Note on an Extension of a Theorem of Saari and Merlin

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Abstract We provide a geometric characterization of the Kemeny Voting Rule in terms of the Euclidean metric and the mean location of a set of points in space.

Introduction

Let A be a set containing a finite number a of alternatives, and suppose that each individual m from a finite set N of n voters casts a vote consisting of a complete, strict, and transitive preference ranking σ_m of the alternatives in A. Such an assignment of rankings to voters is known as a *profile, p*. We'll take the *Hamming distance* $d_H(\sigma, \tau)$ between any two such rankings to be the number of unordered pairs $\{u, v\}$ of alternatives that are ranked differently by σ and τ:

$$d_H(\sigma, \tau) = |\{\{u, v\}: u <_\sigma v \Leftrightarrow v <_\tau u\}|.$$

The *Kemeny voting rule* selects, as the set $T = T_K(p)$ of *Kemeny (social) rankings*, the collection of those τ that minimize the sum $\sum_{m \in N} d_H(\sigma_m, \tau)$ of the distances to the voted rankings. The set $S = S(p)$ of *Kemeny social choices* contains all alternatives atop Kemeny rankings. In the event of a tie among rankings, T contains several τ, each achieving the same minimum distance sum, and S may contain several Kemeny social choices. Because the Kemeny rule outcome is a set of strict rankings, it is a *preference function* in the sense of [Y-L], rather than a social welfare function.

In 1978 Young and Levenglick [Y-L] characterized the Kemeny Rule among preference functions:

> **Theorem A** The Kemeny Rule is the unique consistent and neutral Condorcet preference function.

Here *consistency* is the property that whenever two disjoint electorates N_1 and N_2 yield sets T_1 and T_2 of social rankings that have at least one ranking in common, the combined electorate $N_1 \cup N_2$ yields as its set of social rankings the intersection $T = T_1 \cap T_2$. *Neutrality* requires that whenever some permutation π of the alternatives is applied to each of the rankings submitted as votes, the new outcome consists of π applied to each of the original social rankings; in other words, the voting system treats alternatives equally.

[•] We'd like to thank Vincent Merlin and Alan Taylor for helpful comments.

The intuition behind the Condorcet property for preference functions is related to that of Condorcet Extension for social choice functions, but the details are quite particular. Let's define the net pairwise margin $n_{u,v}$ for two alternatives u and v by

$$n_{u,v} = \left\lfloor \begin{matrix} \textit{the number of voters} \\ \textit{ranking } u \textit{ over } v \end{matrix} \right\rfloor - \left\lfloor \begin{matrix} \textit{the number of voters} \\ \textit{ranking } v \textit{ over } u \end{matrix} \right\rfloor,$$

and define the strict and weak Condorcet relations by

$$u >_C v \text{ iff } n_{u,v} > 0, \text{ and}$$
$$u \geq_C v \text{ iff } n_{u,v} \geq 0.$$

We will say that a ranking σ ranks an alternative v *immediately above* another alternative u if $v >_\sigma u$, with no alternative w satisfying $v >_\sigma w >_\sigma u$. Note that if σ' is obtained from such a σ by switching the positions of u and v then the binary relations $>_\sigma$ and $>_{\sigma'}$ differ on exactly one ordered pair. A preference function f is said to be *Condorcet* if

(i) Whenever $u >_C v$, no ranking σ in T_f ranks v *immediately above* u, and
(ii) Whenever both $u \geq_C v$ and $v \geq_C u$ hold, we have $\sigma \in T_f$ if and only if $\sigma' \in T_f$ for each σ that ranks v *immediately above* u and each σ' obtained from σ by switching the positions of u and v.

These requirements may sound a bit peculiar at first, but [Y-L] supplies some good intuitions.

One effect of the Young and Levenglick result was to establish the Kemeny Rule as being both more important, and more mathematically natural, than had previously been recognized. More recently, Saari and Merlin [SM] provided a very nice geometric characterization:

> **Theorem B** The Kemeny Rule ranking assigned to a profile p is the ranking of the transitive ranking region which has the closest l_1 distance to the outcome q.

Recall that the l_1 metric in $\mathbf{R}^j$ is defined by $\|x-y\|_1 = \sum_{k=1}^{j} |x_k - y_k|$.

To interpret the other terminology of Theorem B, lets begin with an arbitrary choice of reference list $R = ((u_1,v_1), \ldots, (u_j,v_j))$ of ordered pairs of alternatives, having the property that each unordered pair {a, b} of distinct alternatives from A appears exactly once (either as (a,b) or as (b,a)) on the list, so that $j = \binom{a}{2}$. This allows each possible transitive ranking σ to be plotted as a point $R(\sigma)$ of $\mathbf{R}^j$, according to the rule

$$R(\sigma)_k = \begin{cases} +1 & \textit{if} \quad u_k >_\sigma v_k \\ -1 & \textit{if} \quad v_k >_\sigma u_k \end{cases}.$$

Note that each such point $R(\sigma)$ is a vertex of a $2\times2\times\ldots\times2$ hypercube in $\mathbf{R}^j$. We'll refer to this hypercube as PCC_a, the pairwise comparison cube for *a* alternatives. Note that there are $2^j - n!$ vertices of PCC_a that are not of the form $R(\sigma)$, and each of these corresponds to a complete, antisymmetric and intransitive binary relation. For example, suppose $A = \{p, q, r\}$ is a set of three alternatives (so that $j = 3$), with $R = ((p, q), (q, r), (r, p))$. Then if σ is the (transitive) ranking $p > q > r$, we get $R(\sigma) = (+1, +1, -1)$ – one of the eight vertices of the $2\times2\times2$ cube PCC_3 in $\mathbf{R}^3$.

`Put Figure 1 about here.`

Figure 1 shows this cube, with the six possible transitive rankings labeling six of the eight vertices. Notice that PCC_3 consists of eight $1\times1\times1$ sub-cubes, which touch along their two dimensional faces; each of these subcubes contains exactly one of the original vertices. Saari uses the term *transitive ranking region* to refer to any subcube whose vertex $R(\sigma)$ corresponds to a transitive ranking σ. In the general case of PCC_a we will refer to *transitive* and *intransitive vertices*, with associated *transitive* and *intransitive subcubes*. In Figure 1, for example, the two intransitive vertices are $(+1, +1, +1)$ and $(-1, -1, -1)$; these correspond to the two possible cycles for three alternatives: $p > q > r > p$ and $r > q > p > r$. The *outcome point* q is the vector of normalized pairwise margins: the k^{th} component q_k of q is given by

$$q_k = \frac{n_{u_k,v_k}}{n}.$$

The Saari and Merlin theorem adds to our understanding of the relationship of the Kemeny Rule to other voting methods, such as the Condorcet Procedure and the Borda count, that can also be given geometric characterizations. In the same spirit, and in connection with recent work ([Z1], [Z2]) on the role of the mean and median in social choice theory, we consider the following extension of their result:

> **Theorem C** The Kemeny Rule ranking assigned to a profile p is the ranking of the transitive vertex which is closest (using standard Euclidean distance – the l_2 norm) to the outcome q.

Consider the following, alternative description of the outcome point q: suppose that each voter m casts their vote σ_m for the point $R(\sigma_m)$ in $\mathbf{R}^j$. We obtain a multiset of points of $\mathbf{R}^j$, and it is easy to see that the point q described earlier is identical to the mean (average) location of the points $R(\sigma_m)$. Consequently, we obtain a restatement of Theorem C, to which we add a parallel characterization of the Condorcet Procedure:

> **Theorem C′** If we plot each vote σ_m as a transitive vertex $R(\sigma_m)$ of the Pairwise Comparison Cube PCC_a, and determine the vector mean q of the resulting points of $\mathbf{R}^j$, then the Kemeny Rule outcome is the set of ranking(s) corresponding to the closest ***transitive vertices*** of this cube to q, and the Extended Condorcet Procedure outcome is the set of binary relations corresponding to the closest ***vertices*** of the cube to q.

Because of the Theorem C′ context, we have to be a bit careful about what we mean by the Condorcet Procedure, but this is not difficult to sort out.[1] In thinking about Theorem C′, note that it may happen that the mean location q lies strictly inside an *intransitive* subcube. In this case the Condorcet procedure yields as election outcome the intransitive binary relation corresponding to the vertex of PCC_a contained in that subcube, while the Kemeny Rule discards this choice in favor of the closest transitive vertices. (In effect, this divides each intransitive subcube into several subregions, according to which transitive vertex is proximate.) Of course, when q lies in the interior of a *transitive* subcube, Kemeny and Condorcet agree: the closest vertex and the closest transitive vertex are each the vertex of PCC_a lying in that subcube. Here "closest" refers, again, to standard Euclidean distance (the l_2 norm).

Now Hamming distance, the l_1 norm, and the l_2 norm would seem to be quite different from each other, so these results are a bit surprising. But also note that the mean location q of any multiset of points is well-known to be the point that minimizes the sums of the squares of the (Euclidean) distances to the points. Hence, while the original description has led some authors to term the Kemeny Rule a ***median method*** (because it entails minimizing the sums of the Hamming distances), Theorem C′ shows that that it also is a ***mean method*** (entailing, in a somewhat different sense, the minimization of a sum of squared Euclidean distances).[2]

Two proofs of Theorem C

We provide a detailed proof from Theorem A (Saari and Merlin), and then sketch a proof from Theorem B (Young and Levenglick). It is straightforward to check the part of Theorem C′ referring to the Condorcet Procedure, so we leave the details to the reader.

Proof from Theorem A Consider the following lemma:

Lemma 1 The following three conditions are equivalent for any point u in PCC_a, and any two vertices A and B of PCC_a:

a) The point u is at least as close, in the l_1 norm, to A's subcube as to B's subcube.
b) The point u is at least as close, in the l_1 norm, to A as to B.
c) The point u is at least as close, in the l_2 norm, to A as to B.

[1] Traditionally the *Condorcet social ranking* orders alternatives according to the Condorcet relation a $>_C$ b. The result may be a complete, strict, and transitive preference ranking $>_C$, or a weak linear ordering (both $>_C$ and $\geq_C$ are transitive, but $>_C$ fails to be complete because of ties). Of course it is also possible that either $>_C$ or $\geq_C$ are intransitive, or that both are intransitive, and most authors would say that there is no Condorcet social ranking in these scenarios (although there may still be a Condorcet alternative). The "Extended" version of the Condorcet procedure characterized in Theorem C′ is a bit different, because it yields a set T containing one or more complete and antisymmetric binary relations. But the relationship with the traditional version is clear. From the set T we can reconstruct $>_C$ as the intersection of the binary relations in T (and of course $\geq_C$ is implicit in $>_C$), while from $>_C$ we can construct T as the set of all complete and antisymmetric relations extending $>_C$. In this sense, it seems not unreasonable to say that the two versions are morally equivalent.
[2] Kemeny also suggested an alternative voting system, in which the sum of the squared Hamming distances is minimized. That system is different from the Kemeny Rule discussed in this paper.

From Lemma 1 it follows that the set of points of PCC_a that are at least as "close" to some vertex A as they are to each other vertex B is the same set, regardless of whether we interpret "close" to mean in the sense of condition a, condition b, or condition c. Theorem C thus follows immediately from Theorem A plus Lemma 1.

Proof of Lemma 1: Recall that $j = \binom{a}{2}$, so that PCC_a is a subset of $\mathbf{R}^j$.

Let $J = \{1, 2, \ldots, j\}$, and let V_a denote the set of vertices of PCC_a. Let $A = (a_1, \ldots, a_j)$ and $B = (b_1, \ldots, b_j)$ be any vertices in V_a. Define the subsets $+\mathcal{A}$, $-\mathcal{A}$, $+\mathcal{B}$, and $-\mathcal{B}$ of J as follows:

$$+\mathcal{A} = \{i: a_i = +1\}$$
$$-\mathcal{A} = \{i: a_i = -1\}$$
$$+\mathcal{B} = \{i: b_i = +1\}$$
$$-\mathcal{B} = \{i: b_i = -1\}.$$

Let $u = (u_1, \ldots, u_j)$ be a typical point in PCC_a. Then $-1 \le u_i \le 1$ for each i, and we define the subsets $+\mathcal{U}$ and $-\mathcal{U}$ of J as follows:

$$+\mathcal{U} = \{i: u_i \ge 0\}$$
$$-\mathcal{U} = \{i: u_i < 0\}.$$

Let $w_i = |u_i|$ for each i. Then for each i, $0 \le w_i \le 1$ and

$$u_i = w_i \text{ if } i \in +\mathcal{U},$$
$$u_i = -w_i \text{ if } i \in -\mathcal{U}.$$

For any $C = (c_1, \ldots, c_j)$ with $C \in V_a$, let R_C denote C's subcube – the region of PCC_a that is at least as close to C, in the ℓ_2 norm, as it is to any other vertex $D \in V_a$. That is,

$$R_C = \{u \in PCC_a : \forall D \in V_a,\ \|C-u\|_2 \le \|D-u\|_2\}.$$

Claim (1a) Let $\mathcal{E}$ be the set containing all points of the form $(e_1, \ldots, e_j)$, such that for each i either $e_i = c_i$ or $e_i = 0$. Then R_C is equal to the $1\times1\times \ldots \times1$ subcube of PCC_a whose vertices are the 2^j points in $\mathcal{E}$.

proof of claim (1a): easy

Lemma 1 now follows immediately from the following claim.

Claim (2a) The following are equivalent:

(1) $\sum_{i \in +A \cap -B} u_i \ge \sum_{i \in -A \cap +B} u_i$

(2) $\|A-u\|_2 \le \|B-u\|_2$

(3) $\|A-u\|_1 \le \|B-u\|_1$

(4) $\|R_A-u\|_1 \le \|R_B-u\|_1$.

Here by the distance $\|R_A - u\|_1$ between a point u and a subcube R_A we mean, of course, the distance between u and the closest point $\bar{A}$ of R_A to u. We'll proceed by showing each of the conditions (2) - (4) to be equivalent to (1).

(2) ⇔ (1): Note that $A \cdot A - B \cdot B = 0$, as $A \cdot A = 1^2 + 1^2 + \ldots + 1^2 = j = B \cdot B$.
Let D = A - B = (d_1, . . ., d_j) , and observe that

$d_i = 2$, if $i \in +\mathcal{A} \cap -\mathcal{B}$,
$d_i = -2$, if $i \in -\mathcal{A} \cap +\mathcal{B}$,
$d_i = 0$, otherwise.

With these facts in mind, we see that:

$$\begin{aligned}
&\|A-u\|_2 \le \|B-u\|_2 \\
&\Leftrightarrow \left(\|A-u\|_2\right)^2 \le \left(\|B-u\|_2\right)^2 \\
&\Leftrightarrow (A-u)\cdot(A-u) \le (B-u)\cdot(B-u) \\
&\Leftrightarrow A\cdot A - 2A\cdot u + u\cdot u \le B\cdot B - 2B\cdot u + u\cdot u \\
&\Leftrightarrow 2(A-B)\cdot u \ge A\cdot A - B\cdot B \\
&\Leftrightarrow 2(A-B)\cdot u \ge 0 \\
&\Leftrightarrow D\cdot u \ge 0 \\
&\Leftrightarrow 2\left(\sum_{i\in +A\cap -B} u_i - \sum_{i\in -A\cap +B} u_i\right) \ge 0 \\
&\Leftrightarrow \sum_{i\in +A\cap -B} u_i \ge \sum_{i\in -A\cap +B} u_i \,.
\end{aligned}$$

Put Figure 2 about here.

(3) ⇔ (1): Consider the numbered regions S_1, . . ., S_8 of the Figure 2 Venn diagram created by $+\mathcal{A}$, $+\mathcal{B}$, and $+\mathcal{U}$ as subsets of J. Note that

$$\|A-u\|_1 = \sum_{i\in +A}|1-u_i| + \sum_{i\in -A}|-1-u_i| = \sum_{i\in +A\cap +U}(1-w_i) + \sum_{i\in +A\cap -U}(1+w_i) + \sum_{i\in -A\cap -U}(1-w_i) + \sum_{i\in -A\cap +U}(1+w_i)$$

$$= j + \sum_{i\in S_1\cup S_5\cup S_2\cup S_6} w_i - \sum_{i\in S_4\cup S_7\cup S_3\cup S_8} w_i \,.$$

Similarly, $\|B-u\|_1 = j + \sum_{i\in S_3\cup S_5\cup S_4\cup S_6} w_i - \sum_{i\in S_2\cup S_7\cup S_1\cup S_8} w_i$.

So,

$$\begin{aligned}
&\|A-u\|_1 \le \|B-u\|_1 \\
\Leftrightarrow\quad & j + \sum_{i\in S_1\cup S_5\cup S_2\cup S_6} w_i - \sum_{i\in S_4\cup S_7\cup S_3\cup S_8} w_i \le j + \sum_{i\in S_3\cup S_5\cup S_4\cup S_6} w_i - \sum_{i\in S_2\cup S_7\cup S_1\cup S_8} w_i \\
\Leftrightarrow\quad & \sum_{i\in S_1\cup S_2} w_i - \sum_{i\in S_4\cup S_3} w_i \le \sum_{i\in S_3\cup S_4} w_i - \sum_{i\in S_2\cup S_1} w_i \,.
\end{aligned}$$

$$\Leftrightarrow \quad 2\left(\sum_{i\in S_1\cup S_2} w_i\right) \le 2\left(\sum_{i\in S_3\cup S_4} w_i\right) \qquad (\Gamma)$$

$$\Leftrightarrow \quad \sum_{i\in S_4} w_i - \sum_{i\in S_1} w_i \ge \sum_{i\in S_2} w_i - \sum_{i\in S_3} w_i$$

$$\Leftrightarrow \quad \sum_{i\in +A\cap -B} u_i \ \ge \ \sum_{i\in -A\cap +B} u_i \text{ , as desired.}$$

(4) $\Leftrightarrow$ (1): For each i = 1,2, . . ., j let $\bar{a}_i = \begin{cases} u_i, \ \textit{if} \ i\in (+A\cap +U)\cup(-A\cap -U) \\ \quad 0, \ \textit{otherwise} \end{cases}$,

and let $\bar{A} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_j) \in$ R$_A$. Then $\left\|\bar{A} - u\right\|_1 = \sum_{i\in(+A\cap -U)\cup(-A\cap +U)} w_i$. Any point F in R$_A$ has an l_1 distance from u at least as great as that of $\bar{A}$, since $\left\|F-u\right\|_1 = \sum_{i=1,2,\ldots,j} |f_i - u_i|$, and for each i in $(+\mathcal{A} \cap -\mathcal{U}) \cup (-\mathcal{A} \cap +\mathcal{U})$, we have either $f_i \ge 0$ and $u_i < 0$ or $f_i \le 0$ and $u_i \ge 0$, so that $|f_i - u_i| = |f_i| + |u_i| \ge |u_i| = w_i$, making $\sum_{i=1,2,\ldots,j} |f_i - u_i| \ge \sum_{i\in(+A\cap -U)\cup(-A\cap +U)} w_i$. It follows that $\left\|R_A - u\right\|_1 = \sum_{i\in(+A\cap -U)\cup(-A\cap +U)} w_i$, and similarly $\left\|R_B - u\right\|_1 = \sum_{i\in(+B\cap -U)\cup(-B\cap +U)} w_i$.

Hence $\quad \left\|R_A - u\right\|_1 \le \left\|R_B - u\right\|_1$

$$\Leftrightarrow \quad \sum_{i\in(+A\cap -U)\cup(-A\cap +U)} w_i \ \le \ \sum_{i\in(+B\cap -U)\cup(-B\cap +U)} w_i ,$$

$$\Leftrightarrow \quad \sum_{i\in S_1\cup S_5\cup S_2\cup S_6} w_i \ \le \ \sum_{i\in S_3\cup S_5\cup S_4\cup S_6} w_i ,$$

$$\Leftrightarrow \quad \sum_{i\in S_1\cup S_2} w_i \le \sum_{i\in S_3\cup S_4} w_i .$$

This last inequality is equivalent to the earlier line (Γ) and hence, as shown earlier, is equivalent to $\sum_{i\in +A\cap -B} u_i \ \ge \ \sum_{i\in -A\cap +B} u_i$.

This completes the proof of Lemma 1, and of Theorem C ■

<u>Proof Sketch from Theorem B</u> Let K-2 denote the geometric procedure described in Theorem C's statement. It is enough, according to Theorem B, to demonstrate that K-2 is a neutral and consistent Condorcet preference function. Neutrality is clear. To see that K-2 is Condorcet, suppose that σ is any ranking of A in which ranks v immediately before u and that σ′ is obtained from σ by switching the positions of u and v. Then it is easy to see that if u $>_C$ v, R(σ′) is strictly closer to q than is R(σ); this rules out σ ∈ $T_{K\text{-}2}$, verifying clause (i) in the definition of Condorcet preference function. Also, if both u $\ge_C$ v and v $\ge_C$ u then it is clear that R(σ′) and R(σ) are equidistant from q. This shows that σ ∈ $T_{K\text{-}2}$ if and only if σ′ ∈ $T_{K\text{-}2}$, verifying clause (ii).

It remains to show consistency. For any transitive vertex v let TR_v be the corresponding ranking region consisting of all points u of PCC_a that are at least as close to v as they are to every other transitive vertex. Note that TR_v is a convex polytope formed as the intersection of finitely many closed half spaces, and is a proper superset of the subcube R_v. We'll define a *face* to be an intersection of one or more sets of the form TR_v and a *face of TR_v* to be a face that is a subset of TR_v. For each point $r \in PCC_a$ let F(r) denote the face of smallest dimension containing r, and let T(r) be the set of transitive vertices closest to r. Consider elections with (nonempty) electorates N_1, N_2, and $N = N_1 \cup N_2$ respectively, and with mean points (in the sense of Theorem C′) q_1, q_2 and q respectively. Consistency of K-2 now follows from the following sequence of five claims:

Claim 1b The following are equivalent for any $r \in PCC_a$:

(i) $v \in T(r)$

(ii) $r \in TR_v$

(iii) F(r) is a face of TR_v.

Claim 2b The point q lies on the open line segment $(\overline{q_1 q_2})$ joining q_1 and q_2.

Claim 3b If $F(q_1)$ and $F(q_2)$ are both faces of some TR_w, and q lies on the open line segment $(\overline{q_1 q_2})$, then $F(q_1)$ and $F(q_2)$ are both subfaces of F(q).

We leave the proofs of Claims 1b, 2b, and 3b to the reader.

Claim 4b If $w \in T(q_1) \cap T(q_2)$ then $w \in T(q)$.
Proof Assume $w \in T(q_1) \cap T(q_2)$. Then $F(q_1)$ and $F(q_2)$ are faces of TR_w. As q_1 and q_2 are both in TR_w, segment $(\overline{q_1 q_2})$ is entirely contained in TR_w, which is convex. So q is in TR_w, F(q) is a face of TR_w, and $w \in T(q)$.

Claim 5b If $T(q_1) \cap T(q_2) \neq \varnothing$ and $v \in T(q)$, then $v \in T(q_1) \cap T(q_1)$.
Proof If $v \in T(q)$, then F(q) is a face of TR_v by Claim1b. If $T(q_1) \cap T(q_2) \neq \varnothing$ then it follows from Claim 3b that $F(q_1)$ and $F(q_2)$ are both subfaces of F(q), so they must both be faces of TR_v. Thus $v \in T(q_1) \cap T(q_1)$. ■

Concluding Remarks

(1) The equivalence of conditions b and c in Lemma 1 was unexpected. It hinges on the special nature of coordinates for u, A, and B; in particular, these conditions are not equivalent for arbitrary points u, A, B.

(2) Suppose we take condition a of Lemma 1

a) The point u is at least as close, in the l_1 norm, to A's subcube as to B's subcube.

and change l_1 to l_2, obtaining a new ***condition d***:

d) The point u is at least as close, in the l_2 norm, to A's subcube as to B's subcube.

As Saari and Merlin point out, condition d is ***not*** equivalent to condition a. Thus, the new l_2-result necessarily depends on measuring distance from q to a vertex, rather than to a subcube.

(3) It seems that it might be possible to provide an alternative proof of Young and Levenglick's characterization theorem, by showing directly that any consistent and neutral Condorcet preference function must agree with the Theorem C description of the Kemeny Rule. For example, with three alternatives it seems one should be able to show that the only possible convex and appropriately symmetric method for dividing up the intransitive subcubes is that of comparing Euclidean proximity to the transitive vertices.

(4) What are the larger implications of Theorems C and C′? It turns out that a number of voting methods can be represented in essentially the same way: individuals cast votes, in effect, for certain specified points in space, the mean location q of these votes is determined, and the election outcome is given by the point nearest to q. Such *Discretized Mean Voting Systems*, or DMVSs, include the scoring systems (including Borda and Plurality) and approval voting, as well as the Kemeny Rule and Extended Condorcet Procedure discussed here. Thus differences between these voting systems arise solely from changes to the spatial configuration of the points voted for (and, also from changes to the "output points" when these are different from the inputs).

To what extent can properties common to all DMVSs be explained in terms of the fundamental axiomatic character of the mean? Is it possible to construct new voting systems, with very different properties, by replacing the mean with some alternative, such as any one of several generalizations of the median to the multivariate context? In particular, does the relative insensitivity of the median to outliers lead to voting systems that are less manipulable than their mean relatives? We explore these and related issues in two forthcoming papers, [Z1] and [Z2].

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