Groups Can Make a Difference: Voting Power Measures Generalized

Claus Beisbart

21st February 2008

Abstract

The voting power of a voter is often measured in terms of the probability that that voter's vote is critical. This measure is extended to a hierarchy of power measures of different ranks. The higher-rank measures quantify the extent to which a voter can join groups that make a difference as to whether a bill passes or not. It is argued that the hierarchy of measures allows for a more appropriate assessment of voting power, particularly of a posteriori voting power in case the votes are stochastically dependent. Also, the new measures discriminate between voting games that can not be distinguished in terms of the probability of criticality only. The new measures are defined, applied to simple examples, and basic properties are established for them.

1 Introduction

Voting rules assign voters the power to affect the outcome of collective decisions. This is the starting point for a research program at the borderline between political science, social choice theory and political philosophy. The research program aims at measuring the voting power of each voter, i.e. the extent to which her vote can affect the outcome of a collective decision (Felsenthal & Machover 1998 or FM, for short, pp. 2, 35–6). Very often, measures of voting power are calculated for real-world voting systems in order to check whether these systems comply with basic requirements on democracy (see Felsenthal & Machover 2000 for such a study, e.g.).

A very popular measure of voting power is the probability that a voter is critical (pivotal). For calculating this measure, one assumes that the voters have two options only, say yes and no. A voter is critical, if and only if the outcome would have been different, had she voted differently (cf. FM, Ch. 3).

For obtaining the probability that a voter is critical, one needs a probability model over the voting profiles. Very often, the Bernoulli model is assumed.

According to the Bernoulli model, the different votes are cast independently, and, for each voter, the probability of a yes-vote is .5. The probability of being critical under the Bernoulli model is the popular Banzhaf measure of voting power (see FM, Ch. 3 for an introduction). The Banzhaf measure of voting power is often called a measure of *a priori* voting power (FM, p. 20, e.g.). The reason for this name is as follows: If no empiric information on the voting behavior of the voters is available, one has to reason a priori in order to determine the probability model that enters the measure. A very reasonable strategy is to apply the Principle of Insufficient Reason. This strategy yields the Bernoulli model and the Banzhaf measure (Comment 3.1.3 in FM, pp. 37–8).

But the probability of being critical can also be calculated under different, more realistic probability models; under probability models, e.g., that were fitted to empirical data from past votes (see Good & Mayer 1975, Chamberlain & Rothschild 1981 and Gelman et al. 2004 for models different from the Bernoulli model). The resulting measure is often taken to be a measure of *a posteriori* voting power (cf. FM, p. 20 for a posteriori voting power) and has been suggested by Morriss (1987), p. 169, for instance. Beisbart & Bovens (2008) use this measure in order to analyze the U.S. Electoral College. A slight variation of this measure is proposed by Kaniovski & Leech (2007).[1]

However, very recently, doubts have arisen as to whether the probability of being critical provides a completely satisfying measure of voting power. This is particularly vivid in the case of a posteriori voting power. Consider the following example due to G. Wilmers (Wilmers' example, for short; Machover 2007, p. 3).

Example 1.1 *Suppose that five voters vote following simple majority voting. There are 32 voting profiles possible. Assume that the profiles with exactly two or exactly three yes-votes have probability zero, each, and that the other profiles have a probability of 1/12, each. Consider an arbitrary voter. The probability of her being critical is zero, because all profiles under which a voter is critical have zero probability. Thus, everybody has zero voting power.*

But this assignment of a posteriori power seems strange, to say the least. What is particularly offensive is the claim that nobody has voting power. As Machover (2007), p. 3 puts it, "it would be absurd to claim that every voter here is powerless, in the sense of having no influence over the outcome [...]". He concludes that, as a measure of voting power, the probability of being critical "behaves in a strange way [...] At least, [...] [it] doesn't tell the whole story about that influence [a voter's influence]."

Wilmers' example also points to a second, slightly different problem. Probability models can be specified such that the probability of being critical is zero for each voter *even under alternative voting rules for the same electorate.* Accordingly, the related measure of a posteriori voting power assigns every voter zero

[1]Of course, it is possible that empirical data favor the Bernoulli model for some specific setting. Thus, measures of a posteriori voting power may use the Bernoulli model as well.

power for each of the alternative rules. Thus, we can not distinguish between the alternative voting rules in terms of power. This is dissatisfying. Is it really true that the alternative voting rules are completely on par as far as the powers to affect the outcome are concerned? The probability of being critical has not the discriminatory power that we wish it to have.

Even the Banzhaf measure suffers from a similar problem. Although the Bernoulli model assigns each voting profile a non-zero probability, the following is possible under the Bernoulli model. Two voting rules are different – there is at least one voting profile under which one rule yields acceptance, whereas the other rule yields rejection – but for each voter, the probability of being critical, i.e. the Banzhaf measure is identical under both rules. An example and qualifications will be given below in Subsec. 4.3. The question is again whether the alternative voting rules are really completely on par, as far as voting power is concerned.

This paper proposes to go beyond the probability of criticality in order to measure voting power. I look for measures of voting power that fulfill the following requirements:

- R1 When calculated under the assumption of the Bernoulli model, the measures partly coincide with the Banzhaf measure of voting power.
- R2 The measures are conceptually tied to the notion of criticality.
- R3 The measures concern individual voters.
- R4 The contraintuitive results for Wilmers' example are avoided under the measures.

These requirements are not beyond criticism. In particular, R2 might be given up. For instance, in order to measure power, one might start from the Shapley-Shubik index (Shapley & Shubik 1954) and see whether it can be generalized for all kinds of probability models. However, very different ideas underly the Banzhaf measure and the Shapley-Shubik index. This has been captured by the distinction between I- and P-power (FM, p. 36). P-power is based upon the idea that the winners of an election jointly earn a fixed prize. This idea does not seem very realistic for many applications (cf. Coleman 1971, p. 272, after FM, p. 18). I will thus stick to I-power. I take it that it is worthwhile to generalize the popular Banzhaf measure and the related probability of criticality.

According to my proposal, each voter will not only be characterized by the probability of criticality, but by a hierarchy of measures of different ranks. The measures quantify the extent to which a voter can make a difference as a member of a group. As is clear from R1, my measures will not presuppose a particular probability model. I am thus interested in a priori as well as in a posteriori voting power.

My work bears the following relation to the existing research literature. The Banzhaf measure has first been introduced by Penrose (1946). It was independently used by different researchers, e.g. by Banzhaf (1968). The history of the Banzhaf measure is traced by Felsenthal & Machover (1998), pp. 6–10. Recently, Bovens & Beisbart (2007) have suggested a measure that fulfills R1–R4. The proposal put forward in this paper is completely unrelated. In particular, unlike Bovens & Beisbart (2007), my measures do not rely on causal information. I take it that this is an advantage, since causal information is often very difficult to obtain. Moreover, abstracting from causal relations helps me to focus on the power that a voter has because of the voting rule only.

The plan of the paper is as follows. Sec. 2 starts from the notion of voting power and introduces the main idea of this paper. It turns out that my idea requires one to think of the criticality of groups of votes first; group criticality is defined in Sec. 3. In Sec. 4, I provide the definition of my measure. A few mathematical results are proven, and applications are discussed. Finally, discussion points are given in Sec. 5.

2 Voting power

In order to introduce the main idea of this paper, I will first consider the standard notion of voting power, or I-power, more specifically.

Let me start with the notion of power. According to Morriss (1987), pp. 32–35 power-over and power-to have to be distinguished. Voting power is a variety of power-to, but it is not the power to vote, but rather the *power to affect the outcome of a collective decision by voting*. Thus, the reason why it is called *voting* power is that the focus is on the power that a political agent has due to her vote (cf. Morriss 1987, p. 155).

But what, then, does it mean to affect the outcome of a collective decision (the outcome of a vote, for short)? In the simple framework that is often adopted in voting theory, the outcome is either the passage or the failure of the proposal that is voted on (of the *bill*, for short; FM, p. 35). You affect the outcome of a collective decision, if you make a difference as to whether the bill passes or not. And you affect the outcome of a collective decision *as a voter*, if *your vote* makes a difference as to whether a bill passes or not. That is, whether the bill passes or not, depends on whether you vote yes or no.

The recent literature on voting power focuses on the measurement of voting power. What is measured is the extent to which the vote of a political agent can make a difference as to whether a bill passes or not.

What needs to be clarified now is the "can". Morriss (1987), p. 80–83 has another helpful distinction. He differentiates between ability and ableness. Whereas ability is roughly about what a person could do, if the circumstances were appropriate, assignments of ableness additionally take into account the opportunity to exercise one's ability. For instance, John might have the ability to swim hundred meters in less than one minute. But if there is no suitable pool, he lacks the ableness to do so. Whether John has the ableness and thus the opportunity to swim depends on the circumstances and not just on John.

According to Morriss (1987), p. 83, ableness rather than ability is what political philosophers are usually interested in – it is the opportunities of the people that are investigated. Let us therefore focus on ableness.[2] Essential part of what is to be measured, then, is the extent to which a voter has opportunity to affect the outcome of a collective decision by voting. Indeed, it is the only part of what is measured, because there is nothing interesting about skills here. Thus, what is to be quantified for a voter a is the extent

E to which a's vote has the opportunity to make a difference as to whether a bill passes or not.

But what is the opportunity in question? A very natural answer – indeed the answer taken by most theorists – is this: We take the other votes as given and ask whether they leave the opportunity that the outcome of the collective decision depends on a's vote. If and only if the configuration of the other votes is such that, whether a bill passes or not, depends on a's vote, then a has the opportunity to make a difference. Indeed, a *will* make a difference then.[3]

Altogether, what is to be measured is the extent to which the other votes form a configuration in which, whether the bill passes, depends on a's vote. The only way to measure this extent seems to be to calculate the respective probability. Thus, we arrive at the following measure of voting power for a voter: It is the probability that the configuration of the other votes is such that, whether the bill passes or not, depends on her vote. In more technical terms: It is the probability for a coalition wrt which her vote is critical (see FM, Def. 2.1.1 on p. 11 and Def. 2.3.4 on p. 24).

This line of thought substantiates the statement that the probability of being critical is "arguably the *only* reasonable way" of explicating the notion of I-power in mathematical terms (FM, p. 36). However, as Wilmers' example shows, just calculating the probability of being critical can lead to counterintuitive results according to which nobody has power. The challenge is thus to modify this line of

[2] Morriss (1987), Ch. 22, particularly pp. 157–160, also claims that Banzhaf voting power provides a measure of *ability*, but this will not be important in what follows.

[3] This is different from the example with John's swim. John may have the skill and the opportunity to swim hundred meters in less than one minute, but still not do so. For instance, he may decide not to swim the hundred meters at all. On the contrary, I cannot decide not to make a difference with my vote. If the configuration of the other votes is suitable, then I will always make a difference independently what I do (this is true independently on whether abstention is possible or not). "To make a difference regarding X" does not describe an action, but rather compares the consequences that different options for acting have on X. The power to swim hundred meters in less than a minute and voting power do not completely parallel in this respect.

thought slightly in order to have additional measurements of voting power. These measures should enable us to say something interesting in Wilmers' example.

I follow a lead of Machover (2007), p. 4, who suggests that "the very concept of power that is used in the a priori mode is in some sense too individualistic". In more detail, I propose to proceed in two steps. As a first step, I suggest to shift the focus from single voters to groups of voters for a while. Consider an arbitrary subset of the voters. It may be asked: What is the voting power of this group? To what extent does the group have the opportunity to make a difference as to whether the bill passes or not?

As a second step, I suggest to get back to one voter and to ask: To what extent does she have opportunity to form groups that will make a difference as to whether a bill passes? For instance, to what extent will she together with one other voter jointly make a difference as to whether the bill passes? More generally, what will be quantified for a voter a is the extent

E_κ to which the following is true: There are $(\kappa - 1)$ other voters such that there is opportunity for the votes of a and of these other voters to jointly make a difference as to whether the bill passes.

Obviously, the extent E_1 coincides with extent E. Note, also, that the extent E_κ provides information specifically about a fixed voter (a) and not just for groups.

As before E, the extents E_k will be quantified in terms of probabilities. The idea is thus to calculate a hierarchy of probabilities for each voter and for each possible group size κ. Each probability in the hierarchy will tell us about the opportunity to enter a group that makes a difference. These probabilities, I suggest, provide a very natural extension of the probability of being critical. They tell us something about how important a voter is for whether a bill passes or not. And they will also provide us with some non-zero measures for Wilmers' example.

The move that I suggest is very natural. Suppose that I want to buy a particular house. Unfortunately, I lack the opportunity to do so, because I don't have enough money. A natural way out is to look for someone else such that we two have the opportunity (the money, as it were) to buy the house jointly. Suppose, for instance, that I have many friends F such that F and I can jointly buy the house. Then I have some saying on what will happen with the house.

To be sure, the extent E and the extents E_κ are about different things for $\kappa > 2$. E_2 etc. are not just about a's vote making a difference. Correspondingly, whereas, under E, the other votes are taken as given, under E_2 etc., the other votes are not all given. For E_κ, a is given freedom, so to speak, to pick $(\kappa - 1)$ other votes and to command these votes together with her own vote. The question is whether the configuration of votes leaves the opportunity to pick a fixed number of other votes in such a way as to make a difference.

This raises the following question: Can a measurement of E_2 etc. be properly called a measurement of voting power? On my view, at this point, it is very natural to acknowledge that the extent E is no more than one extent in a more general hierarchy of extents E_κ, viz E_1. Moreover, all of these extents concern the very same question how a voter can affect the outcome with her vote. I therefore suggest to broaden the notion of voting power such as to include the extents E_κ. The hope is that this broader notion helps us to characterize voting games in a more satisfying way than the probability of a voter being critical does.

Let me now spell out my idea in mathematical terms. In the next section I will introduce the general framework and consider groups.

3 Voting powers of groups

In order to model voting rules, I will use simple voting games (SVGs, for short; FM, Def. 2.1.1 on p. 11). A *simple voting game* $\mathcal{W}$ is a collection of subsets of a finite set N with $\emptyset \notin \mathcal{W}$ and $N \in \mathcal{W}$; furthermore monotonicity is required (ib.). N is called assembly. The elements of N represent votes or voters.[4] Subsets of N are called coalitions. There is a one-two-one correspondence between coalitions and voting profiles or bipartitions (Def. 2.1.5, p. 14 in FM). Elements $S \in \mathcal{W}$ are called winning coalitions – the bill passes, iff there is an winning coalition $S \in \mathcal{W}$ such that the votes in S are yes and the other votes in $N \setminus S$ are no. Coalitions $S \notin \mathcal{W}$ are called losing coalitions.

A voter a is $\mathcal{W}$-critical wrt a coalition S, iff $S \cup \{a\} \in \mathcal{W}$, but $S \setminus \{a\} \notin \mathcal{W}$. In this case, if the voter had cast another vote than she actually did, the outcome of the collective decision would have been different. If a is critical and additionally part of S, a is $\mathcal{W}$-critical inside; otherwise she is $\mathcal{W}$-critical outside S (FM, Defs. 2.3.4 and 2.3.6, p. 25–6). In the following, we will mostly drop the "$\mathcal{W}$" in "$\mathcal{W}$-critical".

Let us now consider move from voters $a \in N$ to groups of voters $G \subseteq N$. Criticality for a group G can be defined as follows (cf. Beisbart 2008):

Definition 3.1 *Let $\mathcal{W}$ be a simple voting game with assembly N. Let be $G \subseteq N$. Consider a specific coalition S. G is critical wrt S, iff $S \cup G \in \mathcal{W}$, but $S \setminus G \notin \mathcal{W}$. If G is critical wrt S, G is called critical inside (outside) S, iff $S \in \mathcal{W}$ ($S \notin \mathcal{W}$).*

The basic idea is as follows: Let us assume that S comprises exactly those votes that were yes in a specific vote. G is critical wrt S, iff there is *some* way in which the group could have voted differently such that the outcome of the vote would have been different. To be sure, if the group has more than one vote, then there are many ways the group could have voted differently from the way it did. But for group criticality, it is only required that there is at least one way

[4]Whether the elements in the assembly represent voters or votes does not make a difference, if every voter commands exactly one vote. In this paper, I will stick to this assumption. The assumption is relaxed in Edelman (2004) and Beisbart (2008).

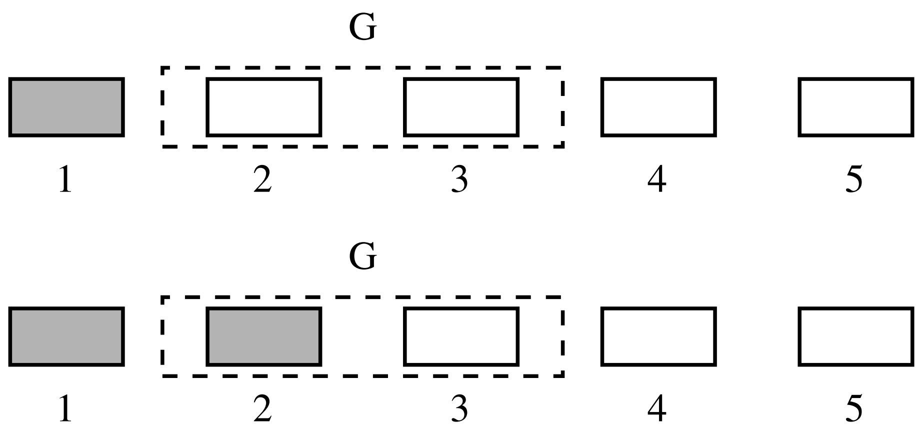

Figure 1: An illustration of group criticality. The voting game is $\mathcal{M}_5$, i.e. a bill is accepted, iff three or more voters vote yes. Each voter is represented with a box. Voters with filled boxes vote yes, whereas voters with empty boxes vote no. Each panel focuses on a particular coalition S that comprises all the voters with a filled box. The group $G = \{2, 3\}$ is critical wrt both coalitions shown.

the group might have voted differently such that the outcome would have been different.[5]

Group criticality can be illustrated using Ex. 1.1 (simple majority voting with five voters). The related simple voting game is called $\mathcal{M}_5$ (Def. 2.3.10 on pp. 25–6 in FM). Label the votes from 1 to 5. Consider the voting profile where 1 votes yes, and the others vote no (i.e. $S = \{1\}$; see Fig. 1). The group $G \equiv \{2, 3\}$ is critical wrt S, because, if both 2 and 3 had voted yes rather no, the outcome of the vote would have been different. Since S is losing, the group is critical without S. G is also critical outside $S = \{1, 2\}$ (second panel). The same group is critical inside $S = \{1, 2, 3\}$, and critical inside $S = \{1, 2, 3, 4\}$, for instance.

Group criticality can be characterized in terms of blocs. For this, start from a SVG $\mathcal{W}$ with assembly N and a group $G \subseteq N$. Let $\&_G$ be the bloc by G (see FM, Def. 2.3.33 on p. 33). Call $\mathcal{W}|\&_G$ the SVG that is obtained from $\mathcal{W}$, if G forms a block (ib.). Very roughly, in $\mathcal{W}|\&_G$, G is forced to cast a block vote, and the decision rule is minimally adapted.

Proposition 3.1 *Let $\mathcal{W}$ be a SVG with assembly N and $S, G \subseteq N$. G is critical wrt S, if and only if $\&_G$ is critical wrt S in $\mathcal{W}|\&_G$.*

The proposition follows immediately from the definitions of group criticality and blocs. Let us now note three consequences of Def. 3.1.

[5]Criticality of groups can only be paraphrased in this way, if monotonicity holds true. If it doesn't, then I suggest to stick to the colloquial description of criticality just given and to change the definition of group criticality as follows: A group G is critical wrt S, iff there are subsets $G', G'' \subset G$ such $(S \setminus G) \cup G' \in \mathcal{W}$ and $(S \setminus G) \cup G'' \notin \mathcal{W}$.

Proposition 3.2 *Let $\mathcal{W}$ be a simple voting game with assembly N. Let $G, S \subseteq N$.*

1. *If G is critical wrt S and G' is an arbitrary subset of G: $G' \subseteq G$, then G is also critical wrt $(S \setminus G) \cup G'$.*
2. *If G is critical wrt S and $G \subseteq G'$, then G' is critical wrt S. In particular, if a group G' contains a vote that is critical wrt S, then G' is critical wrt S.*
3. *If G is not critical wrt S and $G' \subseteq G$, then G' is not critical wrt S either.*

The proofs are again very easy and will not be given here.

Assume now that an arbitrary probability model over the coalitions of a SVG is given.

Definition 3.2 *Let $\mathcal{W}$ be a simple voting game. The measure of voting power for group $G \subseteq N$ is the probability for a coalition S wrt which G is critical (or the probability that G is critical, for short).*

If the probability model on the profiles is represented as a joint probability function $p(\lambda_1, ..., \lambda_n)$, where the λ_is denote the votes of the single voters, then the power of $G = \{1, 2, ..., g\}$ only depends on the marginal probability functions for the non-group votes,

$$\sum_{\lambda_1} ... \sum_{\lambda_g} p(\lambda_1, ..., \lambda_n) \ . \tag{1}$$

Here the sums extend over "yes" and "no", each. Thus, the measure is really about the opportunity that the *other* votes leave to the group.

In order to illustrate the notion of power, let me calculate the power of group $G = \{2, 3\}$ in Ex. 1.1. As a probability model, I assume the model specified in this example. Clearly, only coalitions with a non-zero probability contribute to the measure of voting power. The coalitions with non-zero probability outside which G is critical are $\{1\}$, $\{4\}$ and $\{5\}$. The coalitions with non-zero probability inside which G is critical are $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$ and $\{2, 3, 4, 5\}$. The power measure of the group is thus .5. This shows that, even if no single voter is ever critical and even if the corresponding power measure is zero, a group of voters can still have non-zero power.

Under the Bernoulli model, the following result is useful (cf. Beisbart 2008, Cor. 3.2):

Proposition 3.3 *Let $\mathcal{W}$ be a simple voting game and $G \subseteq N$ a subset of the assembly N. Assume that G has two members. Under the Bernoulli model, the power measure of the group equals the sum of the power measures of its members.*

The proof follows from our Def. 3.2, our Prop. 3.1 and Theorem 3.2.18 in FM, p. 47.

Morriss (1987), pp. 109–114 has a useful section (14.2) on the power of groups. The proposed measurement of group power is partly in agreement with what Morriss says. For instance, as Morriss emphasizes, a group can have power to do something, even if not all of the group members are required for this (p. 112). This is also true of the proposed measure (cf. part 2 of Prop. 3.2). However, there is also some disagreement. For instance, according to Morriss, an assessment of group power to do something has to take into account whether the group is able to coordinate suitably in order to get this done (p. 110). Something like this does not enter my measure. I do not think that this is a problem; in calculating measures of individual power, many aspects of individual power, e.g. whether a voter is physically able to cast a no-vote, are abstracted away; similar things are abstracted away in the case of groups. Also, I'm not interested here in a measurement of group power in general; rather, what interests me is to what extent a SVG leaves opportunity for a group making a difference as to whether a bill passes. `is this paragraph needed?`

4 Higher-rank voting powers for individuals

I have now a power measure for groups at hand. On this base, I can become more individualistic again.

For an illustration what I have in mind, consider Ex. 1.1 and $S = \{1\}$ again. Voter 2 is not critical wrt this coalition. Nevertheless, she can ask: Is there one other voter b such that b and I can switch the vote jointly as a group? Put differently, am I member of a group of size 2 that is critical wrt S? And the answer is yes, as we have already seen: The group $\{2, 3\}$, for instance, does the job. But suppose now that the answer would be no again. Voter 2 can then go on and ask: Are there two other voters such that they and me together can jointly switch the vote? Put differently, am I member of a group of size 3 that is critical wrt S? And so on.

Let us start with an extension of the notion of criticality.

4.1 Criticality of higher ranks

Definition 4.1 *Let $\mathcal{W}$ be a simple voting game and $a \in N$ a voter. Let $n = |N|$. a is critical of rank $\kappa \in \{1, .., n\}$ wrt $S \subseteq N$, iff there is a group G with the following properties:*

1. *$a \in G$;*

2. *G is critical wrt S; and*

3. $|G| = \kappa$.

That is, a is critical of rank κ wrt S, iff there is a group κ members, one of them a, such that the group is critical wrt S.

Let me illustrate this definition in terms of Ex. 1.1. Consider the losing coalition $S \equiv \{1\}$. I will consider the voters 1 and 2 and investigate whether each of them is critical of rank κ wrt S for $\kappa = 1, 2, 3$.

Focus first on voter 2. 2 is not critical of rank 1 wrt S, since the only group that includes 2 and has one member is $\{2\}$, and $S \cup \{2\} = \{1, 2\}$ is not yet a winning coalition. However, 2 is critical of rank 2 wrt S. In order to see this, consider the group $G = \{2, 3\}$, which includes 2 and has two members. As we have seen before, this group is critical wrt S. Thus 2 is critical of rank 2 wrt S. By the way, this does not imply that *any* group with two members and 2 as a member is critical wrt S. For instance, $\{1, 2\}$ is not critical wrt S. Regarding rank 3, clearly, 2 is also critical of rank 3 wrt S. This can be shown using the group $\{1, 2, 3\}$, for instance.

Let me now move on to voter 1. Clearly, 1 is not critical of rank 1 wrt S. 1 is not even critical of rank 2 wrt $S = \{1\}$: For any group G with two members that include 1, we have $|S \cup G| = |\{1\} \cup G| = 2 < 3$, and thus $S \cup G$ is not yet winning, as would be necessary for criticality of the group. However, 1 is critical of rank 3 wrt $S = \{1\}$. This can be shown using the group $\{1, 2, 3\}$, e.g.

What is interesting in this example is that different voters can be distinguished by asking whether they are critical of a certain rank or not. 2 is critical of rank 2 wrt S, but 1 is not.

Let me now note a few characteristics of higher-rank criticality. The first thing to be observed is this: In Def. 4.1, condition could be replaced by the requirement that $|G| \leq \kappa$. This follows from monotonicity.

The following lemma states another useful thing.

Lemma 4.1 *Let $\mathcal{W}$ be a simple voting game and let $S \subseteq N$. Suppose that group $G \subseteq N$ with $|G| = \kappa$ is critical wrt S. Then every voter $a \in G$ is critical of rank κ wrt S. Every voter $a \in N$ is critical of rank $(\kappa + 1)$ wrt S.*

Proof. If $a \in G$, G itself can be taken as the group the existence of which is required in Def. 4.1. If $a \in N$, $G \cup \{a\}$ (which has cardinality $\kappa + 1$) is as required in Def. 4.1. Q.e.d.

Obviously, the well-known notion of criticality coincides with criticality of rank 1. Note, also, that each voter is critical of rank $n = |N|$. Finally, as the second part of Prop. 3.2 implies, if a is critical of rank κ wrt S, then she is also critical of rank $(\kappa + 1)$ wrt S.

4.2 A differential notion of higher-rank criticality

Under the definition of criticality just given, a voter will usually be critical of several ranks wrt one and the same coalition S. This is not always useful. I will therefore introduce a different bookkeeping of the ranks. I start by defining a differential notion of criticality, call it d-criticality.

Definition 4.2 *Let $\mathcal{W}$ be a simple voting game and $a \in N$ be a voter. Let $n = |N|$ and $S \subseteq N$. a is d-critical of rank 1 wrt S, if a is critical of rank 1 wrt S. a is d-critical of rank $\kappa \in \{2, .., n\}$ wrt S, iff a is critical of rank κ wrt S, but a is not critical of rank $(\kappa - 1)$ wrt S.*

Thus, a voter $a \in N$ is d-critical of rank κ wrt S, iff the smallest possible group G that includes a and that is critical wrt S, has cardinality κ.

Obviously, criticality of rank 1 and d-criticality of rank 1 coincide. It also follows immediately that, if a is d-critical of rank κ wrt S, then a is not d-critical of rank κ' wrt S for any $\kappa' \neq \kappa$. On the other hand, for each a, there is exactly one $\kappa \in \{1, .., |N|\}$ such that a is d-critical of rank κ wrt S.

Let me analyze $\mathcal{M}_5$ once more (cf. Fig. 2). Consider voter 1. She is d-critical of rank 3 wrt $\{1\}$ and $\emptyset$, $\{2, 3, 4, 5\}$ and N; she is d-critical of rank 2 wrt $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1, 3, 4, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 3, 5\}$ and $\{1, 2, 3, 4\}$; and she d-critical of rank 1 (critical) wrt all other coalitions.

For determining whether a voter is d-critical of a certain rank, the following result is useful.

Proposition 4.1 *Let $\mathcal{W}$ be a simple voting game with assembly N, $a \in N$ a voter and assume $S \subseteq N$. Let $G \subseteq N$ be a group with $a \in N$ and $|G| = \kappa$ and assume that the group is critical wrt S.*

1. *If S is a losing coalition and $|(G \setminus \{a\}) \cap S| = \nu$, then a is not d-critical of rank κ' wrt S for any $\kappa' > \kappa - \nu$.*

2. *If S is a winning coalition and $|(G \setminus \{a\}) \cap (N \setminus S)| = \nu$, then a is not d-critical of rank κ' wrt S for any $\kappa' > \kappa - \nu$.*

Proof. 1. Let $S \notin \mathcal{W}$ and $|(G \setminus \{a\}) \cap S| = \nu$. Note that $\nu < \kappa$. Consider now $G' = G \setminus (S \setminus \{a\})$. We know that $a \in G'$ and that $|G'| = |G| - |G \cap (S \setminus \{a\})| = |G| - |(G \setminus \{a\}) \cap S| = \kappa - \nu$. Moreover, $S \cup G' = S \cup G$ is a winning coalition, whereas $S \setminus G' \subseteq S$ is a losing coalition. Thus G' is critical wrt S. It follows that a cannot be d-critical of rank κ' wrt S for any $\kappa' > |G'| = \kappa - \nu$.
2. Let $S \in \mathcal{W}$ and $|(G \setminus \{a\}) \cap (N \setminus S)| = \nu$. Note that $\nu < \kappa$. Consider now $G' = G \setminus ((N \setminus S) \setminus \{a\})$. We know that $a \in G'$ and that $|G'| = |G| - |G \cap ((N \setminus S) \setminus \{a\})| = |G| - |(G \setminus \{a\}) \cap (N \setminus S)| = \kappa - \nu$. Moreover, $S \cup G' \supseteq S$ is a winning coalition, whereas $S \setminus G' = S \setminus G$ is a losing coalition. Thus G' is critical wrt S. It follows that a cannot be d-critical of rank κ' wrt S for any $\kappa' > |G'| = \kappa - \nu$. Q.e.d.

What the proposition tells us is this: If we wonder whether a voter a is d-critical of some rank and look for related groups $G \ni a$, then G should not have

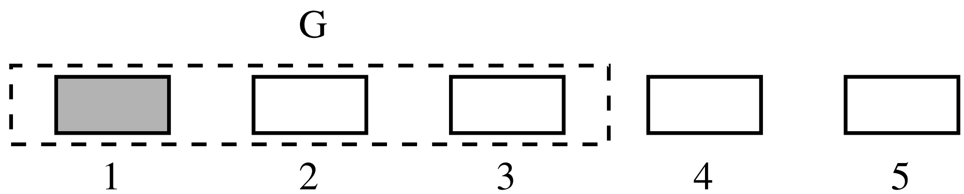

Figure 2: An illustration of d-criticality. $G = \{1, 2, 3\}$ contains 2 and is critical wrt $S = \{1\}$, so it is relevant for 2 being critical of rank 3 wrt S. However, $G' = \{2, 3\}$ would also suffice for a joint action that changes the outcome of the vote. Thus, 2 is not d-critical of rank 3 wrt S. The reason why G is inefficient is that S is losing and that G has overlap with S. Voter 1 votes yes, and the question is whether the outcome of the vote can be changed to yes. For this, 1 need not be part of the group – “she does the right thing, anyway”.

overlap with S (with $N \setminus S$) apart from a itself, if S is winning (losing). The overlap means that the group is inefficient in being critical – a smaller group would do as well. This is illustrated in Fig. 2.

4.2.1 d-criticality and minimal winning/maximal losing coalitions

d-criticality might remind one of minimal winning coalitions and maximal losing coalitions. A coalition S is a minimal winning coalition iff it is a winning one ($S \in \mathcal{W}$) and any proper subset $S' \subset S$ is a losing coalition ($S' \notin \mathcal{W}$; FM, Def. 2.3.2 on p. 23). A coalition S is a maximal losing coalition iff it is a losing one ($S \notin \mathcal{W}$) and and any proper superset $S' \supset S$ is a winning coalition ($S' \in \mathcal{W}$).

Some will suspect that there is a relationship between d-criticality and minimal winning coalitions (or maximal losing coalitions). A first guess starts from the following observation: For a voter to be d-critical of some rank wrt S, a group G is needed such that $S \cup G$ is winning and $S \setminus G$ is losing. The guess then is that $S \cup G$ (1) will be minimal or (2) has to be minimal, if there is d-criticality (or that $S \setminus G$ (1) will be maximal or (2) has to be maximal, if there is d-criticality). But the guess is wrong in all of its versions.

I will first consider the (1) versions and consider minimal winning and maximal losing coalitions in turn. Suppose that a is d-critical of rank α wrt S. There is thus a group $G \subseteq N$ with $|G| = \kappa$ and $a \in G$ such that G is critical wrt S, and a smaller group would not do. Accordingly, $G \cup S$ is a winning coalition. It does not follow that $G \cup S$ is a minimal winning coalition. This is clear from the following example. Consider a weighted voting game with the weights $1, 1, 3$ and a quota of 3 – $[3; 1, 1, 3]$ (see FM, Def. 2.3.15 on pp. 29–30 for the notation). Here and henceforth, I will number the voters consecutively – i.e., in $[3; 1, 1, 3]$, voters 1 and 2 have weight 1, whereas voter 3 has vote 3. Let $S = \{1, 2\}$. $a = 2$ is d-critical of rank 2 wrt S. The related group is $G = \{2, 3\}$. There is no smaller group that does the same thing. But $S \cup G = \{1, 2, 3\}$ is not a minimal winning coalition – $\{3\}$ is also winning. The reason why $S \cup G$ is not minimal, of course, is that the S-part of $S \cup G$, i.e. $S \setminus G$, is not optimal.

I will now consider maximal losing coalitions. Suppose that a is d-critical of rank κ wrt S. There is thus a group $G \subseteq N$ with $|G| = \kappa$ and $a \in G$ such that G is critical wrt S, and a smaller group would not do. Accordingly, $G \setminus S$ is a losing coalition. It does not follow that $G \setminus S$ is a maximal losing coalition. This is clear from the following example. Consider again the simple voting game $[3; 1, 1, 3]$. $a = 2$ is d-critical of rank 2 wrt $S = \{1\}$. The related group is $G = \{2, 3\}$. There is no smaller group that does the same thing. But $S \setminus G = \emptyset$ is not a maximal losing coalition – $\{1\}$ is losing as well, e.g.

Let me now consider the (2) guesses. Suppose, first, that there is a group $G \ni a$ such that $G \cup S$ is a minimum winning coalition and that $G \setminus S$ is a losing coalition (not necessarily a maximal one). It follows that G is critical of rank $\kappa = |G|$ wrt S, but it does not follow that a is d-critical of the same rank wrt S. For a simple illustration, consider the weighted voting game $[5; 1, 2, 2, 4]$, set $S = \{1\}$ and $G = \{1, 2, 3\}$. $S \cup G$ is a minimal winning coalition, $S \setminus G = \emptyset$ is losing; but 1 is not d-critical of rank 3 wrt S, since $G = \{1, 4\} \ni 1$ is also critical wrt S.

Suppose, second, that there is a group $G \ni a$ such that $G \cup S$ is a winning coalition (not necessarily a minimal one) and that $G \setminus S$ is a maximal losing coalition. It follows that G is critical of rank $\kappa = |G|$ wrt S, but it does not follow that a is d-critical of the same rank wrt S. For a simple illustration, consider the weighted voting game $[5; 1, 2, 2, 4]$ again, set $S = \{4\}$ and $G = \{1, 2\}$. $S \cup G = \{1, 2, 4\}$ is a winning coalition, $S \setminus G = \{4\}$ is losing, and it is a maximal losing coalition; but 1 is not d-critical of rank 2 wrt S, since $G = \{1\}$ is also critical wrt S.

Suppose, finally, that there is a group $G \ni a$ such that $G \cup S$ is a minimal winning coalition and that $G \setminus S$ is a maximal losing coalition. It follows immediately that $|G| = 1$ and that 1 is critical wrt S (of rank 1). Thus, no new characterization of d-criticality of higher ranks is provided.

The reason why there is no tight relation between minimal winning coalitions (or maximal losing coalitions) and d-criticality is, of course, that two notions of minimality are in play. One minimality refers to properties of $S \cup G$ (of $S \setminus G$); the minimality that is relevant for d-criticality refers to the size of the group G.

Since d-criticality is not immediately related to minimal winning coalitions, the question arises: What is relevant for measuring power – is it d-criticality or some strengthened notion that involves minimal winning coalitions, e.g.?

On my view, whether the winning coalition $S \cup G$ in Def. 4.2 is a minimal one or not, does not matter for power. As one of the examples above shows, $S \cup G$ might be winning, but not a minimal winning coalition, not because of G, but because of the composition of $S \setminus G$, and this composition is quite irrelevant for the question what a voter can do jointly with others. What counts for power, intuitively, is that a jointly with some other people can make a difference, not how exactly this is done.

However, I will tentatively strengthen the notion of d-criticality in a different way in App. A. I do not think, though, that this strengthening is useful.

4.3 Higher-rank voting powers

Let me now turn to the notion of power. In order to avoid double-counting, it is convenient to define power as a differential notion, i.e. in terms of d-criticality instead of criticality.

Definition 4.3 *Let $\mathcal{W}$ be a simple voting game and the cardinality of its assembly N, $|N| = n$. The measure of voting power of rank $\kappa \in \{1, .., n\}$ for voter a, call it $\beta_a^{\prime;\kappa}$, is the probability for a voting profile wrt which a is d-critical of rank κ (the probability that a is d-critical of rank κ, for short).*

Thus, in Wilmers' example, the following higher-rank powers can be derived: For each voter a, we have $\beta_a^{\prime;1} = 0$, $\beta_a^{\prime;2} = 2/3$, $\beta_a^{\prime;3} = 1/3$ and $\beta_a^{\prime;\kappa} = 0$ for $\kappa > 3$. This follows from the determination of d-criticality on p. 12 and the probability model in the example.

How, then, can we deal with the objection that was based on Wilmers' example? The objection was that, if the voting power of a voter is measured as the probability of criticality, nobody has voting power in the example. Using our extended notions of power, we can reply as follows: In some sense, we are afraid, the voters do not have power in this example – they do not have power of rank 1. But they do have power of higher ranks. Particularly, they have power of ranks 2 and 3. It is never the case that a voter can affect the outcome by switching her vote. But there is always the opportunity to form groups that can switch the vote. As we will presently see, each voter will always have a non-zero voting power of some rank. Thus an assignment of voting power under which no voter has any power is not possible any more using the power hierarchy.

In Wilmers' example, the voting game and the probability model are completely symmetric. Thus, every voter has the same hierarchy of higher-rank powers. Let us therefore consider a variation of Ex. 1.1, such that the symmetry is lost.

Example 4.1 *Assume that the probabilities are as given in Ex. 1.1 except that the probability for $\{1\}$ is $1/12 + \epsilon$ and the probability of $\{2\}$ is $1/12 - \epsilon$ for some $1/12 \geq \epsilon > 0$. What is the effect on the powers of the different ranks? 2 gains power of rank 2, but loses voting power of rank 3 – she has now $\beta_2^{\prime;2} = \frac{2}{3} + \epsilon$ and $\beta_2^{\prime;3} = \frac{1}{3} - \epsilon$. On the contrary, 1 loses power of rank 2 and gains power of rank 3 – she has now $\beta_2^{\prime;2} = \frac{2}{3} - \epsilon$ and $\beta_2^{\prime;3} = \frac{1}{3} + \epsilon$. All other measures are unaffected.*

The question is now what this means. Is there something that we can say about *overall* voting power in this new example?

One idea would be to add up the powers of the different ranks for each voter. But this idea does not work. First, one can prove that, generally,

$$\sum_{\kappa=1}^{|N|} \beta_a'^{,\kappa} = 1 \ . \tag{2}$$

Second, one cannot just add up the powers of the different ranks – they have different meanings. Rather, if one wants to condense the information that is present in the hierarchy of powers, in order to obtain simple comparisons, one has to proceed as follows.

Define first cumulative measures of higher-rank power:

Definition 4.4 *Let $\mathcal{W}$ be a simple voting game, where the assembly N has cardinality n. For each voter $a \in N$ and for each $\kappa \in \{1, .., n\}$ define the cumulative power measure of rank κ, α_a^κ as*

$$\alpha_a^\kappa = \sum_{\iota=1}^{\kappa} \beta_a'^{,\iota} \ . \tag{3}$$

α_a^κ is the probability that voter a is critical of rank κ wrt a coalition (instead of being d-critical of that rank). Clearly, these measures approach 1, as κ approaches $n = |N|$.

We can now define the following preorder: Voter a has overall at least as much voting power as voter b ($a \succcurlyeq b$, for short), if $\alpha_a'^{,\kappa} \geq \alpha_b'^{,\kappa}$ for all $\kappa = 1, .., n$. Additionally, let us say that a has overall the same voting power as b, iff $\alpha_a'^{,\kappa} = \alpha_b'^{,\kappa}$ for all $\kappa = 1, .., n$. Finally, we can say that a has overall more voting power than b, if $a \succcurlyeq b$ and $\alpha_a'^{,\kappa} > \alpha_b'^{,\kappa}$ for some κ. Thus, very roughly, you have overall more voting power than I do, if, as the rank increases, your cumulative measures approach 1 earlier than mine do. The idea is that I am less powerful, because, typically, I need to find more people in order to form a group that is critical.

With these notions in mind, let us return to Ex. 4.1. Voter 2 has overall more voting power than any other voter has. Voter 1 has overall less voting power than anyone else in the example. The reason is, of course, that the coalition $S_1 = \{1\}$ has an increased probability, whereas $S_2 = \{2\}$ has a diminished probability. S_1 is particularly unfortunate for 1, because she is further away from making a difference than anyone else is. Likewise, S_2 is particularly unfortunate for 2.

I would not go as far as to say that these comparisons for Ex. 4.1 match our intuitions – most people are not likely to have many intuitions regarding the example. But I would like to claim that the measures proposed in this paper and the comparisons built upon them provide interesting information that stand further reflection.

We can now pose the following question. Is it ever possible that a has more voting power of rank κ than voter b, but that b has more voting power of rank

$(\kappa+1)$ than a? If this is impossible, then comparability will hold for the preorder $\succcurlyeq$: For any $a, b \in N$, $a \succcurlyeq b$ or $b \succcurlyeq a$ will be true. However, as the following example shows, the above is not impossible.

Example 4.2 *Consider the weighted voting game* $[5; 3, 2, 1, 1, 1]$*. Consider the set* $S_1 = \{1, 3, 4\}$*. Voter 3 is critical (d-critical of rank 1) wrt* S_1*, but 2 is not – she is only d-critical of rank 2 wrt* S_1*. Consider now* $S_2 = \emptyset$*. 2 is d-critical of rank 2 wrt* S_2*, but 3 is not – she is only d-critical of rank 3 wrt* S_2*. Assume now that* $p(S_1) = \epsilon$ *and that* $p(S_2) = 1 - \epsilon$ *for an* $\epsilon \in (0, 1)$*. Then we have:* $\alpha_3^{\prime,1} = \epsilon$ *and* $\alpha_2^{\prime,1} = 0$*, whereas* $\alpha_3^{\prime,2} = \epsilon$ *and* $\alpha_2^{\prime,2} = 1$*. So for* $0 < \epsilon < 1$*, we have* $\alpha_3^{\prime,1} > \alpha_2^{\prime,1}$*, but* $\alpha_3^{\prime,2} < \alpha_2^{\prime,2}$*. Following our definition of "having overall more voting power", we can neither say that 2 has overall more power than 3, nor is it the other way round. Also, neither* $2 \succcurlyeq 3$ *nor* $3 \succcurlyeq 2$ *is true. Furthermore, 1 and 2 have not the same power. Their powers are incommensurable, to take up a notion from practical philosophy (cf. Raz 1999, p. 46 for a definition of incommensurable values).*

Note, however, that the probability distribution is very peculiar in this example. The hope, then, is that, under many realistic probability models, comparability can be established.

There are alternative ways to condense the information in the hierarchy of measures proposed in this paper. I leave a related discussion to future work.

Dummies. A dummy is a voter that is not critical wrt any coalition (Def. 2.3.4 on p. 24 in FM). Obviously, the notion of a dummy relies on the notion of criticality of rank 1. One may ask whether the notion of a dummy can be extended to higher-rank dummies. Can there be a higher-rank dummy who is not critical of a fixed higher-rank wrt any coalition? The answer is no. The reason is that, for any simple voting game, there is a coalition S such that at least one voter is critical wrt S. It follows from Prop. 3.2, Part 2, that every voter is either critical wrt some coalition S or that she is critical of rank 2 wrt to some coalition.

Higher-rank powers under $\mathcal{M}_n$ and the Bernoulli model. As a reference, it is useful to have the higher-rank powers for a very simple example.

Example 4.3 *Let* $\mathcal{M}_n$ *be simple majority voting with* n *voters and adopt the Bernoulli model as probability model. Let* m *the smallest integer that is larger than* $n/2$ *(*m *is the threshold). For each voter* a*, we have*

$$\beta_a^{\prime,\kappa} = \begin{cases} 2^{1-n}\binom{n-1}{m-1} & \text{for } \kappa = 1\ , \\ 2^{1-n}\left(\binom{n-1}{m-\kappa} + \binom{n-1}{m-2+\kappa}\right) & \text{for } \kappa > 1. \end{cases} \tag{4}$$

Here it is assumed that $\binom{n}{k} = 0$ *for* $k > n$ *or* $k < 0$*. Of course, because of the symmetry, each voter has the same hierarchy of measures.*

Proof. I can concentrate on $\kappa \in \{2, .., m\}$ – for $\kappa = 1$ the result is well-known (FM, p. 55), and it is clear that $\beta_a'^{;\kappa}$ is zero for $\kappa > m$. Consider now an arbitrary voter a. For the higher-rank power measures, it does not matter, what the vote of a herself is (first part of Prop. 3.2). I can therefore focus on the other $(n-1)$ voters and on coalitions $S \subseteq N \setminus \{a\}$. In order to calculate $\beta_a'^{;\kappa}$, one has to know: How many coalitions $S \subseteq N \setminus \{a\}$ have the following property: (1) There is a group of κ voters, including a, G such that $S \cup G \in \mathcal{W}$ and $S \setminus G \notin \mathcal{W}$, but (2) there is no group of $(\kappa - 1)$ voters, including a, G' such that $S \cup G' \in \mathcal{W}$ and $S \setminus G' \in \mathcal{W}$. Because of property 1, S needs to have at least $(m - \kappa)$ elements – otherwise, there will be no group with κ voters such that $S \cup G \in \mathcal{W}$. Also, it can at most have $(m - 1 + \kappa - 1)$ elements – otherwise, there is no group with κ members including a such that $S \setminus G \notin \mathcal{W}$ (since we look at coalitions $S \subseteq N \setminus \{a\}$ only, a's vote can't help in destroying an acceptance, and we have to add $\kappa - 1$ instead of κ to $m - 1$). Thus $(m - \kappa) \leq |S| \leq (m - 1 + \kappa)$. Because of property 2, S has to have the highest possible or the lowest possible cardinality. On the other hand, any $S \subseteq N \setminus \{a\}$ with these cardinalities will do. There are $\left(\binom{n-1}{m-\kappa} + \binom{n-1}{m-2+\kappa}\right)$ of them with a probability of 2^{1-n}, each. Q.e.d.

Let me now consider another example. In this example, there is no symmetry between the different voters.

Example 4.4 *Consider the simple voting game $\mathcal{M}_5$. Label the voters in the assembly from 1 to 5. Assume that the voters are ordered according to how liberal they are. 1 is least liberal, and 5 is most liberal. The bills propose measures that are mostly quite liberal. As a consequence, if 1 votes yes, all others vote yes as well. More generally, if a votes yes, all voters $b > a$ will vote yes as well (this can be modeled with a one-dimensional policy space). Thus, only the coalitions $S_0 = \emptyset$, $S_1 = \{5\}$, $S_2 = \{5, 4\}$, $S_3 = \{5, 4, 3\}$, $S_4 = \{5, 4, 3, 2\}$ and $S_5 = \{5, 4, 3, 2, 1\}$ have non-zero probability. Call the probability for S_i, p_i. We assume that the p_is are symmetric, i.e. $p_{5-i} = p_i$ for $i = 0, .., 5$.*

In the upper panel of Table 1, the coalitions S_i are listed, and the ranks of the (differential) power measures to which they contribute are specified for the different voters (which correspond to different columns). In the second panel, the differential power measures are given.

If we consider the relation "having overall more power", it turns out that 3 has overall more power than anybody else. Voters 2 and 4 have overall more power than 1 and 5. This result is very intuitive – after all, 3 occupies the middle position, whereas 1 and 5 are most extreme. Note also, that first-rank power would not allow us to differentiate between 2 and 4 on the one hand, and 1 and 5 on the other hand.

In order to illustrate the usefulness of the proposed measures, I will discuss one more example. The example is constructed in the following way: Two different voting games are compared. The voting games can not be distinguished in terms

coalition	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$
$\emptyset$	$\kappa=3$	$\kappa=3$	$\kappa=3$	$\kappa=3$	$\kappa=3$
$\{5\}$	$\kappa=2$	$\kappa=2$	$\kappa=2$	$\kappa=2$	$\kappa=3$
$\{4,5\}$	$\kappa=1$	$\kappa=1$	$\kappa=1$	$\kappa=2$	$\kappa=2$
$\{3,4,5\}$	$\kappa=2$	$\kappa=2$	$\kappa=1$	$\kappa=1$	$\kappa=1$
$\{2,3,4,5\}$	$\kappa=3$	$\kappa=2$	$\kappa=2$	$\kappa=2$	$\kappa=2$
$\{1,2,3,4,5\}$	$\kappa=3$	$\kappa=3$	$\kappa=3$	$\kappa=3$	$\kappa=3$

rank κ	$\beta_1^{\prime,\kappa}$	$\beta_2^{\prime,\kappa}$	$\beta_3^{\prime,\kappa}$	$\beta_4^{\prime,\kappa}$	$\beta_5^{\prime,\kappa}$
$\kappa=1$	p_2	p_2	p_2+p_3	p_3	p_3
$\kappa=2$	p_1+p_3	$p_1+p_3+p_4$	p_1+p_4	$p_1+p_2+p_4$	p_2+p_4
$\kappa=3$	$p_0+p_4+p_5$	p_0+p_5	p_0+p_5	p_0+p_5	$p_0+p_1+p_5$

Table 1: Ex. 4.4. In the upper panel, the columns refer to the different voters a. Each entry in the table note the rank κ of which a coalition is d-critical. In the lower panel, the (differential) power measures are summarized.

of first-rank power – the voters have the same first-rank powers in both voting games. However, the voting games are different in terms of higher-rank powers. As a probability model, the Bernoulli model is adopted.

Example 4.5 *We will compare two simple voting games with four voters, each. The first voting game is $\mathcal{M}_4$ – i.e. a vote is accepted, iff there are at least three yes votes. The other voting game $\mathcal{W}_2$ is spanned by the following two minimal winning coalitions: $S_1 = \{1,2\}$ and $S_2 = \{3,4\}$ – i.e. $\mathcal{W}_2 = \{X \subseteq N | X \supseteq S_1 \text{ or } X \supseteq S_2\}$. Note that the voting games are not dual to each other (it is well known that voting games that are dual to each other have the same Banzhaf measures for the votes, Theorem 3.2.7 on p. 42 in FM; for duality see Def. 2.3.2 on p. 23, ib.).*

It is clear that both voting games are symmetric: Within each of them, the voters can not be distinguished in terms of power measures. Let us first focus on rank 1 and start with $\mathcal{W}_1$. The Banzhaf measure of each voter is $\frac{3}{8}$ – this follows from Eq. (4). Under $\mathcal{W}_2$, voter 1 is critical inside the following coalitions: $\{1,2\},\{1,2,3\},\{1,2,4\}$. Since the Banzhaf measure equals the probability that a voter is critical inside, given she votes yes, we obtain $\frac{3}{8}$ for the Banzhaf measure. Thus, $\mathcal{W}_1$ and $\mathcal{W}_2$ cannot be distinguished in terms of their Banzhaf measures or power measures of rank 1.

That the voting games can be distinguished in terms of higher-rank power measures can be seen as follows. Under $\mathcal{W}_1$, each voter is d-critical of rank 3 wrt $S = \emptyset$. Consequently, the differential power measure is non-zero for rank 3 – it equals $\beta_a^{\prime,3} = \frac{1}{8}$ for each voter a. It follows that $\beta_a^{\prime,2} = \frac{1}{2}$ for each voter because of the sum rule in Eq. (2). On the other hand, under $\mathcal{W}_2$, no voter is ever d-critical of rank 3 wrt a coalition. For instance, $a = 1$ is critical of rank 2 wrt $S = \emptyset$, *because* $\{1,2\}$ *is winning. Thus, because of the sum rule in Eq.* (2)*, each voter must have a power measure of second rank of* $\beta_a^{\prime;2} = \frac{5}{8}$*. This shows that the voting games can be distinguished in terms of higher-rank power measures.*

The example raises the following research question: Suppose that two simple voting games have exactly the same set of power measures for each voter under the Bernoulli model. Are the voting games identical up to isomorphisms and duality transformations (for isomorphisms, cf. Def. 2.1.7 on p. 15 in FM)? And if the answer is yes, is there an algorithm with which the voting game can be reconstructed from the power measures? I hope to come back to this question in my future research.[6]

Of course, one can also ask whether different voting games can be distinguished in terms of power measures by assuming a probability model different from the Bernoulli model. The answer is clearly no, if all coalitions that are winning under one game, but losing under the other one, have zero probability, each. So there are at least minimal requirements on the probability model. I will not pursue this issue further.

5 Discussion

In collective decision making, not only single voters can make a difference as to whether a bill passes or not. Groups can also make a difference as to whether a bill passes. This fact should make a difference for the way we think of voting power of individuals – or so has been argued in this paper.

In more detail, I have proposed an extension of a well-known measure of voting power. The well-known measure is the probability of being critical. If it is calculated on the basis of the Bernoulli model, the popular Banzhaf measure of (a priori) voting power is obtained. But the probability of being critical can also be calculated for alternative probability models. If they are constrained by empirical data, measures of a posteriori voting power arise.

In the recent literature, it has been argued that the probability of criticality does not provide a suitable measure of a posteriori voting power. As Wilmers' example shows, this measure will assign zero voting power to every voter, if voting profiles under which at least one voter is critical have zero probability. It seems odd that nobody has any power whatsoever, though.

[6] It is well-known that *weighted* voting games are almost fully characterized in terms of the Banzhaf measures only (Dubey & Shapley 1979, p. 127). That is, if two weighted voting games have exactly the same sets of Banzhaf measures, then they are either dual or identical (up to isomorphisms). In fact, in our example, $\mathcal{W}_2$ is *not* a weighted voting game. It is not even a proper voting game (Def. 2.1.1 on p. 11 in FM). Note, also, that the degeneracies between dual voting games are not broken by my measures. It can be proven that the hierarchies of power measures coincide for a voting game $\mathcal{W}$ and its dual $\mathcal{W}^\star$, if the Bernoulli model is adopted.

What is also dissatisfying about the probability of being critical is this: This measure does not have as much discriminatory power as one would like it to have. For instance, calculating the Banzhaf measures for a simple voting game does not allow for a full characterization of a voting game. There are pairs of different simple voting games, such that the Banzhaf measures for the voters are identical (Ex. 4.5; for qualifications see Subsec. 4.3). Put differently, if one moves from a voting game to the Banzhaf measures, information is lost. Extensions of the Banzhaf measures are called for that may compensate this loss of information.

For these reasons, I have proposed to go beyond the probability of being critical for measuring voting power. The probability of criticality quantifies the extent to which a voter has the opportunity to make a difference as to whether a bill passes or not. Likewise, one can calculate the extent to which a voter has the opportunity to find other voters in order to form a group that makes a difference as to whether a bill passes or not. Put differently, for each voter, we look at the opportunities for group actions that involve her.

For going beyond the probability that a voter is critical, I have first defined criticality for a group. Roughly, given a specific coalition, a group is critical, iff the following is true: There is some way in which the group could have voted differently such that the outcome of the vote would have been different. The most important step in extending the probability of criticality was to define higher-rank criticality for single voters. Roughly, a voter is critical of rank κ, iff there is a group of votes including a and with cardinality κ such that the group is critical. The proposed new measures are then the probabilities that a voter is critical of a fixed rank. In order to avoid double-counting, I have introduced a differential counting. As a result, for each voter there is a hierarchy of measures with ranks ranging from 1 to the cardinality of the assembly. The powers of the different ranks add up to 1 for each voter. Roughly, you have overall more voting power than I have, if your measures start growing for smaller ranks than mine do. For rank 1 and the Bernoulli model, the new measure coincides with the Banzhaf measure. For higher ranks, additional information is provided.

On p. 3 three requirements on the suggested extension have been introduced. The requirements are fulfilled in the following sense:

R1 If the measures are calculated under the assumption of the Bernoulli model, the rank 1 measures coincide with the Banzhaf measure of voting power.

R2 Obviously, our measures are conceptually tied to the notion of criticality.

R3 My measures concern individual voters – they quantify to what extent a voter can make a difference as part of groups.

R4 The contraintuitive results for Wilmers' example are avoided under our measures: True, the first-rank measure is zero for every voter, but each voter has non-zero power of second and third rank.

The versatility of our measures has been shown using several examples. In particular, it has been highlighted that their discriminatory power is larger than that of the probability of criticality.

In order to conclude, let me deal with a few objections.

The first objection targets the definition of group criticality and everything else that is built upon it. Consider $\mathcal{M}_5$ and the coalition $S = \{1, 2\}$. Suppose also that 2 and 3 always cast opposite votes – 2 votes yes, iff 3 votes no. Consequently, the probability for coalition $\{1, 2, 3\}$ is zero. The objection is that, in this case, intuitively, the group $G = \{2, 3\}$ is not critical wrt S, because, in order to change the outcome of the vote, the group members would have to cast the same votes, which they never do. However, according to our definition Def. 3.1, the group G *is* critical wrt S.

The objection does not work, though. The intuition according to which the group is not critical rests upon a variation of the exercise fallacy (Morriss 1987, pp. 15–18) and is thus misguided. The exercise fallacy confuses the having of power and the exercising of power. But one can have power and never exercise it. Likewise, one can often face two options that lead to different outcomes, and always take the first option, say. In this case, one does still make a difference, although, given the votes of the other voters, the outcome itself is predictable. In the example, given coalition S, the group G does have the opportunity to change the vote. It could have voted differently. That it never does vote unanimously yes, which would be required to change the outcome of the vote, is just a different matter.[7]

The second objection is about the hierarchy of power measures. Within this hierarchy, it makes a qualitative difference whether I can find two other voters in order to form a group that is critical, or whether three or more voters are needed for this. But, as a matter of fact, this may often not make any difference at all. For instance, it may be the case that, whenever I and some other voter have decided to join forces, other voters will enter the group without further ado. The objection, then, is that the hierarchy of measures is too inflexible as to capture such effects.

My reply is as follows. It is right that certain groups are easier to form than others. Moreover, the easiness of forming a group is not a simple function of group size. But the proposed measure is not about how easy it is to form groups as a matter of fact. The difficulties of forming groups are abstracted away for my measures. Rather, the question is: Is there the opportunity to form groups at all?

In a similar way, the Banzhaf measure abstracts from certain real-world complications. Suppose, for instance, that a voting rule requires unanimity. It may

[7]The fact that the objection rests only upon a variation of the exercise fallacy (rather than the exercise fallacy itself) traces back to the fact that voting power is not a skill and cannot be literally exercised (cf. footnote 3 on p. 5).

be the case that I'm under severe pressure to vote yes, if all others vote yes. Accordingly, one may object that I'm not really critical, if all others vote yes, because the pressure is so high that I will vote yes. But this is not how the Banzhaf measure is set up. The Banzhaf measure does not take into account the costs of voting one way rather than another. It is undeniable that there is an opportunity to let the bill fail, and this is quantified by the Banzhaf measure (what has been said regarding the Banzhaf measure in this paragraph is also true of the probability of criticality as a measure of a posteriori voting power). Likewise, the difficulties of forming real-world groups are abstracted away in my measure (a measure that does not abstract that much is proposed in Bovens & Beisbart 2007).

Morriss (1987) is helpful here again. He distinguishes between two perspectives (p. 86–88). One perspective focuses on an isolated outcome only. The question is: Is it possible for some agent to effect this outcome? The other perspective is broader; the question is: Which combinations of outcomes are *com*possible (p. 87)? That a person can effect some outcome, but only with difficulties or with high costs, means this: The person cannot at the same time effect the outcome and effect many other costly things. For the Banzhaf measure and its extension the first perspective is constitutive.

A final objection is that the proposed measures do not necessarily lead to comparability regarding power. In this work, an example has been provided (Ex. 4.2) where we can neither say that a has more power than b nor that it is the other way round nor that they have the same power. This seems disappointing.

In reply, I want to stress the following: It cannot be expected in the first place that every concept from ordinary language can be explicated as to yield the mathematical structure one would like to have. What is important, I think, is that the explication can account for important things that we do with the concept in question. And it is a matter of fact that we compare the powers of different people in ordinary talk and in political science the like. But this does not imply that it can always be determined whether one person has more power than the other one rather than the other way round.

This is how I see my work on the notion of voting power: We start from a concept well-known from ordinary talk and try to explicate it in a Carnapian way. According to Carnap, the explicans – the concept that does the explication – has to be similar to the explicandum, exact, simple and fruitful (Carnap 1950, p. 5). One crucial task in the explication is to see whether the concept leads to a handy mathematical structure – a preorder, a metric etc. An explication that does so is particularly fruitful. However, the most satisfying structure might not be close enough to the explicandum. Thus, a less satisfying structure is required. It is an open question how satisfying the best explication can be from a mathematical point of view. In any case, I hope that the structure set up in this papers proves fruitful for further research in voting theory.

A An alternative notion of higher-rank power?

So far, I have developed the idea that, roughly, a voter has power to the extent that she has opportunity to make a difference jointly with others. A slight variation of this idea is as follows: A voter has power, if she has opportunity to make a difference jointly with others *and if she is essential for the fact that she and the others make a difference.*

This idea can be rendered more precise in the following two steps.

Definition A.1 *Let $\mathcal{W}$ be a simple voting game with assembly $N \ni a$. Let group $G \subseteq N$ be critical wrt a coalition S. $a \in N$ is essential for G being critical (wrt S), iff $G \setminus \{a\}$ is not critical wrt S.*

Obviously, a vote a can only have this property, if $a \in G$.

Consider $\mathcal{M}_5$ as an example again. Let $S = \{1\}$. The group $G = \{2, 3\}$ is critical wrt S. Voter 2 is essential for G being critical wrt S, and the same is true of voter 3. The reason is that any proper subgroup G' of G, $G' \supset G$, is not critical any more, since $|G| \leq 1$, and thus $|G \cup S| \leq 2$.

As G, $\tilde{G} = \{2, 3, 4\}$ is critical wrt $S = \{1\}$. But voter 2 is not essential any more for G being critical, since $\tilde{G} \setminus \{2\}$ is critical wrt S as well.

Suppose now that $a \in N$ is essential for some group $G \subseteq N$ being critical wrt $S \subseteq N$. There are two cases possible: Either $a \in S$ or $a \notin S$. The next proposition provides alternative characterizations of these cases.

Proposition A.1 *Let $\mathcal{W}$ be a simple voting game with assembly N. Assume $G, S \subseteq N$.*

1. *Suppose $a \in S$. G is critical wrt S and a is essential for G being critical, iff*

   (a) $a \in G$;

   (b) $S \setminus G \notin \mathcal{W}$; and

   (c) $(S \setminus G) \cup \{a\} \in \mathcal{W}$.

   If condition (c) is fulfilled, then $S \in \mathcal{W}$.

2. *Suppose $a \notin S$. G is critical wrt S and a is essential for G being critical, iff*

   (a) $a \in G$;

   (b) $S \cup G \in \mathcal{W}$; and

   (c) $(S \cup G) \setminus \{a\} \notin \mathcal{W}$.

   If condition (c) is fulfilled, then $S \notin \mathcal{W}$.

Proof. 1. a. Suppose that $a \in S$, that G is critical wrt S and that a is essential for G being critical. (a) and (b), i.e. $a \in G$ and $S \setminus G \notin \mathcal{W}$ follow trivially. Furthermore, we have $S \cup G \in \mathcal{W}$. Consider now $G \setminus \{a\}$. Since $a \in S$, $S \cup (G \setminus \{a\}) = S \cup G \in \mathcal{W}$. Thus, $S \setminus (G \setminus \{a\}) = (S \setminus G) \cup \{a\}$ must be a winning coalition – otherwise $G \setminus \{a\}$ would also be critical wrt S, and a would not be essential. Thus, the condition (c) obtains. Since $a \in S$, $(S \setminus G) \cup \{a\} \subseteq S$, and, because of condition (c), S must also be winning.
b. Suppose that the conditions (a)–(c) are fulfilled. We need to show that G is critical wrt S, whereas $G \setminus \{a\}$ is not so. G is critical wrt S, since $S \setminus G \notin \mathcal{W}$ by (b) and $S \cup G$ is a superset of $(S \setminus G) \cup \{a\}$, which in turn is winning according to (c). Thus, $S \cup G$ is winning as well. On the other hand, $G \setminus \{a\}$ is not critical wrt S, since $S \cup (G \setminus \{a\}) = (S \setminus G) \cup \{a\} \in \mathcal{W}$ is winning by hypothesis.

The proof for 2. parallels the proof for 1. Q.e.d.

Following Prop. A.1, the groups that are critical wrt some fixed coalition S and for which a fixed a is essential, can be constructed in a systematic way. Depending on whether S is winning or not, part 1 or part 2 apply. In the first case, one has to check whether $a \in S$. If $a \notin S$, then no group with the required property can be found. Let us therefore assume that $a \in S$. The next step is to look for subsets $S' \subseteq S$ such that $S' \in \mathcal{W}$, but $S' \cup \{a\} \notin \mathcal{W}$. In a final step these sets S' have to be represented as $S \setminus G$ for suitable Gs with $a \in G$ – and these are just the groups we are looking for. If S is losing, part 2 has to be applied in a similar way.

Let us illustrate this procedure using $\mathcal{M}_5$, a = 2 and $S = \{1\}$. S is losing, thus the second part of the proposition applies. Since $a \notin S$, we expect some groups that have the required property. We first look for supersets S' of S, such that $S' \in \mathcal{W}$, but $S' \setminus \{a\} \notin \mathcal{W}$. Clearly, a has to be in S'. The appropriate sets S' are: $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 2, 5\}$. Thus, the related groups are $\{1, 2, b\}$ and $\{2, b\}$ for $b = 3, 4, 5$.

We can now turn to power. It is convenient to add essentiality to the requirements of d-criticality. The strengthened notion of criticality that results will be called D-criticality.

Definition A.2 *Let $\mathcal{W}$ be a simple voting game with assembly N. A voter $a \in N$ is D-critical of rank κ wrt a coalition $S \subseteq N$, iff there is a group $G \subseteq N$ with the following properties:*

1. *$|G| = \kappa$;*
2. *$a \in G$;*
3. *G is critical wrt S;*
4. *there is no smaller group G' with properties 2 and 3;*

5. a is essential for G being critical wrt S.

Of course, D-criticality of rank 1 is criticality of rank 1.

As an illustration, let us examine $\mathcal{M}_5$ again. $a = 2$ is D-critical of rank 2 wrt $S = \{1\}$ because of the groups $\{2, b\}$ for $b = 3, 4, 5$. For the group is critical wrt S, and any smaller group G with $2 \in G$, would not be critical. Moreover, $G \setminus \{2\}$ is not critical, either.

Unfortunately, D-criticality has a disadvantageous property. Whether a voter a is D-critical wrt a coalition depends on a's own vote. This can be illustrated as follows. Consider $\mathcal{M}_5$ again. Compare the voting profiles $S_1 = \{1\}$ and $S_2 = \{1, 2\}$. Consider voter $a = 2$. She is D-critical of rank 2 wrt S_1 and not D-critical of any rank wrt S_2. The reason is this: She could only be D-critical of rank 2 wrt S_2, because, otherwise, the minimality requirement (part 4 in Def. A.2) would not be fulfilled. A group that is compatible with requirements 1–4 is $\{2, 3\}$, e.g. But 2 is not essential for this group being critical; since 2 votes yes anyway, $G = \{3\}$ would be sufficient.

Thus, it is not only the others' votes that determine whether a voter is D-critical of a certain rank. Consequently, the probability of D-criticality is not a measure of the opportunities that the votes of the others leave to a fixed voter. Power measures that are defined on the base of D-criticality are thus no measures of voting power. For this reason I will stop investigating D-criticality and related notions. A similar problem does not arise for d-criticality. Part of the reason is Prop. 3.2, Part 1. There is also an additional reason not to consider the probability of D-criticality as a power measure. The requirement of essentiality has no relation to what a voter can do by casting a vote – and this is what voting power is all about.

Acknowledgements

This work was partly supported by a grant that the Leverhulme Trust awarded the Voting Power and Procedures (VPP) group at the LSE (Grant F/07 004/AJ). I am also grateful to the organizers of the fifth VPP workshop at Warwick University. This paper has greatly benefitted from discussions at that workshop, and I would like to thank the other participants. I owe particularly useful comments and suggestions to L. Bovens, M. Braham, D. Felsenthal, M. Machover, P. Morriss and S. Napel.

References

Banzhaf, J. F. I., *3.312 Votes, A Mathematical Analysis of the Electoral College*, Villanova L. Rev. **13** (1968), 304–332.

Beisbart, C. & Bovens, L., *A Power Measure Analysis of Amendment 36 in Colorado*, Public Choice (2008), in press.

Beisbart, C., *One Person, Several Votes*, submitted to *Public Choice*, 2008.

Bovens, L. & Beisbart, C., *Measuring Influence for Dependent Voters: A Generalisation of the Banzhaf Measure*, submitted to Social Choice and Welfare, 2007.

Carnap, R., *Logical Foundations of Probability*, University of Chicago Press, Chicago, 1950.

Chamberlain, G. & Rothschild, M., *A Note on the Probability of Casting a Decisive Vote*, Journal of Economic Theory **25** (1981), 152–162.

Coleman, J. S., *Control of Collectivities and the Power of a Collectivity to Act*, in: *Social Choice* (Lieberman, B., ed.), Gordon and Breach, New York, 1971, pp. 269–300.

Dubey, P. & Shapley, L. S., *Mathematical Properties of the Banzhaf Power Index*, Mathematics of Operations Research **4** (1979), 99–131.

Edelman, P. H., *Voting power and at-large representation*, Mathematical Social Sciences **47** (2004), 219–232.

Felsenthal, D. S. & Machover, M., *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*, Edward Elgar, Cheltenham, 1998.

Felsenthal, D. S. & Machover, M., *Enlargement of the EU and Weighted Voting in its Council of Ministers*, Voting Power Report 01/00, London School of Economics and Political Science, Centre for Philosophy of Natural and Social Science, London; downloadable from http://www.lse.ac.uk/vp, 2000.

Gelman, A., Katz, J. N., & Bafumi, J., *Standard Voting Power Indexes Don't Work: An Empirical Analysis*, British Journal of Political Science **34** (2004), 657 – 74.

Good, I. J. & Mayer, L. S., *Estimating the Efficacy of a Vote*, Behavioral Science **20** (1975), 25–33.

Kaniovski, S. & Leech, D., *A Behavioural Power Index*, mimeo, URL: http://serguei.kaniovski.wifo.ac.at/fileadmin/files_kaniovski/pdf/bpower.pdf, 2007.

Machover, M., *Discussion Topic: Voting Power when Voters' Independence is not Assumed*, mimeo, URL: http://eprints.lse.ac.uk/2966/, 2007.

Morriss, P., *Power. A Philosophical Analysis*, Manchester University Press, Manchester, 1987, second edition 2002.

Penrose, L. S., *The Elementary Statistics of Majority Voting*, Journal of the Royal Statistical Society **109** (1946), 53–7.

Raz, J., *Engaging Reason*, Oxford University Press, Oxford, 1999.

Shapley, L. S. & Shubik, M., *A Method for Evaluating the Distribution of Power in a Committee System*, American Political Science Review **48** (1954), 787–792.