Power indices taking into account agents' preferences

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1. Introduction

One of the main shortcomings which is mentioned almost in all publications on voting power indices is the fact that known indices do not take into account the preferences of agents [4-6]. Indeed, in construction of those indices, e.g., Shapley-Shubik or Banzhaf power indices [2, 7], all agents are assumed to be able to coalesce. Moreover, none of those indices evaluates to which extent the agents are free in their wishes to create coalition, how intensive are the connections inside one or another coalition.

Consider an example. Let three parties *A*, *B* and *C* with 50, 49 and 1 sets, respectively, are presented in a parliament, and the voting rule is 51 votes for.

Then winning coalitions are *A*+*B*, *A*+*C*, *A*+*B*+*C* and *A* is pivotal in all coalitions, *B* is pivotal in the first coalition and *C* is pivotal in the second one. Banzhaf power index β for these parties is equal to[1]

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[1] Banzhaf power index is evaluated as

$$\beta_i = \frac{b_i}{\sum_j b_j},$$

$$\beta(A) = 3/5,\ \beta(B) = \beta(C) = 1/5.$$

Assume now that parties *A* and *B* never coalesce in pairwise coalition, i.e., coalition *A*+*B* is impossible. Let us, however, assume that the coalition *A*+*B*+*C* can be implemented, i.e. in the presence of `moderator' *C* parties *A* and *B* can coalesce. Then *A* the winning coalitions are *A*+*B* and *A*+*B*+*C*, and *A* is pivotal in both coalitions while *C* is one; *B* is pivotal in none of the winning coalitions. In this case $\beta(a) = 2/3$, $\beta(B) = 0$ and $\beta(C) = 1/3$, i.e., although *B* has half of the seats in the parliament, its power is equal to 0.

Such situations are met in real political systems. For instance, Russian Communist Party in the second parliament (1997-2000) had had almost 35% of seats, however, its power during that period was always almost equal to 0 [1].

We introduce here two new types of indices based on the idea similar to Banzhaf power index however taking into account agents' preferences to coalesce.

First type of these indices consists of four indices. They use the information about agents' preferences about other agents. These preferences are assumed to be linear orders. Since these preferences may not be symmetric, the desire of agent 1 to coalesce with agent 2 can be different than the desire of agent 2 to coalesce with agent1. These indices take into account in a different way such asymmetry of preferences.

In all these four indices the information about preferences is ordinal, i.e. the intensity of preferences is not evaluated.

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b_i is the number of winning coalitions in which agent *i* is pivotal, i.e., if agent *i* expels from the coalition it becomes a loosing one.

In the second type of power index the information about the intensity of preferences is taken into account, i.e., we extend the former type of power index to cardinal information about agents' preferences.

The structure of the paper is as follows. Section 2 gives main notions. In Section 3 we define and discuss ordinal power indices. In Section 4 cardinal indices are introduced. In Section 5 we evaluate power distribution of groups and factions in the Russian Parlament in 2000-2003 using some of new indices. Section 6 provides some axioms for the indices introduced.

2. **Main notions**

The set of agents is denoted as *N*, $N = \{1,\ldots,n\}$, *n*>1. A coalition ω is the subset of *N*, $\omega \subseteq N$.

Each agent has a predefined number of votes, $v_i > 0$, $i = 1,\ldots,n$. It is assumed that a quota *q* is predetermined and as a decision making rule the voting with quota is used, i.e., the decision is made if the number of votes for it is not less than *q*, i.e.,

$$\sum_i v_i \geq q .$$

The model describes a voting by simple and qualified majority, voting with veto (as in the Security Council of UNO), etc.

A coalition ω is called winning if the sum of votes in the coalition is not less than q. An agent *i* is called pivotal in the coalition ω if the coalition $\omega \setminus \{i\}$ is a loosing one.

To solve the problem stated above, two types of indices, ordinal and cardinal, are introduced. Both types are constructed on the following basis: the intensity of connection $f(i,\omega)$ of the agent with other members of ω is defined. Then for such agent *i* the value χ_i is evaluated as

$$\chi_i = \sum_{\omega} f(i,\omega),$$

i.e. the sum of intensities of connections of i over those coalition ω in which *i* is pivotal. Naturally, other functions instead of summation can be considered.

Then the power indices are constructed as

$$\alpha(i) = \frac{\chi_i}{\sum_j \chi_j}.$$

The very idea of $\alpha(i)$ is the same as for Banzhaf index, with the difference that in Banzhaf index we evaluate the number of coalitions in which *i* is pivotal.

The main question now is how to construct the intensity functions $f(i,\omega)$. Below we give two ways how to construct those functions.

Each agent *i* is assumed to have a linear order[2] P_i revealing her preferences over other agents in the sense that *i* prefers to coalesce with agent *j* than with agent *k* if P_i contains the pair (*j, k*). Obviously, P_i is defined on the cartesian product $(N \setminus \{i\}) \times (N \setminus \{i\})$.

Since P_i is a linear order, the rank p_{ij} of the agent *j* in P_i can be defined. We assume that $p_{ij} = |N|$ -1 for the most preferable agent *j* in P_i.

For instance, if *N*={*A,B,C,D*} and $P_A : B \succ C \succ D$, then $P_{AB} = 3$, $P_{AC} = 2$ and $P_{AD} = 1$.

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[2] i.e. irreflexive, transitive and connected binary relation. We often denote it as $\succ$.

Now one can define an average intensity of connection of i with other members of coalition ω as

$$f(i,\omega)=\frac{\sum_{j\in\omega} p_{ij}}{|\omega|}$$

A second way of construction of $f(i,\omega)$ is based in the idea that the values p_{ij} of connection of i with j are predetermined somehow. In general it is not assumed that $p_{ij}=p_{ji}$. Then the intensity function can be constructed as above.

Below we give 4 different ways how to construct $f(i,\omega)$ in ordinal case and 16 ways of construction of cardinal function $f(i,\omega)$.

3. Ordinal indices

For each coalition ω and each agent i construct now an intensity $f(i,\omega)$ of connections in this coalition. In other words, f is a function which maps $A\times\Omega$, $\Omega=2^N\backslash\{\emptyset\}$, into R^1, i.e., $f: A\times\Omega\to R^1$. This very value $f(i,\omega)$ is evaluated using the ranks of members of coalition. We suggest several different ways to evaluate f using different information about agents' preferences:

a) Intensity of i's preferences. In this form only preferences of i's agent over other agents are evaluated, i.e.,

$$f^+(i,\omega)=\frac{1}{\omega}\sum_{j\in\omega} p_{ij}$$

b) Intensity of preferences for i. In this case consider the sum of ranks of i given by other members of coalition ω

$$f^{-}(i,\omega) = \frac{1}{\omega}\sum_{j\in\omega} p_{ji}$$

c) Average intensity with respect to i's agent

$$f(i,\omega) = \frac{f^{+}(i,\omega) + f^{-}(i,\omega)}{2}$$

d) Total average intensity. Consider any coalition ω of size $k \le n$. Without loss of generality we can put $\omega = \{1,\ldots,k\}$. Then consider for each i

$$f^{+}(i,\omega) = \sum_{j\in\omega} p_{ij}$$

Then

$$f(\omega) = \frac{\sum_{i\in\omega} f^{+}(i,\omega)}{|\omega|},$$

We call a voting situation a triple $\{N, v, \vec{P}\}$, where N is a set of agents, $|N| = n, n > 1$, $v = (v_1,\ldots,v_n)$ is a set of votes which agents possess, $\vec{P}$ is a preference profile, where each agent $i \in N$ has a preference (linear order) P_i over $N \setminus \{i\}$.

In fact, in the definition of voting situation one should include a quota and a decision making rule. However, since a simple majority rule will be considered, it will not lead to some ambiguity.

Consider now two voting situations.

Example 1. Let n=3, N={A, B, C}, $\nu(A) = 33$, $\nu(B) = \nu(C) = 33$, q=50. Consider two preference profiles given in Tables 1 and 2.

Table 1

P_A	P_B	P_C

C	C	A
B	A	B

Table 2

P_A	P_B	P_C
B	A	A
C	C	B

For both voting situations there are three winning coalitions in which agents are pivotal. These coalitions are *A+B*, *A+C* and *B+C*.

Let us calculate the functions *f* as above for each agent in each winning coalition.

The preferences from Tables 1 and 2 can be re-written in the matrix form as

$$\|p_{ij}\| = \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & C \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

$$\|p'_{ij}\| = \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & C \end{array} \\ \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

Now, calculate the values $f^+(i,\omega), f^-(i,\omega), f(i,\omega)$ obtained by each agent *i* in each winning coalition ω. These values are given in Tables 3-5. Since the role of each agent in each coalition is the same, we will not divide the corresponding values to the coalition size

	A	B	C
AB	1	1	–
AC	2	–	2
BC	–	2	1

Table 3. The values for $f^+(i,\omega)$

	A	B	C
AB	1	1	–
AC	2	–	2
BC	–	1	2

Table 4. The values for $f^-(i,\omega)$

	A	B	C
AB	2	2	–
AC	4	–	4
BC	–	3	3

Table 5. The values for $f(i,\omega)$

In Table 6 the values for $f(\omega)$ are given for all winning coalitions ω .

AB	AC	BC
1	2	3/2

Table 6. The values $f(\omega)$

Using these intensity functions one can define now the corresponding power indices $\alpha(i)$. Let i be a pivotal agent in a winning coalition ω. Denote as χ_i the number equal to the value of the intensity function for a given coalition ω and agent i. Then the power index is defined as follows

$$\alpha(i) = \frac{\sum\limits_{\substack{\omega, \\ i \text{ is pivotal in } \omega}} \chi_i}{\sum\limits_{j \in N} \sum\limits_{\substack{\omega, \\ j \text{ is pivotal in } \omega}} \chi_j}$$

As it can be readily seen this index is similar to the Banzhaf index the difference is that χ_i in the Banzhaf index is equal to 1, in the case under study v_i represents some intensity value.

It is worth mentioning that the value of the intensity function $f(\omega)$ does not depend on any agent, it evaluates only the average intensity of `connections' inside a given coalition.

We denote the indices $\alpha(i)$ as $\alpha_1(i),\ldots,\alpha_4(i)$ or as $\alpha(i \mid f^+(i,\omega))$, $\alpha(i \mid f^-(i,\omega))$ and $\alpha(i \mid f(i,\omega))$, $\alpha(i \mid f(\omega))$, respectively.

Now we will consider several examples. Let us evaluate now the values $\alpha_1(\cdot)-\alpha_4(\cdot)$ for all agents. Consider first the profile from Table 1.

The agent *A* (as well as agents *B* and *C*) is pivotal in two coalitions; the sum of the values $f^+(i,\omega)$ for each *i* is equal to 3. Then

$$\alpha_1(A) = \frac{3}{3+3+3} = \frac{1}{3} = \alpha_1(B) = \alpha_1(C).$$

In fact, it can be seen the $\alpha_1(\cdot) = \beta(\cdot)$, i.e. α_1 does not depend on preferences of agents.

The value $\alpha_2(\cdot)$ is evaluated differently. The sum of values $f^-(i,\omega)$ from Table 4 for all i and ω is equal to 9. However, for A $\sum_{\omega} f(A,\omega) = 3$, $\sum_{\omega} f(B,\omega) = 2$ and $\sum_{\omega} f(C,\omega) = 4$. Then $\alpha_2(A) = \frac{3}{9} = \frac{1}{3}$; $\alpha_2(B) = \frac{2}{9}$ and $\alpha_2(C) = \frac{4}{9}$.

For the values of $\alpha_3(\cdot)$ from Table 5 one can obtain $\alpha_3(A) = \frac{6}{18} = \frac{1}{3}$, $\alpha_3(B) = \frac{5}{18}$, $\alpha_3(C) = \frac{7}{18}$.

To evaluate $\alpha_4(\cdot)$ let us construct Table 7 using Table 5.

	A	B	C
AB	1	1	–
AC	2	–	2
BC	–	3/12	3/12

Table 7. The values $\frac{\sum_{\omega}\sum_{i} f^+(i,\omega)}{|\omega|}$

Now the values $\alpha_4(\cdot)$ are equal $\alpha_4(A) = \frac{3}{9} = \frac{1}{3}$, $\alpha_4(B) = \frac{5}{18}$ and $\alpha_4(C) = \frac{7}{18}$.

Let us evaluate now the indices $\alpha_1(\cdot) - \alpha_4(\cdot)$ for the preference profile from Table 2. The corresponding values $f^+(\cdot,\cdot)$, $f^-(\cdot,\cdot)$ and $f(\cdot)$ are given in Tables 8-11.

	A	B	C
AB	2	2	–
AC	1	–	2

BC	–	1	1

Table 8. The values for $f^{+}(i,\omega)$

	A	B	C
AB	2	2	–
AC	2	–	1
BC	–	1	1

Table 9. The values for $f^{-}(i,\omega)$

	A	B	C
AB	4	4	–
AC	3	–	3
BC	–	2	2

Table 10. The values for $f(i,\omega)$

	A	B	C
AB	2	2	–
AC	3/2	–	3/2
BC	–	1	1

Table 11. The values for $\sum\sum p_{ij}$

The values of the indices are given in Table 12 as well as the values of Banzhaf index β

	A	B	C
α_1	1/3	1/3	1/3
α_2	4/9	1/3	2/3

α_3	7/18	1/3	5/18
α_4	7/18	1/3	5/18
β	1/3	1/3	1/3

Table 12. Power indices for the preferences given in Table 2.

Example 2. Let N={*A, B, C, D, E*}, each agent has one vote, q=3 and the preferences of agents are given in Table 13.

P_A	P_B	P_C	P_D	P_E	rank
B	A	D	A	B	4
C	C	A	B	A	3
D	D	B	C	D	2
E	E	E	E	C	1

Table 13. Preferences of agents for N={*A, B, C, D, E*}.

The values for indices $\alpha_2(\cdot)$ and $\alpha_4(\cdot)$ are given in Table 14.

	A	B	C	D	E
$\alpha_2(\cdot)$	0.28	0.26	0.18	0.2	0.08
$\alpha_4(\cdot)$	0.218	0.230	0.190	0.200	0.140

Table 14.Values of α_2 and for α_4 Example 2.

4. Cardinal indices

Assume now that the desire of party i to coalesce with party j is given as real number p_{ij}, $0 \le p_{ij} \le 1$, $i, j = 1, \dots, n$. In general, it is not assumed that $p_{ij} = p_{ji}$.

We can call the value p_{ij} as an intensity of connection of i with j.

We define

a) average intensity of i is connection with other members of coalition ω

$$f^+(i,\omega) = \frac{\sum_{j\in\omega} p_{ij}}{|\omega|};$$

b) average intensity of connection of other members of coalition ω with i

$$f^-(i,\omega) = \frac{\sum_{j\in\omega} p_{ij}}{|\omega|};$$

c) average intensity for i

$$f(i,\omega) = \frac{1}{2}(f^+(i,\omega) + f^-(i,\omega));$$

d) average intensity in ω

$$f(\omega) = \frac{\sum_{i,j\in\omega} p_{ij}}{|\omega|}.$$

In contrast to ordinal case now we can introduce several new intensity functions:

e) minimal intensity of i's connections

$$f^+_{\min}(i,\omega) = \min_j p_{ij} \ ;$$

f) maximal intensity of i's connections

$$f^+_{\max}(i,\omega) = \max_j p_{ij} \ ;$$

g) maximal fluctuation of i's connections

$$f_{mf}(i,\omega) = \frac{1}{2}(\min_j p_{ij} + \max_j p_{ij})$$

h) minimal intensity of connections of other agents in ω with i

$$f^-_{\min}(i,\omega) = \min_j p_{ji} \ ;$$

i) maximal intensity of connections of other agents in ω with i

$$f^{-}_{\max}(i,\omega) = \max_{j} p_{ji}$$

j) s-mean intensity of i's connections with other agents in ω

$$f^{+}_{sm}(i,\omega) = \frac{1}{|\omega|}\sqrt[s]{\sum_{j} p_{ij}^{s}}\ ;$$

k) s-mean intensity of connections of other agents in ω with i

$$f^{+}_{sm}(i,\omega) = \frac{1}{|\omega|}\sqrt[s]{\sum_{j} p_{ji}^{s}}\ ;$$

l) max min intensity

$$f_{\max\min}(\omega) = \max_{i} \min_{j} p_{ij}\ ;$$

m) min max intensity

$$f_{\min\max}(\omega) = \min_{i} \max_{j} p_{ji}\ ;$$

n) maximal fluctuation

$$f_{mf}(\omega) = \frac{1}{2}(f_{\max\min}(\omega) + f_{\min\max}(\omega))$$

Note that the intensity functions in the cases d), l), m) and n) do not depend on agent herself but only on coalition ω .

Now the corresponding power indices can be define as above, i.e.

$$\alpha^{\text{card}}(i) = \frac{\sum\limits_{\substack{\omega \text{ is winning,}\\ i \text{ is pivotal in } \omega}} \chi_{i}}{\sum\limits_{j \in N} \sum\limits_{\substack{\omega\ ,\\ j \text{ is pivotal in } \omega}} \chi_{i}(\omega)},$$

where χ_i is one of the above intensity functions.

Example 3. Let $N=\{A,B,C,D\}$, each voter has only one vote, q=3, and the matrix $\|p_{ij}\|$ is given in Table 15. In table 16 the power indices are given for the cases a), b), e), h).

	A	B	C	D
A	–	0.8	0.2	0.1
B	0.4	–	0.7	0.2
C	0.1	0.9	–	0.5
D	0.7	0.2	0.1	–

Table 15. Matrix $\|p_{ij}\|$ for Example 3.

	A	B	C	D
$\alpha_{a)}$	0.23	0.26	0.30	0.20
$\alpha_{b)}$	0.25	0.39	0.20	0.16
$\alpha_{e)}$	0.17	0.35	0.30	0.17
$\alpha_{h)}$	0.23	0.46	0.15	0.15

Table 16. Some Cardinal indices for Example 3.

5. **Evaluation for Russian Parliament**

We have studied a distribution of power among factions in Russian Parliament (1999-2003) using these new indices.

The matrix $\|p_{ij}\|$ is constructed using the consistency index; the latter (the index of consistency of positions (of two groups) is constructed as

$$c(q_1,q_2)=1-\frac{|q_1-q_2|}{\max(q_1,1-q_1,q_2,1-q_2)}.$$

where q_1 and q_2 be the share of “ay” votes in two groups in MPs [1].

We consider the value of consistency index as the value of intensity of connections between agents *i* and *j*. Then we are in cardinal framework, and we can use one of the indices considered in the previous section.

On Fig. 1. the values of $\alpha_{a)}$ index are given for the Russian Parliament from 2000 to 2003 on the monthly basis.

It can be readily seen that index α gives lower values for Communist Party (sometimes up to 3%) and higher values for Edinstvo (up to 1%). It is interesting to note that Liberal-Democrats (Jirinovski’s Party) has almost equal values by both indices, which corresponds to the well-known flexibility of that party position.

Let us note that different ways to use the index α are possible. For instance, following the approach from [1], we may assume that if the consistency value for two factions is less some threshold value δ, then parties do not coalesce, i.e. $p_{ij}=0$.

5. Axioms for power indices

We introduce several axioms which any rensonable power index should satisfy to.

Axiom 1. Under a given quota rule for any agent $i \in N$ there exists an intensity profile $\vec{P}$ such that $\alpha(i) > 0$.

Axiom 2. Consider two voting situations $\{N, v, \vec{P}\}$ and $\{N, v', \vec{P}\}$ such that $\exists A \in N$ s.t. $v'(A) \geq v(A)$, and $\forall B \in N$, $B \neq A$, $v'(B) = v(B)$. Then $\alpha'(A) \geq \alpha(A)$.

Axiom 3. (Symmetry) Let η be a one-to-one correspondence of *N* to *N*. Then

$$\eta(\alpha_1, \ldots, \alpha_n) = (\alpha_{\eta(1)}, \ldots, \alpha_{\eta(n)})$$

Axiom 4. Let $i \in N$ be pivotal in no winning coalition ω. Then $\alpha(i) = 0$.

Axiom 5. Let P' be such that $\forall i \neq i_0$ $P_i^{'} = P_i$ and for i_0 $P_{i_0}^{'} A \subseteq P_{i_0} A$. Then $\alpha'(A) \geq \alpha(A)$.

Axiom 6. Let P' be an intensity matrix such that $p_{ij}^{'} = kp_{ij}$ for every $i, j = 1, \ldots, n$

Then $\alpha'(i) = \alpha(i)$, where α' is the power vector obtained from P'.

References

1. Aleskerov F., Blagoveschenskiy N., Satarov G., Sokolova A., Yakuba V. "Evaluation of power of groups and factions in the Russian parliament (1994-2003)", WP7/2003/01, Moscow: State University "High School of Economics", 2003 (in Russian)

2. Banzhaf, J. F., 1965, Weighted Voting Doesn' t Work: A Mathematical Analysis. Rutgers Law Review 19, 317-343.

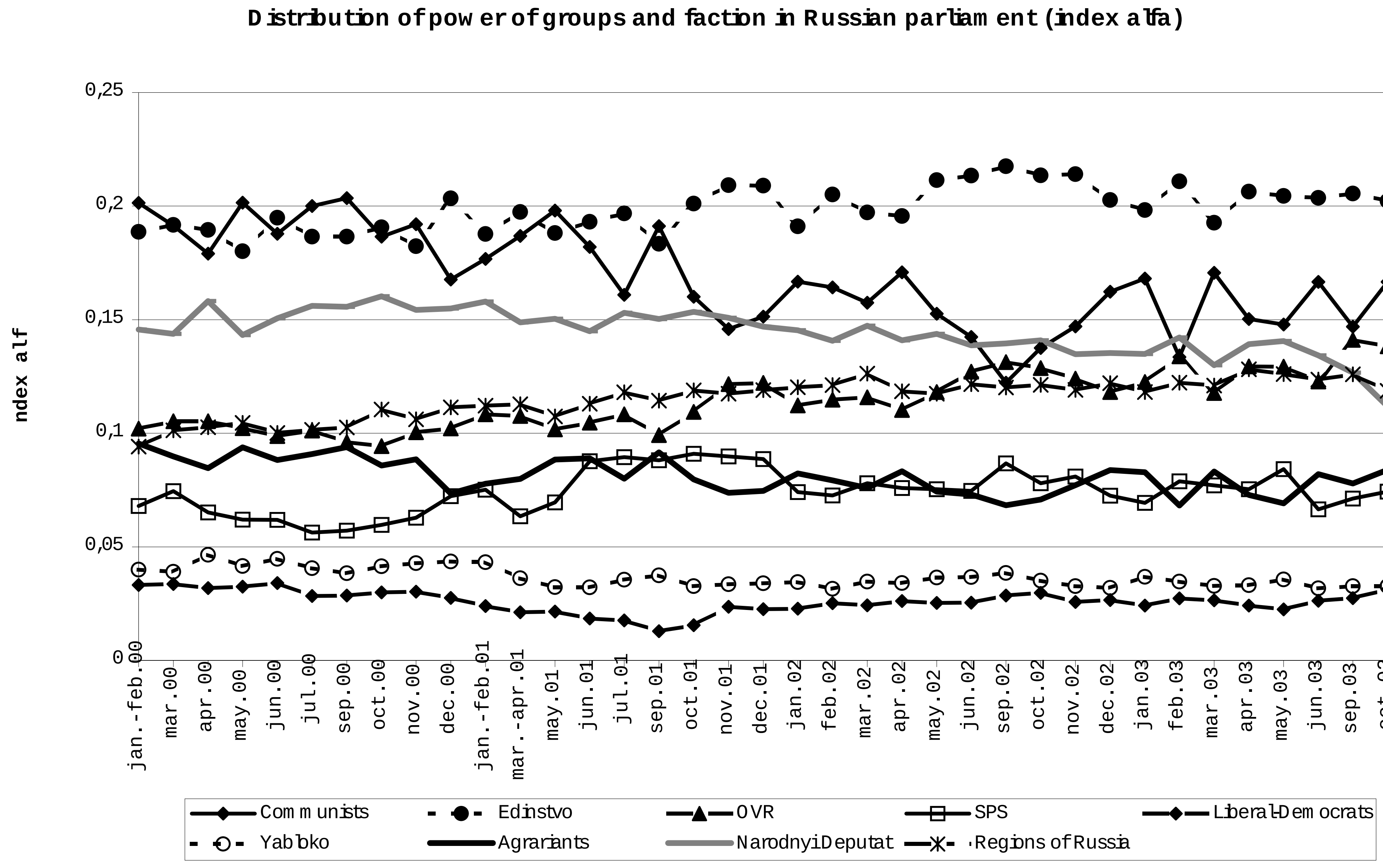
3. Felsenthal D., Machover M 1998 "The Measurment of Voting Power: Theory and Practices, Problems and Paradoxes", Edgar Elgar Publishing House.

4. Grofman, B. and H. Scarrow, 1979, Ianucci and Its Aftermath: The Application of Banzhaf index to Weighted Voting in the State of New York, Brams, S., Schotter A. апй G Schwodiauer (eds.) Applied Game Theory.

5. Heme, K. and H Nurmi, 1993, A Priori Distribution of Power in the EU Council of Ministers and the European Parliament, Scandinavian Journal of Po1ilitical Studies 16, 269-284

6. Leech, D., 2002, Voting power in the governance of the International Monetary Fund, Annals of Operations Research 109, 375-397.

7. Shapley, L.S., and M. Shubik, 1954, A method for Evaluting the Distribution of Power in a Committee System, American Political Science Review 48, 787-792.


Distribution of power of groups and faction in Russian parliament (index alfa)
ndex alf
0,25
0,2
0,15
0,1
0,05
0
jan.-feb.00
mar.00
apr.00
may.00
jun.00
jul.00
sep.00
oct.00
nov.00
dec.00
jan.-feb.01
mar.-apr.01
may.01
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oct.02
nov.02
dec.02
jan.03
feb.03
mar.03
apr.03
may.03
jun.03
sep.03
oct.03
nov.03
Communists
Edinstvo
OVR
SPS
Liberal-Democrats
Yablko
Agrariants
Narodnyi Deputat
Regions of Russia