I-POWER AND P-POWER: SHAPLEY-SHUBIK AND/OR PENROSE-BANZHAF?

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Power indices methodology is widely used to measure an a priori voting power of members of a committee. In this paper we analyse Shapley-Shubik and Penrose-Banzhaf concepts of power measure and classification of so called I power (voter's potential influence over the outcome of voting) and P power (expected relative share in a fixed prize available to the winning group of committee members) introduced by Felsenthal, Machover and Zwicker (1998). We show that objections against Shapley-Shubik power index, based on its interpretation as a P-power concept, are not sufficiently justified. Both Shapley-Shubik and Penrose-Banzhaf measure could be successfully derived as cooperative game values, and at the same time both of them can be interpreted as probabilities of some decisive position (pivot, swing) without using cooperative game theory at all. Problem of relative and absolute power in a committee is discussed: is the power of a given committee constant, or is it a function of quota and allocation of weights?

Keywords: Absolute power, cooperative games, I-power, pivot, power indices, relative power, P-power, swing.

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1. Introduction

Let N = {1, ..., n} be the set of members (players, parties) and ω_i (i = 1, ..., n) be the (real, non-negative) weight of the i-th member such that

$$\sum_{i \in N} \omega_i = 1, \omega_i \geq 0$$

(e.g. the share of votes of party i, or the ownership of i as a proportion of the total number of shares, etc.). Let γ be a real number such that $0 < \gamma < 1$. The (n+1)-tuple

$$[\gamma, \grave{\mathbf{u}}] = [\gamma, \omega_1, \omega_2, ..., \omega_n]$$

such that

$$\sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0, 0 \leq \gamma \leq 1$$

we shall call a committee (or a weighted voting body) of the size n = card N with quota γ and allocation of weights

$$\grave{\mathbf{u}} = (\omega_1, \omega_2, ..., \omega_n)$$

(by *card* S we denote the cardinality of the finite set S, for empty set *card* ∅ = 0)

Any non-empty subset S ⊆N we shall call a voting configuration. Given an allocation **ω** and a quota γ, we shall say that S ⊆ N is a winning voting configuration, if

$$\sum_{i \in S} \omega_i \geq \gamma$$

and a losing voting configuration, if

$$\sum_{i \in S} \omega_i < \gamma$$

(i.e. the configuration S is winning, if it has a required majority, otherwise it is losing).

Let

$$G = \left[(\gamma, \grave{\mathbf{u}}) \in R_{n+1} : \sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0, 0 \leq \gamma \leq 1 \right]$$

be the space of all committees of the size n and

$$E = \left[\mathbf{e} \in R_n : \sum_{i \in N} e_i = 1, e_i \geq 0 \; (i = 1, ..., n) \right]$$

be the unit simplex.

A *power index* is a vector valued function

$$\pi : G \rightarrow E$$

that maps the space G of all committees into the unit simplex E. A power index represents a reasonable expectation of the share of decisional power among the various members of a committee, given by ability to contribute to formation of winning voting configurations. We shall denote by $\pi_i(\gamma, \boldsymbol{\omega})$ the share of power that the index $\boldsymbol{\pi}$ grants to the i-th member of a committee with weight allocation $\boldsymbol{\omega}$ and quota γ. Such a share is called a *power index of the i-th member*.

2. Penrose-Banzhaf and Shapley-Shubik power indices

Two most widely used power indices were proposed by Penrose and Banzhaf (1946, 1965) and Shapley and Shubik (1954). We shall refer to them as PB-power index and SS-power index.

The PB-power measure is based on the concept of swing. Let S be a winning configuration in a committee $[\gamma, \boldsymbol{\omega}]$ and $i \in S$. We say that a member i has a swing in configuration S if

$$\sum_{k \varepsilon S} \omega_k \geq \gamma \quad and \quad \sum_{k \varepsilon S \setminus \{ i \}} \omega_k < \gamma$$

Let s_i denotes total number of swings of the member i in the committee $[\gamma, \boldsymbol{\omega}]$. Then PB-power index is defined as

$$\pi_i^{PB}(\gamma, \boldsymbol{\omega}) = \frac{s_i}{\sum_{k \in N} s_k}$$

In literature this form is usually called a relative PB-index. Original Penrose definition of power of the member i was

$$\pi_i^{P}(\gamma, \omega) = \frac{s_i}{2^{n-1}}$$

which (assuming that all coalitions are equally likely) is nothing else but the probability that the given member will be decisive (probability to have a swing). In literature this form is usually called an absolute PB-index. The relative PB-index is obtained by normalization of the absolute PB-index.

Let the numbers 1, 2, ..., n be fixed names of committee members. Let

$$(i_1, i_2, \ldots, i_n)$$

be a permutation of those numbers, members of the committee, and let member k is in position r in this permutation, i.e. $k = i_r$. We shall say that member k of the committee is in a pivotal situation with respect to a permutation $(i_1, i_2, \ldots, i_n)$, if

$$\sum_{j=1}^{r-1} w_{i_j} < q \quad and \quad \sum_{j=1}^{r} w_{i_j} \geq q$$

The SS-power measure is based on the concept of pivot. Assume that an ordering of members in a given permutation expresses an intensity of their support (preferences) for a particular issue in the sense that, if a member i_s precedes in this permutation a member i_t, then i_s support for the particular proposal to be decided is stronger than support by the member i_t. Then, it is plausible to assume, that if i_t votes YES, is also votes YES, if i_s votes NO, it also votes NO. If i_r has a pivot, then his vote (YES or NO) will coincide with the final outcome of voting. Member in a pivotal situation has a decisive influence on the final outcome. Assuming many voting acts and all possible preference orderings equally likely, under the full veil of ignorance about other aspects of individual members preferences, it makes sense to evaluate an a priori voting power of each committee member by a value proportional to the number of his pivots. The probability of being in pivotal situation is measured by the SS-power index:

$$\pi_i^{SS}(\gamma, \grave{\mathbf{u}}) = \frac{p_i}{n!}$$

where p_i is the number of pivotal positions of the committee member i and n! is the number of permutations of the committee members (number of different orderings).

3. The I-power, P-power and cooperative games

Felsenthal, Machover and Zwicker (1998) introduced concept of so called I-power and P-power.

By I-power they mean “voting power conceived of as a voter’s potential influence over the outcome of divisions of the decision making body: whether proposed bills are adopted or blocked. Penrose’s approach was clearly based on this notion, and his measure of voting power is a proposed formalization of a priori I-power:” By P-power they mean “voting power conceived as a voter’s expected relative share in a fixed prize available to the winning coalition under a decision rule, seen in the guise of a simple TU (transferable utility) cooperative game. The Shapley-Shubik approach was evidently based on this notion, and their index is a proposed quantification of a priori P-power” (in the both cases we are quoting Felsenthal and Machover, 2003, p. 8). Hence, the fundamental distinction between I-power and P-power is in the fact that the I-power notion takes the outcome to be the immediate one, passage or defeat of the proposed bill, while the P-power view is that passage of the bill is merely the ostensible and proximate outcome of a division; the real and ultimate outcome is the distribution of fixed purse - the prize of power – among the victors (Felsenthal and Machover, 2003, p. 9-10). As a conclusion it follows that SS-power index does not measure a priori voting power, but says how to agree on dividing the “pie” (benefits of victory).

As the major argument of this classification the authors provide a historical observation: Penrose paper from 1946 was ignored and unnoticed by mainstream – predominantly American – social choice theorists, and Shapley and Shubik’s 1954 paper was seen as inaugurating the scientific study of voting power. Because the Shapley-Shubik paper was wholly based on cooperative game theory, it induced among social scientists an almost universal unquestioning belief that the study of power was necessarily and entirely a branch of that theory (Felsenthal and Machover, 2003, p. 8). Conclusion follows, that since the cooperative game theory with transferable utility is about how to divide a pie, and SS-power index was derived as a special case of Shapley value of cooperative game, the SS-power index is about P-power and does not measure voting power as such.

We demonstrated above, that one does not need cooperative game theory to define and justify SS-power index. SS-power index is a probability to be in a pivotal situation in an intuitively plausible process of forming a winning configuration, no division of benefits is involved. Incidentally SS-power index originally appeared as an interesting special case of Shapley value for cooperative games with the transferable utility, but in exactly the same way one can handle the PB-index. Let us make a short excursion into the cooperative game theory.

Let N be the set of players in a cooperative game (cooperation among the players is permitted and the players can form coalitions and transfer utility gained together among themselves) and 2^N its power set, i.e. the set of all subsets $S \subseteq N$, called coalitions, including empty coalition. Characteristic function of the game is a mapping

$$v : 2^N \rightarrow R$$

with

$$v(\emptyset) = 0$$

The interpretation of v is that for any subset S of N the number $v(S)$ is the value (worth) of the coalition S, in terms how much "utility" the members of S can divide among themselves in any way that sums to no more than v(S) if they all agree. The characteristic function is said to be super-additive if for any two disjoint subsets $S, T \subseteq N$

$$v(S \cup T) \geq v(S) + v(T)$$

i.e. the worth of the coalition $S \cup T$ is equal to at least the worth of its parts acting separately. Let us denote cooperative characteristic function form by [N, v]. The game [N, v] is said to be super-additive if its characteristic function is super-additive. By a value of the game [N, v] we mean a non-negative vector $\boldsymbol{\varphi}(v)$ such that

$$\sum_{i \in N} \varphi_i(v) = v(N)$$

By

$$c(i,T) = v(T) - v\{T - \{i\})$$

we shall denote marginal contribution of the player $i \in N$ to the coalition $T \subseteq N$. Then in an abstract setting the value $\varphi_i(v)$ of the i-th player in the game [N, v] can be defined as a weighted sum of his marginal contributions to all possible coalitions he can be member of:

$$\varphi_i(v) = \sum_{T\subseteq N, i\in T} \alpha(T)c(i,T)$$

Different weights α(T) leads to different definitions of values.

Shapley (1953) defined his value by the weights

$$\alpha(T) = \frac{(t-1)!(n-t)!}{n!}$$

where t = card (T). He proved that it is the only value that satisfies three axioms: dummy axiom (dummy player, i.e. the player that contributes nothing to any coalition, has zero value), permutation axiom (for any game [N, u] that is generated from the game [N, v] by a permutation of players, the value φ(u) is a corresponding permutation of the value φ(v)) and additivity axiom (for sum [N, v+u] of two games [N, v] and [N, u] the value φ(v+u) = φ(v) + φ(u)).

As Owen (1982) noticed, the relative PB-index is meaningful for general cooperative games with transferable utilities. One can define Banzhaf value by setting the weights

$$\alpha(T) = \frac{v(N)}{\sum_{k\in N, T\subseteq N} c(k,T)}$$

Owen (1982) shows a certain relation between the Shapley value and Banzhaf value of cooperative game with transferable utilities: both give averages of player's marginal contributions, the difference lies in the weighting coefficients (in the Shapley value coefficients depends on size of coalitions, in the Banzhaf value they are independent of coalition size).

The relation between values and power indices is straightforward: A cooperative characteristic function game represented by a characteristic function v such that v takes only the values 0 and 1 is called a simple game. With any committee with quota q and allocation **w** we can associate a super-additive simple game such that

$$v(S) = \begin{cases} 1 \quad if \ \sum_{i\in S} w_i \geq q \\ \\ 0 \quad otherwise \end{cases}$$

(i.e. a coalition has value 1 if it is winning and value 0 if it is losing). Super-additive simple games can be used as natural models of voting in committees. Shapley and Shubik (1954) applied the concept of the Shapley value for general cooperative characteristic function games to the super-additive simple games as a measure of voting power in committees. Here we generalized the Penrose-Banzhaf relative power index as a value for general cooperative characteristic function games. But we do not need cooperative games to model voting in committees.

4. Absolute and relative power

Power indices assign a priori evaluation of voting power to individual committee members. Except of absolute PB index all other indices are expressing relative power.

PB-power measure opens an intriguing question about absolute and relative power: is there something like that? Does power of a committee as an entity depends on quota and distribution of votes among its members or not? The absolute PB-power index

$$\pi_i^P(\gamma,\omega) = \frac{s_i}{2^{n-1}}$$

provides the probability that member i of the committee will be decisive. Then

$$1 - \pi_i^P(\gamma,\omega)$$

is a probability that i will not be decisive, but not the probability that somebody of other committee members will be decisive. What the sum of absolute PB-power indices

$$\sum_{i \in N} \pi_i^P(\gamma,\omega) = \frac{\sum_{i \in N} s_i}{2^{n-1}}$$

means?

To illustrate meaningfulness of the problem of relative and absolute power we shall use a simple example (we shall call it a paradox of abstention): Let there is a committee of five members [q, ω] = [9; 6, 4, 1, 1, 1]. It is easy to see that the distribution of swings of the committee members is 9, 7, 1, 1, 1, vector of absolute PB-indices is (9/16, 7/16, 1/16, 1/16, 1/16) and relative PB-indices are (9/19, 7/19, 1/19, 1/19, 1/.19). Consider now the following situation: member 1 is systematically not using one of his votes (permanent abstention) and quota remains the same, 9 votes. Then we have a new committee [9; 5, 4, 1, 1, 1] with distribution of swings 8, 8, 0, 0, 0, absolute PB-index (8/16, 8/16, 0, 0, 0) and the same relative PB-index (8/16, 8/16, 0, 0, 0). We can observe that the relative index of permanent absentee increases from 9/19 to 8/16, while his absolute index decreases from 9/16 to 8/16. Absolute power of member 1 decreases, while his relative power increases by not using systematically part of the votes. Obvious conclusion is that power of the committee is not constant.

By PB measure it is implicitly assumed that the power of the committee is proportional to number of swings and therefore depends on quota and distribution of votes. Let us accept this position and look for plausible interpretation.

Concept of PB power is based on bipartitions of YES and NO votes. There are 2^n such different bipartitions, for example for N = {1, 2, 3}we have 8 of them:

YYY, YYN, YNY, YNN
NYY, NYN, NNY, NNN

Looking for a probabilistic interpretation we can assume that all of them are equally likely, the probability of each bipartition being $1/2^n$.

We can distinguish between "negative" swings (ability to destroy a winning coalition) and "positive" swing (ability to convert a losing coalition to a winning one). Let s_i^- be the number of negative swings and s_i^+ be the number of "positive" swings of the member i, then clearly $s_i^- = s_i^+$. Then PB-index $s_i/2^{n-1} = 2s_i/2^n$ provides an information about the control of the member i over the outcome in the sense that it gives an average number of swings (positive and negative) per one bipartition (or, a probability that a bipartition will appear in which i will have a swing). Using such an interpretation we can add PB-indices of individual members: e.g. $s_r/2^{n-1} + s_k/2^{n-1}$ gives an average number of (positive and negative) swings of the members r and k acting independently. Then clearly the sum

$$\sum_{i\in N} \pi_i^P(\gamma,\omega) = \frac{\sum_{i\in N} s_i}{2^{n-1}} = \frac{2\sum_{i\in N} s_i}{2^n}$$

gives an average number of (positive and negative) swings of all committee members acting independently per one bipartition. PB power of the committee can be quantified as the average number of swings per one bipartition (to be fair, Felsenthal and Machover are never saying that, but it follows from their argumentation) .

For example, having two committees of the same size, a) [51; 50, 30, 20] and b) [51; 45, 30, 25], the PB power of the committee a) is 10/8, while the PB power of the committee b) is 12/8. The PB power of a committee with dictator is 1, the PB power of a committee with all veto members is $2n/2^n = n/2^{n-1}$.

Power of the committee was never explicitly defined or quantified in numerous discussions on PB-power and SS-power. Only in Holler papers (see e.g. Holler, 2002) the question about the concept of power of a committee is mentioned: is it a public good, a club good or a private good?

What is a power of a committee? Nobody knows. Committee is facing a series of decision making acts when it decides between a move M and the status quo SQ. In each decision making act it selects M or SQ, it takes a decision. Under the veil of ignorance one does not distinguish between M and SQ, they are just two alternatives. Independently on quota and distribution of weights the power of the committee follows from authorization to decide and adding nothing to the model it seems to stay constant. From this point of view there is no difference between absolute and relative power of individual members: power of the committee is a public good shared by individual members.

Nevertheless, distinguishing between relative power and absolute power of committee members one cannot avoid the problem of the power of a committee as an entity. A priori power methodology does not claim to provide an instrument for prediction of “actual” power of individual decision makers, but rather serves for evaluation of rules and institutions. To do that in a coherent way we need to know what the voting power is, whether it is constant for a given committee or a function of quota and weight allocation.

5. Concluding remarks

Using the Felsenthal and Machover classification, there is no reason why not consider Banzhaf value to be a plausible rule for dividing the cake: If Shapley-Shubik expresses P-power, then Penrose-Banzhaf expresses the same. On the other hand, we have demonstrated I-power interpretation of the both indices: they provide the probability of being in a “decisive position”, defined either as pivot, or as swing. Both of the indices can be defined and interpreted in terms of cooperative game theory, both of them can be introduced and analysed without any reference to cooperative games.

Dispute about I-power and P-power is not useless, it raises question about what the voting power is about. In this we see the contribution of Felsenthal and Machover. But discriminating the well defined concept of the SS-power index on the basis of its origin leads to nowhere. We need both PB-power measure and SS-power measure, having nothing better until now.

PB-power measure opens an intriguing question about absolute and relative power: is there something like that? Does power of a committee as an entity depends on quota and distribution of votes among its members or not? Is so called donation paradox for the relative PB-power index (one member of the committee can increase his relative power by transferring part of his votes to another member, while his absolute power is decreasing) a paradox of power as such or a paradox of PB measure? SS-power index does not exhibit such a paradox (see Turnovec, 1998). Another paradox (let us call it a paradox of abstention) says, that by not using systematically part of his votes, when the quota remains to be fixed (systematic abstention) a member i can increase his relative power, while its absolute power is decreasing. Again, nothing like that happens for SS-power index. The message is that we have to be very careful in interpretation of results based on relative PB-power index and not to use it without absolute PB-power index, what is frequently the case in many published studies.

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