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Further reflections on the expediency and stability of alliances

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Revised August 2006

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ABSTRACT

We consider the co-operative non-transferable utility 'power game' $\mathcal{P}_{\mathcal{W}}$ arising from a simple voting game $\mathcal{W}$. A strategy is the formation of an alliance in $\mathcal{W}$, and the payoff of players is their Penrose voting power in the resulting composite voting game. We study in detail the expediency and stability of alliances when $\mathcal{W}$ is a simple majority or a unanimity voting game, as well as pointing out a seemingly paradoxical phenomenon that can occur when $\mathcal{W}$ is a super-majority game. We also discuss some cases (including a real-life historical one) in which a dummy in $\mathcal{W}$ can become empowered by participating in an expedient alliance.

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1 Introduction

The study of the formation and dissolution of alliances of voters aiming to increase their voting power is relatively new. The present note is a sequel to our earlier (2002) paper [6] on this subject. Since the latter's publication, we have obtained some new results which can be viewed also as a complement to some of the results obtained by Gelman [7].

We report here our findings regarding the possibility of forming expedient and stable alliances within an assembly of voters under various decision rules, depending on the size of the assembly. We shall also show that sometimes an alliance containing a dummy can be expedient – a fact which may explain a real-life historical puzzle.

For the reader's convenience, we repeat in the next section the relevant definitions used in our earlier paper, which we shall be using in this paper as well. In sections 3, 4 and 5 we report our new results. Section 6 concludes.

2 Preliminaries

We assume the reader is familiar with the definitions of *simple voting game* (SVG) and *weighted voting game* (WVG) and with the basic definitions and notation pertaining to them, as laid down in [4, Ch. 2]. In particular, we shall use the square-bracket notation for WVGs (see [4, Def. 2.3.4]).[1]

As in [4, Def. 2.3.10], we shall denote by '$\mathcal{M}_n$' the canonical simple majority WVG with n voters. That is,

$$\mathcal{M}_n := \big[\tfrac{n+1}{2}; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}\big]. \tag{1}$$

By '$\mathcal{M}_n^*$' we shall denote the dual of $\mathcal{M}_n$; that is

$$\mathcal{M}_n^* := \big[\tfrac{n}{2}; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}\big]. \tag{2}$$

[1]We use the term 'game' in this connection out of deference to common usage. But although an SVG does have the *formal* structure of a simple co-operative game with transferable utility, we do not treat it here as such but as a plain decision rule. For a fuller discussion of this point, see [4, Comment 2.2.2].

Note that if n is odd, $\mathcal{M}_n = \mathcal{M}_n^*$ ($\mathcal{M}_n$ is self-dual). But if n is even $\mathcal{M}_n \neq \mathcal{M}_n^*$, and the latter WVG is improper.

Further, as in [4, Def. 2.3.10], we shall denote by '$\mathcal{B}_n$' the canonical unanimity WVG with n voters. That is,

$$\mathcal{B}_n := [n; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}]. \tag{3}$$

The dual of $\mathcal{B}_n$ is

$$\mathcal{B}_n^* := [1; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}]. \tag{4}$$

In what follows, the starting point is some given SVG $\mathcal{W}$ whose assembly (ie, set of voters) is N. We shall denote by 'ψ' the Penrose measure of voting power and refer to it simply as 'power', without further qualification. (For its definition see [4, Def. 3.2.2], where it is denoted by 'β'' and referred to as the 'Bz [Banzhaf] measure of voting power'.) We denote by '$\psi_a[\mathcal{W}]$' the power of voter a in $\mathcal{W}$.

By a well-known theorem (see [4, Thm. 3.3.14]), among all SVGs $\mathcal{W}$ with n voters, the sum of the voters' powers attains its maximum when $\mathcal{W} \cong \mathcal{M}_n$. If n is odd, this condition is also necessary for maximizing the sum of the powers. If n is even, there are several isomorphism types of SVG that maximize the sum; among them, those isomorphic to $\mathcal{M}_n$ have the lowest number of winning coalitions, whereas those isomorphic to its dual, $\mathcal{M}_n^*$, have the highest.

Also, among all SVGs $\mathcal{W}$ with n voters, the sum of the voters' powers attains its minimum if $\mathcal{W} \cong \mathcal{B}_n$ or $\mathcal{W} \cong \mathcal{B}_n^*$. For $n > 2$, this condition is also necessary (see [4, Thm. 3.3.11]).

We recall that '$\mathcal{W}|\&_S$' denotes the SVG that results from $\mathcal{W}$ when a coalition $S \subseteq N$ fuses and forms a bloc $\&_S$ (see [4, Def. 2.3.23]). The assembly of $\mathcal{W}|\&_S$ is $(N - S) \cup \{\&_S\}$. If $\mathcal{W}$ is a weighted voting game (WVG), then so is $\mathcal{W}|\&_S$: take the weight of $\&_S$ to be the sum of the weights that the members of S had in $\mathcal{W}$, while the weights of all other voters as well as the quota are kept the same as in $\mathcal{W}$.

In [6, p. 303], an *alliance* is defined as a bloc $\&_S$ together with an SVG $\mathcal{W}_S$ whose assembly is S, and which is referred to as the *internal* SVG, or decision rule, of the alliance. We shall denote this alliance by $(S; \mathcal{W}_S)$. Informally, we think of the alliance as formed voluntarily by the members of S.

When the members of $S \subseteq N$ form an alliance $(S; \mathcal{W}_S)$, this gives rise to a new *composite* SVG, which we denote by '$\mathcal{W}\|\mathcal{W}_S$'. The assembly of $\mathcal{W}\|\mathcal{W}_S$ is N, the same as that of $\mathcal{W}$. The winning coalitions of $\mathcal{W}\|\mathcal{W}_S$ are all sets of the form $X \cup Y$, with $X \subseteq S$ and $Y \subseteq N - S$, satisfying at least one of the following two conditions:

- Y is a winning coalition of $\mathcal{W}$;
- X is a winning coalition of $\mathcal{W}_S$ and $S \cup Y$ is a winning coalition of $\mathcal{W}$.

(For an equivalent definition of $\mathcal{W}\|\mathcal{W}_S$, which shows it to be a special case of the general operation of composition of SVGs, see [6, p. 310].)

Informally speaking, $\mathcal{W}\|\mathcal{W}_S$ works as follows. When a bill is proposed, the members of S decide about it using $\mathcal{W}_S$, the agreed internal SVG of their alliance. Then, when the bill is brought before the plenary, the assembly of $\mathcal{W}$, all the members of S vote as a bloc, in accordance with their internal decision; so that now the final outcome is the same as it would have been in $\mathcal{W}|\&_S$ with the bloc voter $\&_S$ voting according to the internal decision.

From now on, we put:

$$s := |S|. \tag{5}$$

Note that each member of S now has *direct* voting power in the SVG $\mathcal{W}_S$, as well as *indirect* voting power in $\mathcal{W}\|\mathcal{W}_S$, which s/he exercises via the bloc $\&_S$. It is easy to prove:

$$\text{For every } a \in S,\ \ \psi_a[\mathcal{W}\|\mathcal{W}_S] = \psi_a[\mathcal{W}_S] \cdot \psi_{\&_S}[\mathcal{W}|\&_S]. \tag{6}$$

(See [6, Theorem 4.1]).

However, for voters $b \in N - S$, the equality $\psi_b[\mathcal{W}\|\mathcal{W}_S] = \psi_b[\mathcal{W}|\&_S]$ does not always hold; it does hold *provided the number of winning coalitions of the internal SVG $\mathcal{W}_S$ is 2^{s-1}, exactly half of the number of all coalitions.* (For explanation, see [6, p. 304].)

In what follows we take $\mathcal{W}$ to be exogenously imposed. Its voters are allowed to form alliances but not to change $\mathcal{W}$ itself. We shall therefore rule out alliances with $S = N$, because that would amount simply to all members of the assembly agreeing to adopt a new decision rule instead of $\mathcal{W}$.

Using the terminology introduced in [6, p. 303], we say that the alliance $(S; \mathcal{W}_S)$ is *feasible* if

$$\psi_a[\mathcal{W}\|\mathcal{W}_S] \geq \psi_a[\mathcal{W}] \quad \text{for all } a \in S; \tag{7}$$

and *expedient* if

$$\psi_a[\mathcal{W}\|\mathcal{W}_S] > \psi_a[\mathcal{W}] \quad \text{for all } a \in S. \tag{8}$$

Moreover, we say that a bloc $\&_S$ is *feasible* or *expedient* if there exists some internal SVG $\mathcal{W}_S$ such that the resulting alliance $(S; \mathcal{W}_S)$ is feasible or expedient, respectively.

The idea behind these definitions is the following. We may regard a voter's power as a payoff – not in the original $\mathcal{W}$, or indeed in any 'voting game'[2] – but in a new *power game* $\mathcal{P}_{\mathcal{W}}$ *induced* by $\mathcal{W}$. This $\mathcal{P}_{\mathcal{W}}$ is a genuine co-operative game with *non-transferable* utility, in which the players are the voters of $\mathcal{W}$, and forming an alliance (in $\mathcal{W}$) is a strategy that affects the payoffs of all players (not only of members of the alliance).

Games of the form $\mathcal{P}_{\mathcal{W}}$, induced in this way by some SVG, have a rather intricate structure. As far as we know, they have not been investigated in depth. In [6] we merely scratched the surface. Gelman [7] presents a few additional results. In the present paper we add a few observations concerning some games of this form.

In the sequel, we shall make use of the following simple result (see [6, Thm. 4.2]).

2.1 Proposition *A bloc made up of two voters is never expedient. It is feasible iff originally the two voters have equal powers, or at least one of them is a dummy.* ∎

3 Simple and super majority

In this and the following section we shall consider cases where $\mathcal{W}$ is symmetric. In this context, we confine our attention to alliances $(S; \mathcal{W}_S)$ in which all voters are *equipotent*, that is, have equal power. We shall say that the alliance is *optimal* if its voters are equipotent and their power attains its maximal value – which equals the power of a voter in $\mathcal{M}_s$. Of course, if s is odd then the alliance is optimal just in case $\mathcal{W}_S \cong \mathcal{M}_s$.

Gelman [7] considers a simple majority WVG, $\mathcal{W} \cong \mathcal{M}_n$, and within it alliances of m voters with internal decision rule of the form $\mathcal{W}_S \cong \mathcal{M}_m$. He is particularly interested in the case where n as well as m and $\frac{n}{m}$ are large. He concludes that when these numbers are sufficiently large, then although such an alliance will be expedient, it will nevertheless be unstable because the power of the voters left outside the alliance will be smaller than the voting power of those inside it, so the former will try to form another alliance, possibly with some of the members of the first alliance. But 'if all voters form coalitions,[3] they become worse off than if they had stayed apart' [7, p. 9]. This may imply, in turn, an incessant process of forming and disbanding alliances. In fact, Gelman [7, p. 11] conjectures quite explicitly that 'the coalition-formation process is inherently unstable. . . . By this we mean that

[2] See footnote 1 above.

[3] By 'coalition' Gelman means here what we call an 'alliance'.

if voters are in an ongoing process of joining and leaving coalitions, with each decision made myopically to immediately increase voting power, then there is no stable equilibrium coalition structure.'

Although it is true that the formation of some alliances may result in instability, stable alliances do exist even where $\mathcal{W} \cong \mathcal{M}_n$ and $\mathcal{W}_S \cong \mathcal{M}_m$. As we shall see, this happens when m is just larger than $\frac{n}{2}$ – a case not investigated by Gelman [7]. Consider first the following three instructive examples.

3.1 Example Let $\mathcal{W} \cong \mathcal{M}_5$. Here the power of each voter is $\frac{3}{8}$. In view of Prop. 2.1, we need only consider alliances $(S; \mathcal{W}_S)$ of three or four voters. Any such alliance will make the bloc $\&_S$ a dictator in $\mathcal{W}|\&_S$. Hence by (6)

$$\text{for every } a \in S, \;\; \psi_a[\mathcal{W}\|\mathcal{W}_S] = \psi_a[\mathcal{W}_S].$$

For reasons of symmetry, we need consider only alliances whose voters are equipotent. In fact, it is sufficient to consider optimal alliances, because in this way the members of S maximize their (direct and indirect) power.

For $s = 4$, an optimal $(S; \mathcal{W}_S)$ gives each $a \in S$ $\psi_a[\mathcal{W}\|\mathcal{W}_S] = \psi_a[\mathcal{W}_S] = \frac{3}{8}$. So no alliance of four voters is expedient.[4]

But for $s = 3$, we have $\mathcal{W}_S \cong \mathcal{M}_3$, which gives each $a \in S$ $\psi_a[\mathcal{W}\|\mathcal{W}_S] = \psi_a[\mathcal{W}_S] = \frac{1}{2}$. Such an alliance is expedient, and, once formed, it is stable: no member wishes to defect, and the two voters left out can do nothing to improve their position.

3.2 Example Let $\mathcal{W} \cong \mathcal{M}_9$. Here the power of each voter is $\frac{35}{128}$. In view of Prop. 2.1, we need only consider alliances $(S; \mathcal{W}_S)$ with $3 \leq s \leq 8$; and as in the preceding example we may assume that $\mathcal{W}_S$ is optimal.

An alliance with $s = 8$ is not expedient, as in this case $\psi_a[\mathcal{W}\|\mathcal{W}_S] = \psi_a[\mathcal{W}_S] = \frac{35}{128}$.

On the other hand, simple calculations show that alliances with $s = 3, 4, 5, 6, 7$ are expedient, giving each member of S indirect power $\frac{25}{64}$, $\frac{45}{128}$, $\frac{3}{8}$, $\frac{5}{16}$ and $\frac{5}{16}$, respectively.

However, alliances with $s = 3, 4, 6$ and 7 are unstable. In the case of $s = 3$ or 4, the voters excluded from the alliance, whose powers decrease as a result of its formation,[5] can retaliate by forming a counter-alliance, reducing

[4] Note that for all n the power of a voter in $\mathcal{M}_{2n}$ is the same as in $\mathcal{M}_{2n+1}$.

[5] This can be seen without any detailed calculation: it follows from the power-maximizing property of $\mathcal{M}_n$. If $\mathcal{W} \cong \mathcal{M}_n$ and an expedient alliance is formed, then by definition the powers of its members increase. Hence the total power of all voters excluded from the alliance must decrease, and by symmetry the power of each of them also decreases.

the powers of the members of S and even turning them into dummies. In the case of $s = 6$ or 7, five of the six or seven members will be tempted to eject the remaining one or two, thereby increasing their (direct and indirect) power from $\frac{5}{16}$ to $\frac{3}{8}$.

If voters behave rationally, they will anticipate the adverse consequences of forming alliances with $s = 3, 4, 6$ and 7, and hence such alliances will not be formed, although they are expedient according to our definition (see Section 2).[6]

But an alliance with $s = 5$ is stable. The four members left out of it become dummies and can do nothing about it. Note that although three of the five alliance members may be tempted to form an internal sub-alliance (in order to raise their indirect power from $\frac{3}{8}$ to $\frac{1}{2}$), they are in fact deterred from doing so because they know that, in reaction, the two members of S left out of the three-member cabal will surely dissolve the original five-member alliance, and may even form a new (dictatorial) alliance with three of the four members that were left out of S.

An interesting situation arises when $\mathcal{W} \cong \mathcal{M}_n$ with even n, and an alliance that contains exactly half of the members.

3.3 Example Let $\mathcal{W} \cong \mathcal{M}_8$. Here the power of each voter is $\frac{35}{128}$. Now consider an alliance $(S; \mathcal{W}_S)$ with $s = 4$. As before, we may assume that the alliance is optimal. Then its members' direct power is $\frac{3}{8}$ and their indirect power is $\frac{45}{128}$. So this alliance is expedient.

However, the remaining four voters, whose power is much reduced, can retaliate by forming their counter-alliance, $(T; \mathcal{W}_T)$, where $T = N - S$. Now the indirect power of members of S will depend on the choice of $\mathcal{W}_T$. This indirect power will be greatest if the probability of the bloc $\&_T$ voting 'yes' is greatest, because only if $\&_T$ votes 'yes' will the vote of members of S make any difference. But the probability of $\&_T$ voting 'yes' will be greatest iff the number of winning coalitions of $\mathcal{W}_T$ is as high as possible – which is the case precisely if $\mathcal{W}_T \cong \mathcal{M}_4^*$ (that is, two votes sufficient to approve a bill). In this case, which is the best that members of S can hope for, their indirect power will still be only $\frac{33}{128}$ – less than their original power in $\mathcal{W}$. As for members of T: they too will have power $\frac{33}{128}$ at most, depending on the rule $\mathcal{W}_S$ chosen by the first alliance. In any case, all voters are now worse off than originally. This seems to give rise to a kind of prisoner's dilemma:[7] if one of

[6]The definition assumes that formation of the alliance is the only change that takes place.

[7]Cf. Gelman's [7] discussion of situations of this kind. As our analysis shows, the analogy with a true prisoner's dilemma is incomplete.

the two alliances dissolves, its members will be in the worst possible position; but if neither of them dissolves, all voters will still be worse off than in the original $\mathcal{W}$. So the alliances will not dissolve unless they can agree to do so simultaneously. However, this apparent dilemma can be resolved in a more radical way. Every member of each alliance will be tempted to defect and join the opposite camp – thereby making the latter dictatorial, and reducing the remaining members of the former to dummies. Thus alliances of size 4 are unstable, and if voters behave rationally they will anticipate this and will not form such alliances.

The following theorem surveys in general the expediency and stability of optimal alliances when $\mathcal{W} \cong \mathcal{M}_n$.

3.4 Theorem *Let $\mathcal{W} \cong \mathcal{M}_n$, with $n > 3$. Consider optimal alliances $(S; \mathcal{W}_S)$ where $S \subsetneq N$. We distinguish four cases, according to the residue of n modulo 4.*

Case 0: $n = 4m$. *Then the alliance is expedient iff* $3 \leq s \leq n-1$, *and expedient as well as stable iff* $s = 2m+1$.

Case 1: $n = 4m+1$. *Then the alliance is expedient iff* $3 \leq s \leq n-2$, *and expedient as well as stable iff* $s = 2m+1$.

Case 2: $n = 4m+2$. *Then the alliance is expedient iff* $3 \leq s \leq n-1$, *and expedient as well as stable iff* $s = 2m+2$ *or* $s = 2m+3$.

Case 3: $n = 4m+3$. *Then the alliance is expedient iff* $3 \leq s \leq n-2$, *and expedient as well as stable iff* $s = 2m+2$ *or* $s = 2m+3$.

Proof The arguments used in the preceding examples can easily be extended and seen to apply here. As far as the expediency claims are concerned, for small values of n one can perform manually the simple, albeit somewhat tedious, computation for each alliance of size s. For larger values of n, say $n \geq 13$, the calculation is still quite simple for extreme values of s (close to 3 or to n). For the general case, see Gelman [7, Subsection 3.3].[8]

An alternative way to prove the expediency claims is to use the so-called Rae index ρ (where, for any SVG $\mathcal{W}$, $\rho_a[\mathcal{W}]$ is the probability that voter a of $\mathcal{W}$ is successful; i.e., that the outcome of a division accords with a's vote). Since by Penrose's identity $\psi_a = 2\rho_a - 1$ (see [4, Theorem 3.2.16]), it is enough to show that, for the s and n specified by our theorem,

$$\rho_a[\mathcal{W} \| \mathcal{W}_S] > \rho_a[\mathcal{W}] \quad \text{for all } a \in S. \tag{9}$$

[8]By the way, Gelman shows that as n increases, the alliance size that maximizes its members' indirect power asymptotically approaches a value of approximately $1.4\sqrt{n}$.

This inequality can be proved using the fact that $\rho_a[\mathcal{W}_S]$ is sufficiently greater than $\frac{1}{2}$.[9]

As for the stability claims, the following observations are in order.

In Case 0, if $m > 1$ an alliance of size $2m$ is unstable for the reason illustrated for $m = 2$ by Example 3.3. The same kind of thing happens for all $m > 1$.

Case 1 is unproblematic: here Examples 3.1 and 3.2 are entirely typical.

In Case 2, an alliance of size $2m + 1$ is expedient, but the internal power of its members is less than twice their original power. The $2m + 1$ remaining voters – whose power is reduced – can retaliate by forming a counter-alliance. Now all voters' indirect power is half of their internal power – which is less than their original power – so the situation is unstable. This is similar to the situation in Case 0 with alliances of size $2m$, except that now, since $2m + 1$ is odd, each alliance of this size can only choose the (self-dual) majority rule as its internal SVG.

In Case 2 and Case 3, an alliance of size $2m + 3$ contains a 'redundant' member, because even an alliance of size $2m+2$ is dictatorial. However, ejecting one of the $2m + 3$ members will not change the powers of the remaining $2m + 2$,[10] so they have nothing to gain by this. ∎

We know that if $\mathcal{W} \cong \mathcal{M}_n$ and an expedient alliance is formed, then all the voters excluded from the alliance lose power (see footnote 5). The following example shows that this need not happen if $\mathcal{W}$ is a super-majority WVG: somewhat surprisingly, formation of an alliance can be Pareto optimal!

3.5 Example Let $\mathcal{W} \cong [4; 1, 1, 1, 1, 1]$. Here the power of each voter is $\frac{1}{4}$, so the total power is $\frac{5}{4}$.

Now suppose an alliance $(S; \mathcal{W}_S)$ is formed, where $s = 3$ and $\mathcal{W}_S \cong \mathcal{M}_3$. This alliance is expedient, because the indirect power of each of its members is $\frac{3}{8}$. However, the power of each of the two excluded voters is still $\frac{1}{4}$, so they are not worse off than before. The total power is now $\frac{13}{8}$.

Moreover, this alliance is stable. On the one hand, no member of the alliance would wish to defect because a defector would lose power (even if the two members left behind were to maintain a – necessarily inexpedient – two-member alliance!). On the other hand, members of the alliance have no incentive to admit one of the two excluded voters, because this cannot increase the powers of the three old members.

[9] Intuitvely speaking, this means that a's vote has a sufficiently good chance of 'carrying along with it' all the votes of the other $s - 1$ members of S. This boosts a's probability of belonging to the majority camp in $\mathcal{W}\|\mathcal{W}_S$ compared to that in $\mathcal{W}$.

[10] See footnote 4.

Nevertheless, it is worth noting that an optimal four-member alliance is also expedient, as the power of each of its members is $\frac{3}{8}$ – the same as in the preceding case. Such an alliance is also clearly stable. Since the single excluded voter is a dummy, the total power is $\frac{3}{2}$.

So if four of the assembly members are vindictive towards the fifth, they will be inclined to form a four-member alliance; otherwise a three-member alliance seems more likely to form. As we have seen, this will maximize the total power, and is in fact Pareto optimal.

This somewhat counter-intuitive example shows that Gelman's assertion [7, p. 5] (clearly made with $\mathcal{W} \cong \mathcal{M}_n$ in mind) that '[f]orming coalitions[11] is beneficial to those who do it but is negative to 'society' as a whole, at least in terms of average voting power' does not hold in general.

4 Unanimity

In this section we shall assume $\mathcal{W}$ is an n-voter unanimity WVG: $\mathcal{W} \cong \mathcal{B}_n$. The power of each $a \in N$ in $\mathcal{W}$ is

$$\psi_a[\mathcal{W}] = \frac{1}{2^{n-1}}. \tag{10}$$

We consider alliances $(S; \mathcal{W}_S)$ where $2 < s < n$; and, as before, for reasons of symmetry we can confine our attention to cases where the members of S are equipotent. We rule out the cases $\mathcal{W}_S \cong \mathcal{B}_s$ and $\mathcal{W}_S \cong \mathcal{B}_s^*$ in which $\mathcal{W}_S$ itself is a unanimity WVG or its dual, as the alliance would then be inexpedient.[12]

On the other hand, any other alliance in which the voters are equipotent is expedient. To see this, let $\eta[\mathcal{W}_S]$ be the Banzhaf score in $\mathcal{W}_S$ of any $a \in S$ (that is, the number of winning coalitions in which a is critical). Then the direct power of a is $\eta[\mathcal{W}_S]/2^{s-1}$, and a's indirect power is

$$\psi_a[\mathcal{W} \| \mathcal{W}_S] = \frac{\eta[\mathcal{W}_S]}{2^{s-1}} \cdot \frac{1}{2^{n-s}} = \frac{\eta[\mathcal{W}_S]}{2^{n-1}}. \tag{11}$$

As we have ruled out $\mathcal{W}_S$ being a unanimity WVG or its dual, it is clear that $\eta[\mathcal{W}_S] > 1$, so by (10) the alliance is expedient, as claimed.

[11] See footnote 3.

[12] In both cases the indirect power of a member of the alliance is the same as in $\mathcal{W}$. Moreover, forming an alliance with an internal unanimity rule is vacuous, as the composite SVG $\mathcal{W} \| \mathcal{W}_S$ would be identical to the original $\mathcal{W}$.

However, voters excluded from the alliance gain even more power than those inside it. To show this, let us denote by '$\omega[\mathcal{W}_S]$' the number of winning coalitions in $\mathcal{W}_S$. Then it is easy to see that the power of each $b \in N - S$ in $\mathcal{W}\|\mathcal{W}_S$ is

$$\psi_b[\mathcal{W}\|\mathcal{W}_S] = \frac{\omega[\mathcal{W}_S]}{2^s} \cdot \frac{1}{2^{n-s-1}} = \frac{\omega[\mathcal{W}_S]}{2^{n-1}}. \tag{12}$$

Since $\mathcal{W}_S$ is not a unanimity WVG, it is impossible for every $a \in S$ to belong, let alone be critical in, every winning coalition. So clearly $\omega[\mathcal{W}_S] > \eta[\mathcal{W}_S]$; hence $\psi_b[\mathcal{W}\|\mathcal{W}_S] > \psi_a[\mathcal{W}\|\mathcal{W}_S]$ for every $b \in N - S$ and $a \in S$, as claimed.[13]

Thus, if considerations of envy were admitted, they would imply that no single expedient alliance in $\mathcal{W}$ can be stable. In what follows, we shall ignore envy.

From now on let us make the reasonable assumption that only optimal alliances are formed.

We can specify $\eta[\mathcal{W}_S]$ precisely for optimal alliances, distinguishing two cases, according as s is even or odd. As is well known,[14]

$$\eta[\mathcal{W}_S] = \begin{cases} \dfrac{1}{2}\dbinom{2m}{m} & \text{if } s = 2m, \\ \dbinom{2m}{m} & \text{if } s = 2m+1. \end{cases} \tag{13}$$

By the way, from (11) and (13) it is easy to see that whether s is even or odd, the power of a member of an optimal alliance of size s is greater than that of a member of any expedient alliance of size $s - 1$. Thus Gelman's assertion [7, p. 7] (clearly made with $\mathcal{W} \cong \mathcal{M}_n$ in mind) that 'it is never a good idea to have a coalition[15] with an even number of members: if m is even, it is always as good or better to be in a coalition of size $m - 1$' does not hold in general.

As for $\omega[\mathcal{W}_S]$, we can specify it precisely for odd s. For even s we can specify the two extreme values: for $\mathcal{W}_S \cong \mathcal{M}_s$ and $\mathcal{W}_S \cong \mathcal{M}_s^*$. It is easy to

[13] We have ruled out $\mathcal{W}_S \cong \mathcal{B}_s^*$, as in this case the alliance, albeit feasible, is not expedient. If such an alliance were formed, then $\eta[\mathcal{W}_S] = 1$ and $\omega[\mathcal{W}_S] = 2^s - 1$. So by (11) the alliance members gain no power, while by (12) those excluded from it do gain power. As power is the (only) payoff in the power-game $\mathcal{P}_\mathcal{W}$ we are considering in this paper, there is no reason – other than pure altruism – for forming such an alliance. This is why we assume it will not form.

[14] See [4, Thm. 3.3.8].

[15] See footnote 3.

see that

$$\omega[\mathcal{W}_S] = \begin{cases} 2^{2m-1} - \dfrac{1}{2}\dbinom{2m}{m} & \text{if } s = 2m \text{ and } \mathcal{W}_S \cong \mathcal{M}_s, \\ 2^{2m-1} + \dfrac{1}{2}\dbinom{2m}{m} & \text{if } s = 2m \text{ and } \mathcal{W}_S \cong \mathcal{M}_s^*, \\ 2^{2m} & \text{if } s = 2m+1. \end{cases} \tag{14}$$

What happens if a member of an optimal alliance $(S; \mathcal{W}_S)$ leaves it? Of course, if the alliance were then to disband, all voters would lose power. But we now make the reasonable assumption that if $s > 3$ the $s - 1$ remaining members will re-form an optimal alliance. The possible exception is $s = 3$, in which case the remaining two members cannot form an expedient alliance, so they may not form one at all.

It is easy to see that if a member of an optimal alliance is expelled from it, the remaining members will lose power. So we can rule out expulsion as a source of instability. On the other hand, we saw that when an expedient alliance $(S; \mathcal{W}_S)$ is formed, the voters excluded from it gain more power than those included in it. Would this tempt a member of an optimal alliance to defect?

To see whether defection from $(S; \mathcal{W}_S)$ is advantageous, let us first consider the case where s is even, $s = 2m$, where $m > 1$. From (11) and (13) it follows that before the defection the power of the prospective defector is $\binom{2m}{m}/2^n$. After defecting, the defector will be one of the voters excluded from an optimal alliance of size $2m - 1$, so by (12) and (14) her power will be $2^{2m-1}/2^n$. It is easy to prove by induction on m that $2^{2m-1} > \binom{2m}{m}$ for all $m > 1$. So defection is indeed advantageous. It follows that optimal alliances of even size are unstable, being vulnerable to defection.

Now let us turn to the somewhat more tricky case where s is odd, $s = 2m + 1$, where $m \geq 1$. In this case the defector deserts $2m$ partners who will form an optimal alliance, except perhaps in the special case $m = 1$. Let us first assume that these $2m$ voters form an alliance with internal decision rule isomorphic to $\mathcal{M}_{2m}^*$. By (11) and (13), before defecting the power of the prospective defector is $\binom{2m}{m}/2^{n-1}$. By (12) and (14), his power after defecting will be $> 2^{2m-1}/2^{n-1}$. By induction on m it is easy to see that $2^{2m-1} \geq \binom{2m}{m}$ for all $m \geq 1$, so the defector will gain power, and the defection is advantageous.

But a prospective defector cannot depend on this calculation, because the $2m$ deserted partners may not adopt a decision rule isomorphic to $\mathcal{M}_{2m}^*$ but one with a smaller number of winning coalitions. The worst-case scenario for defection is that the rule they choose is isomorphic to to $\mathcal{M}_{2m}$.[16] From (11), (12), (13) and (14) it can be seen that in case they do so, the defector will gain power if $2^{2m-1} - \frac{1}{2}\binom{2m}{m} > \binom{2m}{m}$, or, tidying up:

$$2^{2m} > 3\binom{2m}{m}; \tag{15}$$

and the defector will lose power if the opposite inequality holds.

It is easy to prove (15) by induction for all $m \geq 3$. However, for $m = 1$ and $m = 2$ the opposite inequality holds. It follows that all alliances except those of size 3 or 5 are unstable, as a member of such an alliance will be better off defecting. Only alliances of size 3 and 5 can be stable, as defection of a single member from them is ill-advised.

For brevity, we shall call an alliance of size 3 with rule $\cong \mathcal{M}_3$ a *trio*, and an alliance of size 5 with rule $\cong \mathcal{M}_5$ a *quintet*. We shall call a voter who is not a member of any alliance a *singleton*.

Once a trio or a quintet is formed, then from the viewpoint of the other voters it behaves as a single voter. For example, if $n > 3$ and one trio is formed, then from the viewpoint of the remaining $n - 3$ voters it looks as if they are in a unanimity WVG with $n - 2$ voters.

If there are three singletons left, they may form a new trio. Similarly, five singletons may form a new quintet. Also, two singletons may join a trio and turn it into a quintet. Ultimately there will emerge a configuration in which the original n voters form quintets, trios and singletons.[17] We shall refer to these as '5–3–1' configurations.

Which 5–3–1 configurations are stable and therefore likely to form? Of course, if $n = 4$ or $n = 5$, there is only one stable option: the formation of one trio.

For $n > 5$ we must distinguish five cases, according to the residue of n modulo 5.

Case 1: If $n = 6$, there are two Pareto optimal 5–3–1 configurations:

First, one quintet and one singleton. Each member of the quintet has indirect power $\frac{3}{16}$ and the singleton has power $\frac{1}{2}$.

Second, two trios, each of whose members has indirect power $\frac{1}{4}$.

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[16]This may even be most likely, because, among all optimal alliances this is the only one with a proper SVG – which has some advantages. If $m = 1$, the two deserted members have no reason at all to form an alliance (see Thm. 2.1); and forming an alliance with decision rule isomorphic to $\mathcal{M}_2$ is vacuous, because $\mathcal{M}_2 = \mathcal{B}_2$ (see footnote 12).

[17]We rule out the formation of 'super-alliances' one or more of whose members are themselves a trio or quintet, because such a super-alliance would not be optimal as an alliance in $\mathcal{W}$.

Any other 5–3–1 configuration is Pareto inferior to one of these two, but neither of them is Pareto superior to the other. So we cannot say for certain which of the two is more likely to form.

Similarly, for all $n = 5m + 1$, where $m \geq 1$: we can have m quintets and a singleton; or $m - 1$ quintets and two trios.

Case 2: If $n = 7$, there are two Pareto optimal 5–3–1 configurations:

First, one quintet and two singletons. Each member of the quintet has indirect power $\frac{3}{32}$ and each singleton has power $\frac{1}{4}$.

Second, two trios and a singleton. Each member of a trio has indirect power $\frac{1}{8}$ and the singleton has power $\frac{1}{4}$.

Again, any other 5–3–1 configuration is Pareto inferior to one of these two, but neither of them is Pareto superior to the other. So we cannot say for certain which of the two is more likely to form.

Similarly, for all $n = 5m + 2$, where $m \geq 1$: we can have m quintets and two singletons; or $m - 1$ quintets, two trios and a singleton.

Case 3: If $n = 8$, there is just one Pareto optimal 5–3–1 configuration: a quintet and a trio. Each member of the former has indirect power $\frac{3}{16}$ and each member of the latter has $\frac{1}{4}$. This 5–3–1 configuration is Pareto superior to any other, and is therefore the one likely to form.

Similarly, for all $n = 5m + 3$, where $m \geq 1$: we have m quintets and one trio.

Case 4: If $n = 9$, there are two Pareto optimal 5–3–1 configurations:

First, one quintet, one trio and a singleton. Each member of the quintet has indirect power $\frac{3}{32}$, each member of the trio has indirect power $\frac{1}{8}$, and the singleton power $\frac{1}{4}$.

Second, three trios, each of whose members has indirect power $\frac{1}{8}$.

Again, any other 5–3–1 configuration is Pareto inferior to one of these two, but neither of them is Pareto superior to the other. So we cannot say for certain which of the two is more likely to form.

Similarly, for all $n = 5m + 4$, where $m \geq 1$: we can have m quintets, one trio and one singleton; or $m - 1$ quintets and three trios.

Case 5: If $n = 10$, there is just one Pareto optimal 5–3–1 configuration: two quintets, each of whose members has indirect power $\frac{3}{16}$. This 5–3–1 configuration is Pareto superior to any other, and is therefore the one likely to form.

Similarly, for all $n = 5m$, where $m > 1$: we have m quintets.

5 Alliances with a dummy

In this section we wish to explore the possible role of a dummy in the formation of an alliance.

Somewhat surprisingly, a dummy can become empowered via a feasible alliance, as the following example shows.

5.1 Example During the first period of the European Union (1958–73), the so called 'qualified majority' (QM) decision rule prescribed for its six-member Council of Ministers by the Treaty of Rome (1957) was a WVG that assigned to France, Germany, Italy, Belgium, The Netherlands and Luxembourg weights $4, 4, 4, 2, 2$ and 1, respectively; and the quota required for passing resolutions on most issues was 12. In this WVG, which is isomorphic to $[12; 4, 4, 4, 2, 2, 1]$, the powers of the members were $\frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{3}{16}, \frac{3}{16}$ and 0, respectively.

Although the Treaty stipulated that this decision rule would only become effective in 1966, and in practice it was rarely invoked from 1966 to 1973, as decisions were normally made by consensus, the rule is quite puzzling. What was the point of giving Luxembourg a useless weight of 1, and why did Luxembourg agree to be a dummy? This puzzle has often been mentioned in the literature on decision making in the EU.

Of course, it is possible that the original signatories of the Treaty of Rome, including the government of Luxembourg, were simply unaware of the QM anomaly. But there may be another explanation. As is well known, Belgium, The Netherlands and Luxembourg had operated a Benelux Customs Union since 1947 (replaced in 1966 by the Benelux Economic Union). It is quite possible that the Benelux countries agreed informally to act as an alliance within the EU. A reasonable internal decision rule could be $\cong [3; 2, 2, 1]$ – which is in fact a simple majority rule. It is easy to verify that under this alliance the indirect power of each of the three members would be $\frac{3}{16}$. Thus the alliance, albeit inexpedient, is feasible – and it would empower Luxembourg![18]

Note that, somewhat surprisingly, a Benelux alliance would benefit the three non-Benelux members: each would now have power $\frac{3}{8}$.

Is it also possible to form an *expedient* alliance with a dummy? Moreover, is it possible to form such an alliance which will also increase the power of all those excluded from the alliance? The answer to both these questions is positive, as shown by the following two examples.

[18]We are indebted to Simon Hix for this explanation. We are also grateful to Frank Steffen and Matthew Braham for independently raising with us the question as to whether a dummy can be empowered via an alliance.

5.2 Example Let $\mathcal{W} = [13; 5, 5, 3, 3, 3, 1]$. Here the power of each of the first two voters is $\frac{7}{16}$, that of each of the next three voters is $\frac{3}{16}$ and the last voter is a dummy.

Now suppose the last four voters form an alliance with internal decision rule $\cong \mathcal{M}_4$. Then the indirect power of each of these four is $\frac{9}{32}$. So the alliance is expedient.

True, this alliance is not stable, for if the voter with weight 1 is ejected, and the remaining three form an alliance with internal rule $\cong \mathcal{M}_3$, then the indirect power of each of these three will be $\frac{3}{8}$.[19]

5.3 Example Let $\mathcal{W} = [10; 2, 2, 2, 2, 2, 1]$. Here each of the first five voters has power $\frac{1}{16}$ and the last voter is a dummy.

Now suppose the last five voters form an alliance with internal decision rule $\cong \mathcal{M}_5$. Then each of these five has indirect power $\frac{3}{16}$ and the first voter, excluded from the alliance, has power $\frac{1}{2}$. So the alliance is not only expedient, but also beneficial to the voter excluded from it.

Note that in this case ejecting the voter with weight 1 from the alliance will not benefit the remaining four: the best they can do is form an alliance with internal rule isomorphic to $\mathcal{M}_4$ or $\mathcal{M}_4^*$ – which will still give them indirect power $\frac{3}{16}$.

However, neither of these two alliances is stable. In the case of the alliance of size 5, which includes the [former] dummy, a voter with weight 2 will be better off defecting. Following such defection, the remaining three voters with weight 2 will be better off if they eject the dummy. In the case of the alliance of four voters with weight 2, it will also be advantageous for any of them to defect.

6 Discussion

The results obtained in this paper constitute a modest advance on the ground covered by us in [6] and by Gelman [7].

The main result of Section 3, Thm. 3.4, is not too surprising. It confirms – as well as making more precise – what is intuitively almost obvious: if $\mathcal{W} \cong \mathcal{M}_n$, then only [dictatorial] alliances whose size is just over $\frac{n}{2}$ can be both expedient and stable. A smaller alliance is vulnerable to annihilating

[19]However, this alliance too is unstable: the two voters with weight 5 excluded from it – whose loss of power would be substantial as a result of its formation – would be able to tempt one of the three alliance members to join them in forming a 3-member dictatorial alliance with an internal decision rule $\cong \mathcal{M}_3$, thereby increasing their power to $\frac{1}{2}$. This alliance would be stable.

retaliation by those excluded from it, whereas a larger alliance is, in a sense, too large for its own good. However, in the cases where n is even, and particularly where it is divisible by 4, the problem of stability of an alliance of size $\frac{n}{2}$ does require some careful scrutiny.

Example 3.5 illustrates a seemingly paradoxical phenomenon that can occur in a super-majority WVG: a stable expedient alliance may also benefit those excluded form it. This of course depends on the fact that power games are not constant-sum games.

Section 4 dealing with the case $\mathcal{W} \cong \mathcal{B}_n$, provides a more extreme illustration of this phenomenon. But the main surprise in this section is the exceptional behaviour of quintets and trios, and the unique stability of some 5–3–1 configurations. We think that these results could hardly have been anticipated without detailed analysis.

Finally, in Section 5 we illustrate the seemingly paradoxical fact that a feasible, and even expedient, alliance may empower a dummy while at the same time benefiting the voters excluded from it. This is a possible explanation for the acquiescence of Luxembourg in its dummy status under the QM rule of the original 6-member EU Council of Ministers. Nevertheless this paper should not be viewed as an attempt to provide an explanation of actual alliance formation in terms of Penrose power. If this were the case, then it would be important to compare this with explanations based on other measures of power. Rather, our idea was to do a "what if" exercise: what would happen if voters were to play a cooperative "power game", which is played by forming alliances and in which the resulting Penrose power of the players-voters were regarded by them as the payoff (which, by the nature of Penrose power must be a non-transferable "utility"). Of course, one could similarly invent other games, based on some other payoff, which could well be some other measure of voting power (e.g., the Shapley–Shubik [10] or the Penrose–Banzhaf power index as modified by Owen [8] for voting games "with a priori coalitions"), and which could well lead to different results than we obtained.

Admittedly, the ground covered so far in the study of power games is rather limited. In this paper we have confined ourselves entirely to WVGs rather than dealing with SVGs that may not be weighted. Moreover, we have concentrated for the most part on symmetric WVGs of the simplest kinds: those isomorphic to $\mathcal{M}_n$ or $\mathcal{B}_n$. A more general theory of power games $\mathcal{P}_{\mathcal{W}}$ remains to be developed.

There is also need for studies of the extent to which considerations of a priori voting power can explain the formation, dissolution and [in]stability of real-life alliances, as well as defection of voters from these alliances. So far – apart from a claim by Aumann [3], unsupported by any detailed data

– there are only two proper studies of this kind known to us – one by Chua and Felsenthal [2], and the other by Andjiga, Badirou, and Mbih [1]– both of which are concerned with the formation of governmental alliances, i.e., ruling coalitions within legislatures, and both arrive at negative or sceptical conclusions.[20]

Thus, for example, we re-analysed the 77 governmental alliances examined by [2], and found that in 51 of these the largest party was a dictator, that in additional 19 governmental alliances at least one member became a dummy or lost some power by joining the alliance, and that only seven of these alliances were either feasible or expedient in comparison to a situation where no alliance was formed. These results clearly indicate that considerations of feasibility and expediency (in the technical sense of the present paper) play no significant role in the formation of governmental alliances.

However, in contrast to governmental alliances – which *must* form in legislatures where no party controls an absolute majority of the votes – considerations of feasibility and expediency in forming alliances may play a more significant role in international organizations or corporate boards of directors, where alliances may but need not form. To verify this one would need to conduct empirical research. But this will not be easy, because the formation of an alliance in such bodies is often tacit and hence difficult to detect.

[20] Two additional studies of this kind known to us are those by Riker [9], and by Felsenthal and Machover [5], which aim to ascertain whether inter-party migrations of delegates in the French National Assembly during the period 1953–54 can be explained by considerations of voting power. However, both these studies seem to us inappropriate, as they ignore in their calculations the existence of a (dictatorial) governmental alliance.

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