where $Ord(X)' = Ord(X)$, $\#X = 3$ and $n = 3$, the number of such functions is $13^{13^3}$, which is a number of the order of the positive integer 1 followed by more than 2400 zeros. Of course, some of these functions are unappealing. Some of these unappealing characteristics are excluded by a set of conditions. Arrow's theorem asserts, in a way, that a given set of appealing conditions had so drastically reduced the set of possible social welfare functions that there is no one left! In the standard version of Arrow's theorem there are four conditions.

**Condition U.** $Ord(X)' = Ord(X)$.

This means that the domain of the function is unrestricted or universal (the only restriction being that the individual preferences are given by complete preorders; there is no supplementary rationality assumed from the individuals). This condition will be discussed in Section 3.

**Condition I.** Let $(\succeq_1, ..., \succeq_n)$ and $(\succeq'_1, ..., \succeq'_n) \in Ord(X)'^n$ and $a$ and $b \in X$. Suppose that for each $i \in N$ $\succeq_i |\{a, b\} = \succeq'_i |\{a, b\}$, then $\succeq_S |\{a, b\} = \succeq'_S |\{a, b\}$ where $\succeq_S = f(\succeq_1, ..., \succeq_n)$ and $\succeq'_S = f(\succeq'_1, ..., \succeq'_n)$.

For a simple explanation, suppose that $X$ is finite. Then, in the individual rankings of the alternatives erase all alternatives except $a$ and $b$. If what is left in the two $n$-lists coincide, then the social ranking restricted to $a$ and $b$ must also coincide. Many rules, including majority rule or for this matter, all rules where the social preference over two alternatives is defined from the individual preferences over these same two alternatives will satisfy Condition $I$. On the other hand, all rules which are based on scores (the simplest one being plurality rule used as a voting procedure in many countries, in particular in the United States and United Kingdom) violate this condition. This condition is often called 'independence of irrelevant alternatives,' but it is rather unfortunate that a condition of the same name with another meaning is also used in individual decision theory.

**Condition P.** Let $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$ and $a$ and $b \in X$. If $a \succ_i b$ for all $i \in N$, then $a \succ_S b$, where $\succ_S$ is the asymmetric component of $\succeq_S = f(\succeq_1, ..., \succeq_n)$.

This says that the social preference respects unanimity. An interesting aspect of the condition is that it excludes that the social welfare function be constant.

**Condition D$^A$.** There is no individual $i$ such that for any $n$-list $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$ and any $x, y \in X$, $x \succ_i y \Rightarrow x \succ_S y$, where $\succ_S$ is the asymmetric component of $\succeq_S = f(\succeq_1, ..., \succeq_n)$.

Such an individual is called a *dictator*. Note that to be a dictator you have to impose (all) your *strict* preference to the society, i.e., $\succ_i \subseteq \succeq_S$

**Arrow's Impossibility Theorem.** *If $n \geq 2$ and $\#X \geq 3$, there is no social welfare function satisfying Conditions U, I, P and $D^A$.*

In his book, Arrow introduced two other conditions. The following conditions are slight modifications of the conditions he introduced in the 1951 version of his book with no consequences on the results.

**Condition M$^A$.** Let $(\succeq_1, ..., \succeq_n)$ and $(\succeq'_1, ..., \succeq'_n) \in Ord(X)'^n$ and $a, b \in X$. Suppose that for each $i \in N$, $a \succeq_i b \Rightarrow a \succeq'_i b$ and $a \succ_i b \Rightarrow a \succ'_i b$. Then $a \succ_S b \Rightarrow a \succ'_S b$, where $\succ_S$ is the asymmetric component of $\succeq_S = f(\succeq_1, ..., \succeq_n)$ and $\succ'_S$ is the asymmetric component of $\succeq'_S = f(\succeq'_1, ..., \succeq'_n)$.

This is a monotonicity assumption meaning that if $a$ was socially preferred to $b$ and if $a$ does not decline vis-à-vis $b$ in the preference scale of every individual, then $a$ remains socially preferred to $b$.

**Condition NI$^A$.** For all $x, y \in X$, there is an $n$-list $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$ for which $x \succ_S y$, where $\succ_S$ is the asymmetric component of $\succeq_S = f(\succeq_1, ..., \succeq_n)$.

$NI^A$ means 'non-imposition.' This condition entails that a (strict) social preference cannot be commanded by a moral or religious code. It is not difficult to see that, given Conditions $U$ and $I$, Conditions $M^A$ and $NI^A$ imply Condition $P$.

In Voting Procedures (1984), Michael Dummett proposes another Arrovian impossibility theorem. He keeps condition $U$ and $I$ and introduces variants (strengthenings for the first two and weakening for non-imposition) of Conditions $D^A$, $M^A$ and $NI^A$. The following is a slight modification of Dummett's principle (I) (page 52), taking Condition $I$ into consideration.

**Condition M$^D$.** Let $(\succeq_1, ..., \succeq_n)$ and $(\succeq'_1, ..., \succeq'_n) \in Ord(X)'^n$ and $a, b \in X$. Suppose that for each $i \in N$, $a \succeq_i b \Rightarrow a \succeq'_i b$ and $a \succ_i b \Rightarrow a \succ'_i b$. Then $a \succeq_S b \Rightarrow a \succeq'_S b$ and $a \succ_S b \Rightarrow a \succ'_S b$, where $\succ_S$ is the asymmetric component of $\succeq_S = f(\succeq_1, ..., \succeq_n)$ and $\succ'_S$ is the asymmetric component of $\succeq'_S = f(\succeq'_1, ..., \succeq'_n)$.

Dummett's monotonicity assumption differs from Condition $M^A$ only by requiring that if the social preference was $a \sim_S b$, then it must remain so or become $a \succ_S b$.

**Condition NI$^D$.** There are two alternatives $a, b \in X$ and an $n$-list $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$ for which $a \succ_S b$, where $\succ_S$ is the asymmetric component of $\succeq_S = f(\succeq_1, ..., \succeq_n)$.

This condition is quite weak and its violation means that for every $n$-list and every $x$ and $y$ the function socially ranks $x$ and $y$ at the same level: whatever the individual preferences, there is a general social indifference.

**Condition $\mathbf{D}^D$.** There is no individual $i$ such that for some $a, b \in X$, for any $n$-list $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$, $a \succ_i b \Rightarrow a \succ_S b$, where $\succ_S$ is the asymmetric component of $\succeq_S = f(\succeq_1, ..., \succeq_n)$.

Such individuals could be called *partial dictators*. It is interesting to note that the condition is quite stronger than Arrow's non-dictatorship condition (it is probably very difficult to find a 'real' Arrovian dictator; even the worst recent historic figures such as Hitler, Stalin, Amin Dada, Bokassa or Milosevic were not Arrovian dictators, but one can easily imagine that they were Dummettian partial dictators). Moreover, this kind of condition can be found in Sen (1970) to describe *liberalism* (see Brunel and Salles (1998) and Salles (2000)). Dummett's no-partial-dictatorship condition is somewhere between anonymity and Arrow's no-dictatorship condition. Anonymity means that individuals are treated equally. Mathematically if $\sigma$ is a permutation over the set $N$ of individuals, $f(\succeq_{\sigma(1)}, ..., \succeq_{\sigma(n)}) = f(\succeq_1, ..., \succeq_n)$.

**Dummett's Impossibility Theorem.** *If $n \geq 2$ and $\#X \geq 3$, there is no social welfare function satisfying Condition U, I, $NI^D$, $M^D$ and $D^D$.*

A very nice feature of Dummett's impossibility theorem is the simplicity of its proof. There is no need to prove what Sen (1985) calls the field expansion lemma, i.e., the neutrality (equal treatment of alternatives) property generated by decisive sets (a subset of $n$ is decisive for $a$ against $b$ if $a \succ_S b$ whenever we have $a \succ_i b$ for all $i$ in the subset).

## 2. Strategy-proof voting procedures

In my view, the six major pieces of modern social choice theory are at this time Arrow's theorem, Black's single-peaked preferences analysis, Nash's bargaining solution, Harsanyi's utilitarianism (1955), Sen's liberalism theorem and Gibbard-Satterthwaite strategy-proofness theorem (1973, 1975). Regarding this last theorem, a remarkable feature is that it was very clearly stated (conjectured) in the *Econometrica* (1961) paper of Dummett and Farquharson. Precisely, one can read

> We cannot assume that each voter's actual strategy will be determined uniquely by his preference scale. This would be to assume that every voter votes "sincerely," whereas it seems unlikely that **there is any voting procedure in which it can never be advantageous for any voter to vote " strategically," i.e., non-sincerely**.

A general proof of this conjecture was given by Gibbard only in 1973 and by Satterthwaite in 1975. In *Voting Procedures*, Dummett provides a very interesting and (again) 'different' proof of this major result. I will give here the standard version of the theorem and indicates how Dummett's version differs from it.

First, $X$ will be assumed to be finite and to simplify the presentation, I will consider that the individuals' preferences are given by linear orderings over $X$ rather than complete preorders. A linear order is an anti-symmetric complete preorder. $\succeq$ is anti-symmetric if for all $x, y \in X$, $x \succeq y$ and $y \succeq x \Rightarrow x = y$. This means that the alternatives are ranked by the individuals without ties, like points on a line. Let $Lin(X)$ be the set of linear orderings over $X$.

**Definition 3.** A *social choice function* is a surjective function $f$ from $Lin(X)^n$ to $X$.

The problem here is the selection of a single alternative. I need not in this essay to introduce any restriction on the linear orderings that the individuals will indicate in the selection procedure. The fact that $f$ is surjective means that $f(Lin(X)^n) = X$, i.e., for any alternative there is a $n$-list $(\succ_1, ..., \succ_n) \in Lin(X)^n$ [3] whose value is this alternative. Surjectivity is only here to remain as simple as possible. It is not a necessary condition. Incidentally, the absence of surjectivity is rather difficult to interpret in a voting context.

**Definition 4.** Individual $i$ *manipulates* the social choice function in $(\succ_1, ..., \succ_n) \in Lin(X)^n$ if there exists $\succ'_i$ over $X$ such that $f(\succ_1, ..., \succ_{i-1}, \succ'_i, \succ_{i+1}, ..., \succ_n) \succ_i f(\succ_1, ..., \succ_{i-1}, \succ_i, \succ_{i+1}, ..., \succ_n)$.

If we suppose that the linear preferences $\succ_1, ..., \succ_i, ..., \succ_n$ are sincere, the definition means that $i$ by reporting a non-sincere preference can make the outcome be preferable to him according to his sincere preference to the outcome that would have been selected had he reported his sincere preference.

**Condition NM.** A social choice function $f$ is said to be *non-manipulable* if there is no $i$ and no $n$-list $(\succ_1, ..., \succ_n) \in Lin(X)^n$ such that $i$ manipulates $f$ in $(\succ_1, ..., \succ_n)$.

**Condition D$^G$.** There is no individual $i$ for which for all $n$-list $(\succ_1, ..., \succ_n) \in Lin(X)^n$, $f(\succ_1, ..., \succ_n) \succ_i x$ for all $x \in X - \{f(\succ_1, ..., \succ_n)\}$.

Such an individual could be called a Gibbardian dictator. In the case of a Gibbardian dictator, the function selects systematically the alternative ranked first by the dictator (whatever the others' preferences are).

**Gibbard-Satterthwaite's Theorem.** *If $n \geq 2$ and $\#X \geq 3$, there is no social choice function satisfying conditions $NM$ and $D^G$.*

Dummett introduced a stronger form of non-dictatorship, again somewhere between anonymity and the Gibbard-Satterthwaite's version.

**Condition D$^{D'}$.** There is no individual $i$ and no alternative $a$ such that for all $n$-list $(\succ_1, ..., \succ_n) \in f^{-1}(\{a\})$, $f(\succ_1, ..., \succ_n) \succ_i x$ for all $x \in X - \{f(\succ_1, ..., \succ_n)\}$.

$f^{-1}(\{a\}$ is the inverse image of $\{a\}$ under $f$. Essentially, in the Gibbard-Satterthwaite framework, a dictator imposes as the collective choice the alternative that is ranked first in his preference ordering, whatever this alternative is. Dummett excludes this possibility even for a single alternative, i.e., for every $i$ and every alternative $x$, $i$ is not able to impose $x$ to the society.

**Dummett's Strategy-Proofness Theorem.** *If $n \geq 2$ and $\#X \geq 3$, there is no social choice function satisfying conditions NM and $D^{D'}$.*

A remarkable characteristic of Dummett's version is the proof. It is rather surprising that, coming from a logician, this proof has a geometrical (more precisely, graph-theoretical) and topological aspect, with notions such as 'region,' 'boundary,' 'path' being defined and used systematically.

## 3. The core of voting games

As mentioned previously, the modern rebirth of social choice theory in the 1940's is due to the works of Arrow and Black. The main result due to Black concerns the majority rule and the existence of a transitive social preference generated by this rule when the individual preferences are appropriately restricted.

[3] Since indifference is excluded in linear orderings, I use the notation $\succ$ rather than $\succeq$.

**Definition 5.** The *majority rule* is an aggregation function for which for all $x, y \in X$ and all $n$-list $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$, $x \succ_S y \Leftrightarrow \#\{i : x \succ_i y\} > \#\{i : y \succ_i x\}$ and $y \succeq_S x$ otherwise.

Suppose there are three individuals 1, 2, and 3, and three alternatives $a$, $b$ and $c$ with the following preferences: $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$. This means that individual 1 prefers $a$ to $b$, $b$ to $c$ and $a$ to $c$. It is obvious that the majority rule generates $a \succ_S b$, $b \succ_S c$ and $c \succ_S a$. This is the Condorcet paradox. It indicates that the majority rule is not a social welfare function if Condition $U$ is satisfied (all the other conditions introduced in Section 1 are obviously satisfied). Black introduced a condition on the set of individual preferences called *single-peakedness.* Black considered that the set of alternatives was the real line (or more exactly a part of it). Individuals had a preference represented by a curve with a unique maximum strictly increasing up to the maximum and strictly decreasing from the maximum. He demonstrated that the median maximum was the alternative selected by the majority rule, i.e., using this rule, was a point socially preferred to every other point. This is a famous result of *Public Choice* called 'the theorem of the median voter' (not always attributed to Black!). Arrow translated this condition in his set-theoretic framework.

**Definition 6.** Let $\{a, b, c\} \subseteq X$. A set of complete preorders $\succeq$ over $X$ satisfy the condition of *single-peakedness over* $\{a, b, c\}$ if either $a \sim b$ and $b \sim c$ or there is one of the three alternatives, say $b$, such that $b \succ a$ or $b \succ c$ .

Let $BL(X)$ denote the set of complete preorders over $X$ such that the condition of single-peakedness is satisfied for all $\{x, y, z\} \subseteq X$.

**Black's Theorem.** *If* $Ord(X)' = BL(X)$ *and if, for any* $\{x, y, z\} \subseteq X$, *the number of individuals for which* $\neg(x \sim_i y$ *and* $y \sim_i z)$ *is odd, the majority rule is a social welfare function.*

This only means that the social preference is transisitive. [4] With three alternatives, there are 13 complete preorders and 8 single-peaked complete preorders. Single-peakedness essentially means that among the three alternatives there is one which is never the (strictly) worst. There is an interesting and intuitively meaningful geometrical representation. If the three alternatives $a$, $b$ and $c$ are on a line with $b$ between $a$ and $c$, we have the following possibilities:

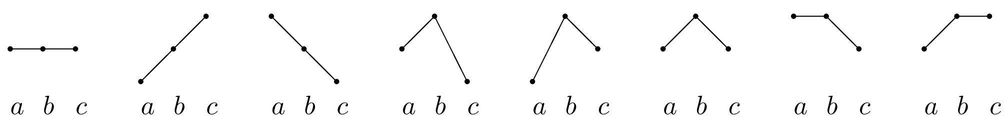


FIGURE 1. Black's single-peakedness condition over $\{a, b, c\}$

When the alternatives $a$, $b$ or/and $c$ are at the same horizontal level, this means that there is an indifference between them, and when one of the alternatives $x \in \{a, b, c\}$ is vertically above $y \in \{a, b, c\}$, this means $x \succ y$. $a$, $b$ and $c$ are linearly ordered $a$ being on the left, $b$ in the center and $c$ on the right. It is then very easy to interpret the admissible (single-peaked) preferences from a political viewpoint, for instance when $a$, $b$ and $c$ are candidates to an election.

[4]Incidentally, we can avoid this condition of oddity if we only require that the asymmetric component of $\succeq_S$, $\succ_S$, be transitive (see Sen (1970) and Sen and Pattanaik (1969)).

In voting games, coalitions, i.e., non-empty subsets of the set of individuals $N$ are a priori endowed with power. This power can be, as in the case of symmetric (anonymous) games, defined by a number of individuals, called a quota $q$ (for instance $q > n/2$).

**Definition 7.** A *voting game* is an ordered pair $G = (N, \mathbb{W})$ where $\mathbb{W} \subseteq 2^N - \emptyset$ and $\mathbb{W}$ satisfies the following monotonicity property:
$C_1 \in \mathbb{W}$ and $C_1 \subseteq C_2 \Rightarrow C_2 \in \mathbb{W}$.
A voting game $G$ is *proper* if $C \in \mathbb{W} \Rightarrow N - C \notin \mathbb{W}$. It is *strong* if $C \notin \mathbb{W} \Rightarrow N - C \in \mathbb{W}$.

$\mathbb{W}$ is the set of winning (powerful) coalitions. A voting game with quota $q \leq n$ (or $q$-game) is defined by $C \in \mathbb{W}$ if $\#C \geq q$. With $n = 9$, the $q$-game with $q = 5$ is proper and strong and the $q$-game with $q = 6$ is only proper.

We can associate to a voting game $G$ an aggregation function $f$.

**Definition 8.** A *voting game of aggregation* is an aggregation function $f$ for which for all $x, y \in X$ and all $n$-list $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$, $x \succ_S y \Leftrightarrow$ there exists a $C \in \mathbb{W}$ such that for all $i \in C$, $x \succ_i y$, and $y \succeq_S x$ otherwise. [5]

Given a voting game of aggregation, we can now define the core associated with a $n$-list $(\succeq_1, ..., \succeq_n)$.

**Definition 9.** The *core* of the voting game of aggregation $f$ associated with the $n$-list $(\succeq_1, ..., \succeq_n) \in Ord(X)'^n$, denoted $Cor(f, (\succeq_1, ..., \succeq_n))$, is the set of the maximal elements of $X$ for the binary relation $\succ_S$, i.e., $Cor(f, (\succeq_1, ..., \succeq_n)) = \{x \in X : (\nexists y \in X) y \succ_S x\}$

In 1961, Dummett and Farquharson introduced an extended version of single-peakedness.

**Definition 10.** Let $\{a, b, c\} \subseteq X$. A set of complete preorders $\succeq$ over $X$ satisfy the condition of *extended single-peakedness over* $\{a, b, c\}$ if there is one of the three alternatives, say $b$, such that $b \succeq a$ or $b \succeq c$ .

Let $DF(X)$ denote the set of complete preorders over $X$ such that the condition of extended single-peakedness is satisfied for all $\{x, y, z\} \subseteq X$.

**Dummett and Farquharson's Theorem.** *Let $X$ be finite and $f$ be any proper voting game of aggregation. Then for all $n$-list $(\succeq_1, ..., \succeq_n) \in DF(X)^n$, $Cor(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$.*[6]

As seen in Figure 2, for a three-alternative subset, the extended single-peakedness condition adds two complete preorders to the 8 complete preorders of Figure 1. This could appear as only a slight amelioration, contrary to what Dummett and Farquharson wrote in the abstract of their paper:

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[5]There is no need to have a *complete* social preference. It is supposed here mainly because we previously defined aggregation functions as having values that are complete binary relations.

[6]In fact, Dummett and Farquharson considered a majority game defined by $x \succeq_S y$ if $\#\{i \in N : x \succeq_i y\} > n/2$ or $\#\{i \in N : x \succeq_i y\} = n/2$ and $x \succeq_1 y$. In case of equality, individual 1 plays a specific rôle (like a president in a committee, for instance). This defines a proper and strong voting game of aggregation with $\succeq_S$ obviously complete. Then they proved the existence of a maximum element (called a top), i.e., the existence of a $x$ such that $x \succeq_S y$ for all $y \in X$. The formulation I gave is essentially due to Nakamura (1975). Salles and Wendell (1977) used the similarity between Dummett and Farquharson's condition and quasi-concavity of utility functions in dimension 1 to extend some of Nakamura's results. On the other hand, Pattanaik (1971) used an analysis very similar with Dummett and Farquharson's analysis, with the same kind of proof method, for conditions extending Black-type of conditions—for instance look at Figure 1 upside down.

A condition on the preferences, **substantially weaker** than one postulated by D. Black, is shown to be sufficient for "stability" in such games.

But I can explain why they wrote this. Firstly, a reduction from 13 to 8 is quite different from a reduction from 13 to 10. Secondly, when we try to give an intuitive meaning to single-peakedness, as in the case of the left-right political spectrum mentioned above, the elimination of these two complete preorders is totally unrealistic. These two preferences do make sense in a political context.

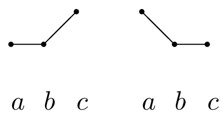


FIGURE 2. Supplementary preferences for extended single-peakedness

On the other hand, the three missing complete preorders, as shown in Figure 3, do not really make sense, in this political context, unless we imagine that the voters can be totally irrational.

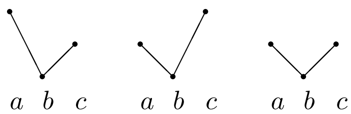


FIGURE 3. Excluded preferences

## 4. Voting procedures in practice

Nowadays, social choice theorists favour two voting methods that are rarely used in practice: the *Borda count* and *approval voting*. Both have interesting features and both have active and famous proponents, Steve Brams and Peter Fishburn for approval voting (see Brams and Fishburn (1983)), Peter Emerson, Donald Saari, and, May I venture?, Michael Dummett for the Borda count (see Emerson (1998), Saari (1995, 2001). Both procedures are simple (can be easily understood by the voters) and need not difficult computations. In the most satisfying theoretical version of the Borda count, the voters give in their ballot paper a complete ranking of the candidates with no ties. If there are $k$ candidates, $k-1$ points are given to the candidate ranked first, $k-2$ to the candidate ranked second, etc. and no point to the candidate ranked last. For each candidate, the points are added and the social preference is based on the points attributed by the voters to the candidates: the candidate ranked first is the candidate who has obtained the greater total of points etc. With approval voting, voters vote for as many candidates as they wish. If there are $k$ candidates they can vote for all $k$ candidates (though this will have no effect on the collective result, exactly as if they voted for none), or for $k-1$ candidates or... for only one candidate. The candidates are then collectively ranked according to the total number of votes they got. Of course, as these procedures give a collective ranking, they can easily be used to select one or more candidates. Surprisingly, many people interested in voting procedures, believing that it is a rather easy mathematical exercise and imagining that they are good at elementary mathematics, either do not know these procedures—but I cannot believe it—or have the greatest contempt for them. Many of these people are actively trying to impose the *alternative vote* in the case of the selection of a single outcome or the *single transferable vote* in the case of the selection of several outcomes.[7]

[7]After the April and May 2002 Presidential elections in France, several articles appeared in *Le Monde* in which a French economist at the MIT I have never heard of as yet proposed a variant of the alternative

When voters have to rank all the candidates, the alternative vote is similar with the positive elimination procedure. In the case of positive voting, an outcome is eliminated at the first stage if it stands at the head of the ranking on the fewest ballots. Then, the rankings are reduced by deleting the eliminated candidate and the same process is used for the reduced rankings. The process is continued until we are left with a unique outcome. In case of equality at some stage, some tie-breaking procedure is used (for instance, the eliminated outcome is that which stands lowest on some individual's list). A dual procedure, the negative elimination procedure, eliminates outcomes which most frequently stands at the bottom of a ranking. Advocates of the alternative vote procedure generally favoured it on the grounds of fairness to the voters (these advocates speak of *wasted vote* (Dummett (1997, Chapter 10)). Incidentally, Dummett emphasizes the fact that the fairness criteria should concern first outcomes, rather than voters. Regarding fairness to the outcomes, he considers three principles. (1) If $x$ is a unique majority maximum (considered as least as good as every other outcome by a majority), $x$ is the fairest outcome. (2) If there is a maximum, no outcome can be fair unless it is a maximum. (3) No outcomes can be fair if it has a lower majority number than some other outcomes (the majority number of any outcome $x$ is the number of other outcomes $y$ for which a majority considers $x$ as least as good as $y$). Consider the following example: there are three candidates $a$, $b$ and $c$; 35% of the voters ranked them $acb$ (meaning $a$ first, $c$ second and $b$ third), 33% ranked them $bca$, and 32% $cab$. Under the alternative vote procedure (and the French run off system), $c$ is eliminated. But in pairwise contests, we can see that $c$ is preferred to $a$ (the winner) by 65% of the voters, and preferred to $b$ by 67%. It is a very large 'Condorcet winner' (a maximum in my previous terminology, or a top in Dummett's words). Furthermore, the alternative vote procedure (and again the French run-off system) is a non-monotonic procedure. Consider the same kind of example. Suppose 35% of the voters rank the three candidates $abc$, 33%, $bca$, and 32% $cab$. $a$ is the winner. Now suppose that for some reason (for instance, an irregularity), the vote has to be started again and the ranking are 37% $abc$, 31% $bca$, and 32% $cab$ (there has been, for instance, a change of mind of some of the people who had previously the ranking $bca$ in favour of the ranking $abc$. Then the winner is $c$ in spite of the fact that more voters ranked $a$, the former winner, first (Dummett (1997, pp 99-103). One might wonder how the alternative vote procedure can still have advocates, once these major flaws are revealed.

Regarding the supposed fairness to the voters of the alternative voting procedure, Dummett (1984, pp 173-174) rightly observes:

> It is no excuse for having ignored the later choices of the supporters of the eventually successful outcome that those supporters can have no complaint... The second and later choices of a voter who has the misfortune to rank first an outcome that remains live up to the final stage of the assessment, but is then defeated, are likewise neglected; and in this case there is not even a fallacious argument to be offered in justification.

The Borda count has a major drawback (or is it?): it is not a majority procedure. This is related to the fairness mentioned above and to the condition of independence of Section 1. Though this condition is implicitly included in the fairness to the outcomes I just described, Dummett is not certain of its relevance. He writes (Dummett (1984, p. 52)):

> All but one of (Arrow's) conditions is a minimal requirement for the rule embodied in the social welfare function to be in the least reasonable. One of them, however, is not: Arrow calls it the principle of 'independence of irrelevant alternatives'.

vote as a solution to the voting problems met in France. He apparently did not notice the major defects of alternative vote, which are comparable with the major defects of the used French procedure (majority with a run off).

Consider an election with ten candidates $x_1, ..., x_{10}$ and suppose that 50.0001% of the voters ranked these candidates from $x_1$ to $x_{10}$ in the natural order and 49.9999% ranked them $x_2$ first, then $x_3$,..., then $x_{10}$, the last being $x_1$. According to the fairness criteria $x_1$ must be chosen. But it is rather obvious (at least to me) that $x_2$ is the best candidate. Among the new rules proposed by Dummett, one is a composite which he calls the *majority number procedure* (Dummett(1984),p 178):

> The tellers will compute, from the voters' lists, the majority numbers of all the outcomes... Having announced these, they will declare successful an outcome having a higher majority number than any other. If two or more outcomes tie as having the highest majority number, they will then compute and announce the preference scores of these outcomes, that one with the highest preference score being declared successful.

On the one hand, given the fairness criteria, this procedure is the best that one can imagine. On the other hand, it has the same drawback as the preference score procedure (the Borda count): it can select a majority-dominated outcome. Furthermore, in this case, if the tellers have to divulge all the information, the voters will know that the chosen outcome is majority-dominated. Consequently, Dummett thinks that the majority number procedure is to be avoided, "except by voters who are highly self-disciplined and appreciate the importance of knowing no more about how the voting has gone than they need to know".

It is well known and easy to see that the Borda count is highly manipulable. But, of course, in large elections, manipulation by a single individual (i.e., the successful misrepresentation of his preference) is nearly null, and manipulation by a group necessitates transfers of information by communication that are rather hypothetical. Dummett (1998) addresses another type of manipulation with the Borda count: the agenda manipulation. A classical example is the introduction, in order to favour a candidate $x$, of a new candidate $y$ ranked on every voter's preference scale immediately below $x$. Dummett proposes two ways to correct this.

In the case of elections with many possible outcomes, Dummett (1984, 1997) proposes new procedures. He is particularly interested in rules insuring the representation of minorities. He is rather critical of the single transferable vote procedure (STV). This procedure is in use in Eire, Northern Ireland, Malta, and, I must add, the American Mathematical Society. The rationale for this procedure rests again on the notion of wasted vote. But Dummett clearly explains why

> Most of the advantages advertised for STV are illusory... Its disadvantages are great: it is complex for the tellers to operate, it is almost impossible to explain accurately, and its effects upon the outcome are often arbitrary. Its outstanding merit is the protection that it gives to minorities (Dummett (1984), p 284).

He then proposes a new method called QPS (Q for quota, and PS for preference score). Voters have not to rank all the candidates, but they do provide a ranking. A voter is said to be solidly committed to a set of candidates if he includes every candidate in the set in his ranking, and ranks each of them higher than any candidate not in the set. The assessment proceeds by stages and will terminate as soon as $k$ seats are filled.

> At stage 1, the tellers will determine whether there are any candidates listed first by more than $1/(k+1)$ of the total $n$ of voters: if so, they immediately qualified for election. If seats remain to be filled, the preference scores of all candidates not qualifying at stage 1 will then be calculated. At stage 2, the ballot papers will be scrutinized to see if there is any pair of candidates, neither of whom qualified at stage 1, to whom more than $n/(k+1)$ voters are solidly committed: if so, that member of the pair with the higher preference score now qualifies for election. If seats remain to

be filled, the tellers will proceed to stage 3, at which they will consider sets of three candidates, none of whom has already qualified. If more than $n/(k+1)$ voters are solidly committed to such trio, that one with the highest preference score qualifies for election... If there still remains seats to be filled after all the qualifying stages have been completed, they will be filled at the final stage by those candidtaes having the highest preference scores out of those who have not yet qualified (Dummett (1984, p 284)).

Dummett shows that QPS retains the quality of STV in minorities representation, but is superior to STV in every other respect, in particular, simplicity and fairness. He does not pretend it is the best voting method, but that it is better than STV. " The question what is the best electoral system is of great complexity, and does not belong exclusively to the theory of voting".

## 5. Conclusion

It is very difficult to convey the richness of the contribution of Michael Dummett to social choice and voting theory and to the practical voting procedures. This essay is only a brief assessment of this richness. What I think is particularly remarkable is that he contributed to three of what I considered above are the six main topics in the area and that he is mainly, after all, a great philosopher and not primarily a social choice theorist. In French 'hobby' is *violon d'Ingres.* I do not know whether Ingres, surely a great painter, was a good violinist. However, I am sure that Michael Dummett is a great social choice theorist.

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