NONCOOPERATIVE FOUNDATIONS OF BARGAINING POWER IN COMMITTEES*

By Annick Laruelle† and Federico Valenciano‡

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Abstract

In this paper we explore the non cooperative foundations of the bargaining power that a voting rule confers to its users in a 'bargaining committee'. That is, a committee that bargains in search of consensus over a set of feasible agreements under a voting rule. Assuming complete information, we model a variety of bargaining protocols whose stationary subgame perfect equilibria are investigated. It is also shown how previous results obtained by us from a cooperative approach, which provided axiomatic foundations for an interpretation of the Shapley-Shubik index and other power indices as measures of 'bargaining power' appear in this light as limit cases.

Key words: Bargaining in committees, voting power, bargaining power.

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†Departamento de Fundamentos del Análisis Económico, Universidad de Alicante, Campus de San Vicente, E-03071 Alicante; laruelle@merlin.fae.ua.es.

‡Departamento de Economía Aplicada IV, Universidad del País Vasco, Avenida Lehendakari Aguirre, 83, E-48015 Bilbao, Spain; elpvallf@bs.ehu.es.

1 INTRODUCTION

In Laruelle and Valenciano (2005b) we address the question of the influence of the voting rule used to settle agreements in a bargaining committee on the outcome of negotiations. By a 'bargaining committee' we mean a set of agents that bargain in search of consensus on a set of feasible alternatives 'in the shadow' of a voting rule. That is, if any agreement can be enforced if a 'winning coalition' supports it, what *general* agreements are likely to arise? This is a frequent real world situation, as it is often the case that a committee's capacity goes beyond that of the dichotomous acceptance/rejection of external proposals. In this case, when a committee has the capacity to adjust the proposal to gather consensus, the question of the influence of the voting rule in the final agreement arises. In search of an answer we adopted in that paper a cooperative axiomatic point of view à la Nash. That is, we addressed the issue as an extension of Nash's (1950) seminal bargaining question, model and answer. The model consisted of the basic ingredients that specify a bargaining committee: the set of feasible payoff vectors, as in Nash (1950), *plus* a second ingredient, the voting rule under which negotiations take place. And the answer was based on a set of reasonable conditions (or 'axioms') on the expectations of rational agents in such a situation.

Thus we provided axiomatic foundations for an interpretation of the Shapley-Shubik index and other 'power indices' as measures of the 'bargaining power' (in the precise game theoretic sense) that a voting rule confers to its users in a bargaining committee. In this setting, assuming the conditions that result from adapting those used by Nash (efficiency, anonymity, independence of irrelevant alternatives, and invariance with respect to positive transformations) and one from Shapley (1953) (null-player) a family of solutions is characterized. Namely, a 'solution' satisfies these conditions if and only if it is the weighted Nash bargaining solution[1] with weights (i.e., 'bargaining powers') given by an anonymous function of the voting rule that gives weight zero to null players. In formula, if B denotes the bargaining element and W the voting rule, the solution has the form:

$$\Phi(B,W) = Nash^{\varphi(W)}(B), \tag{1}$$

where $Nash^w$ denotes the w-weighted Nash bargaining solution. But the conditions on φ (anonymity and null player) are satisfied by many 'power indices'. In particular if the 'transfer' condition is also assumed[2] then the unique solution is given by

$$\Phi(B,W) = Nash^{Sh(W)}(B), \tag{2}$$

[1]Nonsymmetric or weighted Nash bargaining solutions were introduced by Kalai (1977).

[2]In Laruelle and Valenciano (2005c) we provide some foundation to this condition introduced by Dubey (1975).

where $Sh(W)$ denotes the Shapley-Shubik (1954) index of voting rule W.

In this paper we switch to the noncooperative approach according to the Nash program. As Binmore (2005) puts it: *"Cooperative game theory sometimes provides simple characterizations of* what *agreement rational players will reach, but we need noncooperative game theory to understand* why."[3] This is the point of this paper. That is, in this paper we explore the non cooperative foundations of these results. We use as primitives the same ingredients as in the previous model. In this setting, assuming complete information, we model a non cooperative bargaining process, or more precisely a variety of them, that provide non cooperative foundations for the above described results, all of them obtained from a cooperative approach, which appear in this light as limit cases. In this way the results are consistent with the ones obtained from the cooperative axiomatic approach, thus providing each others mutual support. In fact the results obtained extend Binmore's (1987) results connecting Nash (1950) and Rubinstein's (1982) alternating offers model, and differ from those of Hart and Mas-Colell (1996).

A close model in the literature is the one considered by Baron and Ferejohn (1989). They address the question of the division of one dollar in a legislature by majority rule when players are risk neutral (that is, the preference profile is transferable utility like). They consider a probabilistic protocol: a proposer is chosen at random (all players being equally probable), and proposes a division. If a majority accepts it, the game ends with this distribution. Otherwise the procedure recommences, but there is a discount factor. They show that any distribution can be sustained as a subgame perfect equilibrium. While in stationary subgame perfect equilibrium, without delay, the proposer proposes to $\frac{n-1}{2}$ players their discounted continuation value, 0 for the others, and the rest for him/herself. They also consider the case in which amendments are possible.

With respect to Baron and Ferejohn's (1989) model, the situation we model differs in a crucial feature. We consider the case of a committee (with non transferable utility profile) in search of (unanimous) consensus, thus avoiding a 'yes'/'no' premature final vote that, according to the voting rule under which negotiations take place, could leave rejecters outside the agreement. But if a unanimous agreement is desired, what is the relevance of the voting rule? The solution to this paradox lies in the choice of the proposer, or more precisely in the bargaining procedure. In our model the voting rule is determinant for the chances of each player playing the role of proposer. If the voting rule is not symmetric, players are not assumed to have the same probability of playing the role of proposer.

The basic idea for the bargaining 'protocols' we consider is the following: A player,

[3] See also the Introduction in Binmore, Osborne and Rubinstein's (1992) for a brief and clear discussion about the cooperative and noncooperative game theoretic shared goals from different approaches, whose philosophy we fully subscribe.

with the support of a 'winning' coalition to play the role of proposer, makes a proposal that if it is accepted by all players ends the negotiations. If some player rejects it, then with some probability negotiation ends in a failure (i.e., the *status quo*), otherwise a new proposer and a winning coalition supporting him/her are chosen. Thus the negotiating process ends either when a proposal is accepted or, if failure occurs, in the *status quo*. A variation in which with some probability, instead of total failure, only the rejecters are left out of the game is also considered. As uses to be the case with bargaining protocols, all is in the details. In this case all depends on the way the proposer and the supporting coalition are chosen. In particular this model accounts for the non-uniqueness of the answer provided by (1). Different specifications concerning this point yield different outcomes. A stationary subgame perfect equilibrium is proved to exist for a variety of protocols, which in the limit, when the probability of failure tends to zero, approaches a solution within the family given by (1). As to the particular case of (2) it appears as a special case with a sort of 'focal' appeal by virtue of the simplicity of the particular protocol associated that confers it some normative value as a term of reference.

The rest of the paper is organized as follows. In section 2 we introduce some basic notation and the basic ingredients in a model of a bargaining committee. In section 3 we briefly review the results in Laruelle and Valenciano (2005b) based on a cooperative axiomatic approach. In section 4 we provide a non cooperative model of a bargaining committee and obtain the main results. Section 5 examines some related work. Finally section 6 contains some concluding remarks.

2 A BARGAINING COMMITTEE

By a *bargaining committee* we mean a committee of n members or *players* that bargain in search of consensus 'in the shadow of a voting rule' in the following sense. They can reach any alternative within a specified set as well as any lottery over them, as long as a winning coalition (according to the specifications of the voting rule) supports it, and no player is imposed upon an agreement worse than the *status quo* (where they will remain if no winning coalition supports any agreement). It is assumed that every player has expected utility (von Neumann and Morgenstern, 1944) (vNM) preferences over this set of lotteries. Thus the situation can be summarized by two elements: the set of feasible utility vectors or *payoffs*, and the voting rule to enforce agreements.

Let the set $N = \{1, ..., n\}$ label the n members of the committee, and let $D \subseteq R^N$ be the set of feasible payoffs $D \subseteq R^n$, and $d \in D$ the disagreement or *status quo* payoff. Thus the pair $B = (D, d)$ summarizes the situation concerning the players' decision. We denote by ∂D the boundary of D. We assume that D is a *closed, convex, comprehensive* (i.e.,

$x \leq y \in D \Rightarrow x \in D)$[4] set containing d, such that there exists some $x \in D$ s.t. $x > d$, and $D_d := \{x \in D : x \geq d\}$ is *bounded* and *nonlevel* (i.e., $\forall x, y \in \partial D \cap D_d$, $x \geq y \Rightarrow x = y$). For any $S \subseteq N$ $(S \neq \emptyset)$, let $B^S = (D^S, d^S)$ denote the S-person bargaining problem in which $D^S = pr_S(D)$ and $d^S = pr_S(d)$, where pr_S denotes the S-projection $pr_S : R^N \rightarrow R^S$. Note the pair $B = (D, d)$ is (the only ingredient of) a classical n-person bargaining problem, to which here we will refer to as the *preference profile* of the committee to emphasize the relevance of the committee members' preferences in the model, in this case represented via utility functions. $\mathcal{B}$ denotes the set of all such pairs. For any permutation $\pi : N \rightarrow N$, $\pi B := (\pi(D), \pi(d))$ denotes the preference profile that results from B by π-permutation of its coordinates, so that for any $x \in R^N$, $\pi(x)$ denotes the vector in R^N s.t. $\pi(x)_{\pi(i)} = x_i$. A bargaining problem B is *symmetric* if $\pi B = B$, for any permutation π.

The second element in the model is the *voting rule*. The set N labels also the *seats*, so that each $i \in N$ labels a seat and the member occupying it. As only yes/no voting is considered, a *vote configuration* can be represented by the set of 'yes'-voters. Thus, depending on the context any $S \subseteq N$ will be referred to either as a vote configuration (of seats) or as a coalition (of players), and s will denote S's cardinal. An N-voting rule is specified by a set $W \subseteq 2^N$ of *winning* (i.e., which would lead to passing a decision) vote configurations such that (i) $N \in W$; (ii) $\emptyset \notin W$; (iii) If $S \in W$, then $T \in W$ for any T containing S; and (iv) If $S \in W$ then $N \backslash S \notin W$. $\mathcal{W}$ denotes the set of all such N-voting rules. For voting rule W, $M(W)$ denotes the set of *minimal winning* configurations, i.e., those that do not contain any other winning configuration. For any $S \in M(W)$ $(S \neq N)$, W_S^* denotes the voting rule $W_S^* := W \setminus \{S\}$. For any permutation $\pi : N \rightarrow N$, πW denotes the voting rule $\pi W := \{\pi(S) : S \in W\}$. A voting rule W is *symmetric* if $\pi W = W$, for any permutation π. We will also speak of *winning* (or *minimal winning*) *coalitions* for a given N-rule, with an obvious meaning. A player $i \in N$ is said to be *decisive* in a coalition $S \in W$ if $S \setminus i \notin W$. A seat/player $i \in N$ is said to be a *null seat/player* in W if, for any S, $S \in W$ if and only if $S \setminus \{i\} \in W$ (in other words the player occupying seat i will never be decisive).

Thus, the whole situation is summarized by a pair (B, W), where $B = (D, d)$ is a classical n-person bargaining problem that represents the configuration of preferences in the committee over the feasible agreements, and W is the N-voting rule to enforce agreements. Thus, any pair $(B, W) \in \mathcal{B} \times \mathcal{W}$ will be referred to as an *N-bargaining committee.*

Classical bargaining problems and simple transferable utility (TU) games can be seen as particular cases of this model[5]. The n-person classical bargaining problem corresponds

[4] We will write for any $x, y \in R^N$, $x \leq y$ $(x < y)$ if $x_i \leq y_i$ $(x_i < y_i)$ for all $i = 1, .., n$.

[5] As shown in Laruelle and Valenciano (2005b), $\mathcal{B} \times \mathcal{W}$ is isomorphic to a subclass of NTU models, so that these equivalences are exact because this subclass contains classical bargaining problems and simple

to the case of a committee bargaining *under the unanimity rule* (i.e., $W = \{N\}$). While when the bargaining element B is the TU-bargaining problem $\Lambda := (\Delta, 0)$, where $\Delta := \left\{x \in R^N : \sum_{i \in N} x_i \leq 1\right\}$, the pair (Λ, W) is equivalent to the *simple* TU game associated with the voting rule, v_W, given by

$$v_W(S) := \begin{cases} 1 & \text{if } S \in W, \\ 0 & \text{if } S \notin W. \end{cases}$$

3 A COOPERATIVE SOLUTION FOR A BARGAINING COMMITTEE

In this section we briefly review the axiomatic cooperative game theoretic results in Laruelle and Valenciano (2005b).

3.1 Rationality conditions

In (2005b) we impose the following conditions on a map $\Phi : \mathcal{B} \times \mathcal{W} \rightarrow R^N$, for vector $\Phi(B, W) \in R^N$ to be considered as a reasonable expectation of utility levels of a *general* agreement in a bargaining committee (B, W). We impose as prerequisites: $\Phi(B, W) \in D$ (feasibility), and $\Phi(B, W) \geq d$ (individual rationality), if $B = (D, d)$. In addition to this we require:

1. *Efficiency (Eff)*: For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, there is no $x \in D$, s.t. $x > \Phi(B, W)$.

2. *Anonymity (An)*: For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, and any permutation $\pi{:}N \rightarrow N$, and any $i \in N$, $\Phi_{\pi(i)}(\pi(B, W)) = \Phi_i(B, W)$, where $\pi(B, W) := (\pi B, \pi W)$.

3. *Independence of irrelevant alternatives (IIA)*: Let $B, B' \in \mathcal{B}$, with $B = (D, d)$ and $B' = (D', d')$, such that $d' = d$, $D' \subseteq D$ and $\Phi(B, W) \in D'$, then $\Phi(B', W) = \Phi(B, W)$, for any $W \in \mathcal{W}$.

4. *Invariance w.r.t. positive affine transformations (IAT):* For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, and all $\alpha \in R^N_{++}$ and $\beta \in R^N$,

$$\Phi(\alpha * B + \beta, W) = \alpha * \Phi(B, W) + \beta,$$

where $\alpha * B + \beta = (\alpha * D + \beta, \alpha * d + \beta)$, denoting $\alpha * x := (\alpha_1 x_1, .., \alpha_n x_n)$, and $\alpha * D + \beta := \{\alpha * x + \beta : x \in D\}$.

5. *Null player* (*NP*): For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, if $i \in N$ is a null player in W, then $\Phi_i(B, W) = d_i$.

The readers can see for themselves the precise correspondence of axioms 1 to 5 with some of Nash's (1950) and Shapley's (1953) axioms.

superadditive games.

3.2 Characterizations

Denote by $Nash(B)$ the Nash (1950) (pure) bargaining solution of an n-person bargaining problem $B = (D, d)$, that is,

$$Nash(B) = \arg\max_{x \in D_d} \prod_{i \in N} (x_i - d_i), \tag{3}$$

and by $Nash^w(B)$ the w-weighted asymmetric Nash bargaining solution (Kalai, 1977) of the same problem for a vector of nonnegative weights $w = (w_i)_{i \in N}$, given by

$$Nash^w(B) = \arg\max_{x \in D_d} \prod_{i \in N} (x_i - d_i)^{w_i}. \tag{4}$$

Nash (1950) characterized (3) as the unique (pure) bargaining solution (i.e., map $\mathcal{B} \to R^N$) satisfying the conditions of 'efficiency', 'anonymity', 'invariance w.r.t. positive affine transformations' and 'independence of irrelevant alternatives'[6]. While Kalai's (1977) solutions emerge basically by dropping anonymity in Nash system.

The following theorem generalizes Nash's characterization.

Theorem 1 *(Laruelle and Valenciano, 2005b) Let $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$ be a solution that satisfies Eff, An, IIA, IAT and NP, then*

$$\Phi(B, W) = Nash^{\varphi(W)}(B), \tag{5}$$

where $\varphi : \mathcal{W} \to R^N$ satisfies efficiency, anonymity and null player[7].

Therefore (4), any map $\varphi : \mathcal{W} \to R^N$ that satisfies efficiency, anonymity and null player would fit into formula (5) and yield a solution $\Phi(B, W)$ that satisfies the five rationality conditions. A solution is singled out by adding to these conditions the adaptation to this setting of the condition by means of which Dubey (1975) characterized the Shapley value on the domain of simple games (i.e., the Shapley-Shubik (1954) index). This condition can be equivalently stated as follows (Laruelle and Valenciano, 2001):

6. *Transfer (T):* For any two rules $W, W' \in \mathcal{W}$, and all $S \in M(W) \cap M(W')$ $(S \neq N)$:

$$\Phi(\Lambda, W) - \Phi(\Lambda, W_S^*) = \Phi(\Lambda, W') - \Phi(\Lambda, W_S'^*). \tag{6}$$

[6] Be aware these conditions in Nash's setting, unlike their homonymous in the previous subsection, refer to a map $\mathcal{B} \to R^N$.

[7] In Laruelle and Valenciano (2005b) we did not assume nonlevelness of D_d. This forced us to assume there a stronger version of *NP*, in which *only* null players have null expectations, and to impose in (4) the disagreement payoff for players whose weight is zero. Thus this is an alternative version of the result proved there that can be proved assuming, as we do here, nonlevelness of D_d.

Denote by $Sh(v)$ the Shapley (1953) value of a TU game v, given by

$$Sh_i(v) = \sum_{S:S\subseteq N} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S\backslash i)),$$

and by $Sh(W)$ the Shapley-Shubik (1954) index of a voting rule W, i.e., the Shapley value of the associated simple game v_W. We have the following result.

Theorem 2 *(Laruelle and Valenciano, 2005b) There exists a unique solution/value* $\Phi : \mathcal{B} \times \mathcal{W} \rightarrow R^N$ *that satisfies Eff, An, IIA, IAT, NP and T, and it is given by*

$$\Phi(B, W) = Nash^{Sh(W)}(B). \tag{7}$$

Note that $\Phi(B, W)$ as given by (7) becomes Nash's classical solution of B when W is the unanimity rule, and it becomes the Shapley-Shubik index of v_W when $B = \Lambda$. In other words, Theorem 2 integrates Nash's and Shapley-Dubey's (Shapley (1953), Dubey (1975)) characterizations into one, which extends both and yields a cooperative 'solution' to the problem of bargaining under a voting rule given by (7).

4 A NON COOPERATIVE MODEL OF A BARGAINING COMMITTEE

Formulae (5) and (7) have a clear interpretation. The asymmetric Nash solutions can be justified as reflecting the different 'bargaining power' of the players *"determined by the strategic advantages conferred on players by the circumstances under which they bargain"* (Binmore, 1998, p. 78). In the case of a bargaining committee the voting rule, possibly nonsymmetric, is the only possible source of differences in 'strategic advantages' that can be justified endogenously within the model. Then, according to formulae (5) and (7), under the rationality conditions assumed in either case, either vector $\varphi(W)$ or $Sh(W)$ gives the 'bargaining power' that the voting rule confers to each member of the committee.

Now the point is to provide some noncooperative justification or interpretation for these formulae. A noncooperative analysis requires to specify the 'game' within the black-box summarily described as "a set of agents bargaining in search of consensus 'in the shadow' of a voting rule". Consensus requires the acceptance of all the players of a proposal. This involves the specification of a procedure of acceptance/rejection of proposals, which entails two basic issues: (i) How the proposer is chosen?, and (ii) How to deal with disagreements? To this end we will specify several bargaining protocols for a committee with a given preference profile and a given voting rule, and investigate the resulting stationary subgame perfect equilibria.

To proceed systematically we consider first in 4.1 two bargaining protocols strictly probabilistic, that is, with no voting rule entering the model, in order to examine the effect of the likelihood of being the proposer and the effect of the way of dealing with disagreement. Then in 4.2 consider several ways of deriving a protocol from the voting rule.

4.1 Probabilistic bargaining protocols

The following strictly probabilistic protocol for a committee with a given preference profile $B = (D, d)$ will play an important role in the sequel.

Protocol 1: A proposer $i \in N$ is chosen with probability p_i (s.t., $\sum_{i \in N} p_i = 1$) and makes a (feasible) proposal $x \in D_d$.

(i) If all the players accept it the game ends with payoffs x.

(ii) If any player does not accept it:

-with probability r $(0 < r < 1)$ the process recommences,

-with probability $1 - r$ the game ends in failure or 'breakdown' and payoffs d.

We have the following result for this protocol or family on them, one for each probability distribution p, that will play a central role in the sequel.

Theorem 3 *Let an N-person committee with preference profile $B = (D, d)$ satisfying the conditions specified in section 2. (i) Under protocol 1 there exists a stationary subgame perfect equilibrium (SSPE). (ii) In the limit when $r \to 1$, the SSPE payoffs tend to the w-weighted Nash bargaining solution of B with weights given by $w_i = p_i$.*

Proof: (i) A stationary strategy profile should specify for every player i the proposal that s/he will make whenever s/he is chosen to be the proposer, and what proposals will s/he accept from others. A proposal of i can be specified by a vector $(y_i, (x^i_j)_{j \in N \setminus i}) \in D_d$, where y_i is the payoff i will propose for him/herself, and x^i_j the payoff i will propose for $j \neq i$. Acceptance and refusal by i of a proposal of another player should depend only on the utility s/he receives. This can be specified by the minimal level of utility for which s/he will accept it. In *SSP* equilibrium every player should be proposed at least what s/he expects if s/he refuses. Thus (w.l.o.g. we assume $d = 0$, and consistently we write in the sequel D_0) it should be, for all i and all $j \neq i$,

$$x^i_j \geq (1 - r)0 + r p_j y_j + r \sum_{k \in N \setminus j} p_k x^k_j. \qquad (8)$$

As the proposer will seek the biggest payoff compatible with this condition, by D_0's non levelness, we can assume equality. And note that the right hand side of the equation does not depend on i. Thus we can drop the superindex in x^i_j and x^k_j, and rewrite the above condition as an equation:

$$x_j = rp_j y_j + r \sum_{k \in N \setminus j} p_k x_j = rp_j y_j + r(1 - p_j)x_j,$$

that can be rewritten for all j as

$$rp_j y_j = (1 - r + rp_j)x_j. \tag{9}$$

Note that if $p_j = 0$ then $x_j = 0$, while if $p_j \neq 0$ (9) can be rewritten

$$y_j = \theta_j x_j \quad (\text{where} \quad \theta_j := \frac{1 - r + rp_j}{rp_j} \; (> 1)). \tag{10}$$

Observe that in this case (i.e., if $p_j \neq 0$) $y_j > x_j$. That is, being the proposer is desirable. In fact the proposer would make the best of this advantage maximizing his/her payoff under the constraint of feasibility, that is, for all j

$$y_j = max \left\{ y \in R : (x_{\hat{j}}, y) \in D \right\}, \tag{11}$$

where $(x_{\hat{j}}, y)$ denotes the point whose j-coordinate is y and all others coordinates equal to those of x (this maximum exists by D_d compactness). As players with probability 0 of being the proposer will receive 0 according to equation (9), we can constrain our attention to those players with a positive probability of being the proposer. To simplify the notation instead of dealing with this subset as $N' = \{i_1, ..i_{n'}\} \subseteq N$, we just take for the sequel $N' = N$. We have then a system with $2n$ equations ((10) and (11)) with $2n$ unknown (($x_1, .., x_n$) and ($y_1, .., y_n$)) specifying a stationary strategy profile: Each j whenever chosen as proposer will propose y_j for him/herself and x_i for each $i \neq j$, and accept only proposals that give him/her at least x_j. That a stationary strategy profile satisfying these equations is a subgame perfect equilibrium is immediate. The problem is to prove that a solution for this system exists. If B is the normalized TU bargaining problem $\Lambda = (\Delta, 0)$, equation (11) becomes

$$y_j = 1 - \sum_{k \in N \setminus j} x_k,$$

so that a linear system results that, as can be easily proved, yields as unique solution: $x_j = rp_j$, and $y_j = 1 - r + rp_j$. But in the general case in order to prove the existence of solution of the system given by (10) and (11) we will need a fix point argument. For each $i \in N$, let $\tau_i : D_0 \to R$ be the (well-defined, by D_0 compactness) map

$$\tau_i(x) := max \left\{ y \in R : (x_{\hat{j}}, y) \in D \right\}.$$

In this notation $x = (x_1, .., x_n)$ and $\tau(x) = (\tau_1(x), .., \tau_n(x))$ provide a solution of the system if and only if for each i it holds $\tau_i(x) = \theta_i x_i$. In other terms, denoting by μ^i, for each $i \in N$, the map $\mu^i : D_0 \to D_0$ defined by

$$\mu^i_j(x) := \begin{cases} x_j, & \text{if } j \neq i, \\ \frac{\tau_i(x)}{\theta_i} & \text{if } j = i, \end{cases}$$

we are done if we prove the existence of some $x \in D_0$ such that $\mu^i(x) = x$ for all i, for in this case $(x_1, .., x_n)$ and $y = (\tau_1(x), .., \tau_n(x))$ provide a solution of the system. To this end we will define a continuous map $\varphi : D_0 \to D_0$ a fix point of which satisfies this condition. Let φ be the map given by

$$\varphi(x) := \frac{1}{2}x + \frac{1}{2}\sum_{i \in N} \frac{1}{n}\mu^i(x).$$

Claim 1: $\varphi(x) = x$ if and only if $\mu^i(x) = x$ for all i. In effect, as x and $\mu^i(x)$ may only differ on their i-coordinate, we have

$$(\varphi(x) = x) \Leftrightarrow (x = \frac{1}{2}x + \frac{1}{2}\sum_{i \in N} \frac{1}{n}\mu^i(x)) \Leftrightarrow (x = \sum_{i \in N} \frac{1}{n}\mu^i(x))$$

$$\Leftrightarrow (x_i = \frac{n-1}{n}x_i + \frac{1}{n}\mu^i_i(x),\ \forall i \in N) \Leftrightarrow (x_i = \mu^i_i(x),\ \forall i \in N)$$

$$\Leftrightarrow (x = \mu^i(x),\ \forall i \in N).$$

Thus there only remains to prove the following.

Claim 2: $\varphi : D_0 \to D_0$ is a well-defined continuous map. Note first that, by D_0 compactness and convexity, τ_i is well-defined and continuous. Thus $\mu^i(x)$ is also well-defined and continuous, and, by D comprehensiveness and as $\theta_i > 1$, belongs to D_0. Thus finally $\varphi(x)$ is also well-defined and continuous, and, by D_0 convexity, belongs to D_0.

Thus Brower's fix point theorem ensures the existence of a fix point of $\varphi : D_0 \to D_0$, which provides the desired *SSPE*.

(ii) Now let $r(0 < r < 1)$, and denote by $x(r)$ the point in D_0 such that $x(r) = (x_1(r), .., x_n(r))$ and $\tau(x(r)) := (\tau_1(x(r)), .., \tau_n(x(r)))$ provide a solution of the system, whose existence have been proved in (ii), specifying a *SSPE*. Denote by $\tau^i(x(r))$ the point in $\partial D \cap D_0$ (the case $n = 2$ is represented in Fig.1) given by

$$\tau^i_j(x(r)) := \begin{cases} x_j(r), & \text{if } j \neq i, \\ \tau_i(x(r)) & \text{if } j = i. \end{cases}$$

Claim 3: The corresponding *SSPE* payoffs for r are given by $\frac{1}{r}x(r)$. The (expected *ex ante*) *SSPE* payoffs are given by $\sum_{i \in N} p_i \tau^i(x(r))$. And observe that for each coordinate i (denoting by pr_i the i-projection) we have

$$pr_i(\sum_{j \in N} p_j \tau^j(x(r))) = \sum_{j \neq i} p_j x_i(r) + p_i \tau_i(x(r)) = (1 - p_i)x_i(r) + p_i \theta_i x_i(r)$$

$$= (1 - p_i)x_i(r) + p_i \frac{1 - r + rp_i}{rp_i} x_i(r) = \frac{1}{r} x_i(r),$$

so the claim is proved.

Claim 4: $\frac{1}{r}x(r) = Nash^p(H(r), 0)$, where the ('hyperplane') bargaining problem whose feasible set's boundary is the hyperplane containing the n points $\tau^1(x(r)), .., \tau^n(x(r))$. This is in fact an immediate consequence of the equality proved in the previous claim.

Claim 5: $lim_{r \to 1} \frac{1}{r} x(r) = Nash^p(B)$. As for all i

$$lim_{r \to 1} \theta_i(r) = lim_{r \to 1} \frac{1 - r + rp_i}{rp_i} = 1, \qquad (12)$$

and $\tau_i(x(r)) = \theta_i(r)x_i(r)$, then when $r \to 1$, $\|x(r) - \tau(x(r))\| \to 0$. Therefore the distances of the n points $\tau^1(x(r)), .., \tau^n(x(r))$ that determine the hyperplane-boundary of $H(r)$ to $x(r)$ tend to zero. Thus in the limit a supporting hyperplane of D_0 contains $lim_{r \to 1} \frac{1}{r} x(r)$ and this point is the p-convex combination of the intersections of each axis with this hyperplane. In other words, this point is $Nash^p(B)$.■

Remarks 1:

(i) Protocol 1 and Theorem 3 have a clear interpretation. In Protocol 1, r represents the patience or readiness of the committee to look for consensus. The bigger r the smaller the risk of breakdown, and the bigger the readiness to continue bargaining in search of consensus after a disagreement.

(ii) According to Theorem 3 the relative advantage of the proposer diminishes as r increases. Namely, according to (12), $\theta_i(r) \to 1$ when $r \to 1$, where recall $\theta_i(r)$ is the proportion between player i's expected payoff when s/he is the proposer and when the proposer is someone else. Nevertheless the probability of being the proposer has a determinant impact on the expected *SSPE* payoffs, which approach $Nash^p(B)$ as $r \to 1$.

(iii) Note that the *SSPE* payoffs for each r, given by $\sum_{i\in N} p_i\tau^i(x(r))$, are not 'efficient' in general (i.e., they are in the interior of D, though the bigger the r the closer to ∂D), for they are the p-weighted average of n points in ∂D, namely, the continuation *SSPE* payoffs after the choice of a proposer corresponding to the n different possible proposers. On the contrary, as for every proposer the continuation *SSPE* payoffs after the choice of a proposer are in ∂D, in the case of B being the TU-bargaining problem $\Lambda = (\Delta, 0)$ the *SSPE* payoffs are 'efficient' (i.e., they are in ∂D) and *the same for every* r, and given by (Claim 4) $Nash^p(H(r), 0) = Nash^p(\Lambda) = p$.

(iv) It is implicit in the model that players with a null probability of being the proposer accept their irrelevant role, and do not hinder agreement. That is, they accept a zero payoff, the same that they would receive if they reject.

In Protocol 1 it may seem somewhat implausible that both the players accepting the proposal and those refusing it face the same risk of failure (disagreement payoff). We consider the following alternative protocol.

Protocol 2: At every stage there is a set of 'active players' $S \subseteq N$ with preference profile $B^S = (D^S, d^S)$. Initially this set is N. For each $S \subseteq N$, let $(p_i^S)_{i\in S}$, s.t., $\sum_{i\in S} p_i^S = 1$, specify a random procedure to choose a proposer from S. As a condition of consistency we assume that if $i \in T \subseteq S$, then: $p_i^S = 0 \Rightarrow p_i^T = 0$. At a stage in which the set of players is S the game proceeds as follows:

(i) A proposer $i \in S$ is chosen with probability p_i^S and makes a (feasible) proposal $x_S \in D^S_{d^S}$.

(ii) If all players in S accept it the game ends with payoffs $(x_S, d_{N\setminus S})$.

(iii) If any player in S does not accept it and $T \subseteq S$ is the set of 'rejecters':

-with probability r $(0 < r < 1)$ the process recommences with S as the set of active players,

-with probability $1 - r$ players in T drop out and the process recommences with $S \setminus T$ as the set of active players.

So now there is no risk of failure in the sense of Protocol 1. Only rejecters risk being left out of the game. Nevertheless the situation is basically almost the same as with Protocol 1 for r sufficiently close to one.

Theorem 4 *Let an N-person committee with preference profile $B = (D, d)$ satisfying the conditions specified in section 2. For all $\epsilon > 0$: (i) Under protocol 2 and for r sufficiently close to 1 there exists a stationary subgame perfect ϵ-equilibrium (ϵ-SSPE). (ii) In the limit when $r \to 1$, the ϵ-SSPE payoffs tend to the weighted Nash bargaining solution of B with weights given by $w_i = p_i^N$.*

Proof: (i) Now a stationary strategy profile should specify for every player i and any possible set S of active players containing i, the proposal that s/he will make if s/he is chosen from S to be the proposer, and what proposals will s/he accept from other proposers in S. A proposal (w.l.o.g. we assume $d = 0$) of i when S is the set of active players, can be specified by a vector $(y_i^S, (x_j^{S,i})_{j\in S\setminus i}) \in D_{0^S}^S$, where y_i^S is the payoff i will propose for him/herself, and $x_j^{S,i}$ the payoff i will propose for $j \in S\setminus i$. In *SSP* equilibrium every player should be proposed at least what s/he expects if s/he refuses. Thus it should be for all i, all S containing i, and all $j \in S \setminus i$,

$$x_j^{S,i} \geq (1-r)0 + rp_j^S y_j^S + r \sum_{k\in N\setminus j} p_k^S x_j^{S,k}. \tag{13}$$

And $y_j^S = max\left\{y \in R : (x_{\bar{j}}^S, y) \in D_{0^S}^S\right\}$. Thus we have a system entirely similar to the one given by (9) and (11), but now *one for each possible subset of active players.* Thus by similar steps as in the proof of Theorem 3 we can conclude that all of them have a solution that together define a stationary strategy profile. But now these conditions are only *necessary* for it to be a *SSPE*. In general they are not sufficient. In fact in this strategy profile a strategy may not be optimal against the others' in the same profile. The reason is that now a proposer may find it advantageous to make a proposal that offers some players less that they would accept and forcing their rejection and their risk of being dropped out if this probability is big enough. More precisely, assume S is the set of active players and j is the proposer. S/he would be interested in deviating from the specified strategy by causing a rejection by a subset of players $S \setminus T$ ($j \in T$) if j's expected payoffs from following it are smaller that the continuation payoffs after provoking $S\setminus T$'s rejection. This will not be so if

$$y_j^S(r) \geq (1-r)\frac{1}{r}x_j^T(r) + r\frac{1}{r}x_j^S(r),$$

or equivalently

$$y_j^S(r) - x_j^S(r) \geq \frac{1-r}{r}x_j^T(r).$$

Which, in view of $y_j^S(r) = \frac{1-r+p_j^S}{rp_j^S}x_j^S(r)$, is equivalent to

$$x_j^S(r) \geq p_j^S x_j^T(r).$$

But this condition is not guaranteed in general. Nevertheless, for any $\epsilon > 0$, by similar steps inequation

$$y_j^S(r) \geq (1-r)\frac{1}{r}x_j^T(r) + r\frac{1}{r}x_j^S(r) - \epsilon,$$

is equivalent to

$$x_j^S(r) \geq p_j^S x_j^T(r) - \frac{r}{1-r}\epsilon,$$

which for r sufficiently close to one holds for all S, all T, and all j, s.t. $j \in T \subseteq S$. Then an induction argument (for $s = 1$ is trivially true) shows that the resulting stationary strategy profile is an ϵ-*SSPE*.

(ii) As in this ϵ-*SSPE* (for r sufficiently close to 1) proposals are immediately accepted in the first round, the payoffs are determined as in the case of Protocol 1 by (9) and (11). Thus for $r \rightarrow 1$ we have the same result in the limit as for Protocol 1.■

At first sight it may seem surprising the little difference that this protocol makes with respect to the first one in terms of *SSPE*, but on closer examination the explanation is clear. In Protocol 2 only rejecters risk being left out of the game, but condition (13) is entirely similar to (8), although now there is one for each i and each coalition S containing i. As in ϵ-*SSPE* proposals are immediately accepted in the first round, only the probabilities of being a proposer when all players are active do matter for r big enough. In other words, when the probability of dropping out rejecters is sufficiently small general consensus is secured. In this respect, the limit results for either protocol are consistent with the intuition in real world situations, where tension between negotiating parties or lack of patience in a negotiation makes consensus difficult.

4.2 Bargaining protocols under a voting rule

No voting rule enters either of the two precedent protocols, both entirely specified in probabilistic terms. In fact, Theorems 3 and 4 can be seen as providing noncooperative foundations to Kalai's (1977) nonsymmetric Nash bargaining solutions. On the other hand, confronting the results given by Theorems 3 and 4 with formula (5) suggests a way of bridging these results belonging to different realms, the noncooperative and cooperative approaches. The basic idea is linking the probabilities of being the proposer, which is the source of bargaining power in protocols 1 and 2, with the voting rule, which is the only element of the bargaining environment in a bargaining committee included in the model. In principle there are infinite imaginable protocols in which the voting rule plays a role. In other words, there are infinite ways of mapping voting rules into probability distributions over players. Thus the question is whether there are some specially simple and reasonable protocols based on the voting rule within the plethora of possibilities consistent with formulae (5) and (7).

A general principle that seems reasonable is the following: Assume that in order to play the role of proposer the support of a winning coalition s/he belongs to is needed. In order to consider in full generality ways of going from voting rules to probabilities respecting this principle we consider maps: $P : \mathcal{W} \rightarrow \mathcal{P}_{N\times 2^N}$, where $\mathcal{P}_{N\times 2^N}$ denotes the set of probability distributions over $N \times 2^N$, and use the notation p_W to denote $P(W)$. If we want $p_W : N \times 2^N \rightarrow [0,1]$ to respect the stated principle it should be required

$$(p_W(i,S) \neq 0) \;\Rightarrow\; (i \in S \in W).$$

As we are interested in protocols that yield (5), the map $P : W \longmapsto p_W$ should satisfy some conditions. First, if the null player principle is to be satisfied it must be

$$(p_W(i,S) \neq 0) \;\Rightarrow\; (i \in S \in W \;\&\; S \setminus i \notin W). \tag{14}$$

That is, the proposer has to be decisive in the coalition that supports him/her. In order to preserve the principle of anonymity it must be required that for any permutation π,

$$p_W(i,S) = p_{\pi W}(\pi i, \pi S). \tag{15}$$

Then any $p : \mathcal{W} \rightarrow \mathcal{P}_{N \times 2^N}$ satisfying (14) and (15) 'abstracts' a protocol determined by the voting rule in a bargaining committee which gives probabilities of being the proposer given by

$$p_i^W := \sum_{S : i \in S} p_W(i,S).$$

Any such protocol will yield in the limit (in the sense of Theorems 3 and 4) a particular case of (5). But, as has been stated, any map satisfying these conditions just 'abstracts' a protocol, and we are interested in the explicit protocols, not in their abstract summary by a p.

Still a great variety of protocols are compatible with the above conditions. We consider first two general and relatively simple ways of selecting a player i to play the role of proposer and a winning coalition S containing him/her such that $i \in S \in W$, and $S \setminus i \notin W$. 1 ((i-S)-protocols): Choose first i, then choose S; and 2 ((S-i)-protocols): Choose first S, then choose i.

It may seem at first sight more natural that the formation of a coalition in support of a proposer should precede the choice of the proposer. What seems to advocate for (S-i)-protocols. But this entails a coalition formation process just encapsulated in a black-box-like probability distribution, which seems alien to noncooperative modelling. In fact from a noncooperative point of view (i-S)-protocols seem more natural than (S-i)-protocols. They can be interpreted in terms of players taking the initiative in search of a coalition that supports him/her as proposer, and a probability distribution over coalitions describing their likelihood of getting it. Nevertheless both generate the same outcomes as we will see. Let p denote a probability distribution over coalitions described by a map $p : 2^N \rightarrow [0,1]$ that, in order to be consistent with the anonymity assumption, assigns the same probability to all coalitions of the same size. In other words $p(S)$ depends only on s. Thus we will write p_s instead of $p(S)$.

(i-S)**-Protocols** (Choose first i, then choose S): Assume a given probability distribution over coalitions p, and the following protocol: Choose at random a player i. Choose a coalition S containing i according to p. If $S \in W$ and $S \setminus i \notin W$, player i is the proposer, otherwise recommence until a proposer is chosen.

The probability of player i being the proposer after the first two steps is given by

$$\frac{1}{n} Prob_p \ (i \text{ is decisive in } S \mid S \ni i) = \frac{1}{n} \frac{\sum\limits_{\substack{S:i \in S \in W \\ S \setminus i \notin W}} p_s}{Prob_p \ (S \text{ contains } i)}. \tag{16}$$

But in general after the first two steps no proposer is chosen. This occurs only if i happens to be decisive in the chosen S containing i (i.e., $S \in W$ and $S \setminus i \notin W$). The probability of player i being the proposer is a conditional probability, which is the result of dividing (16) by the probability of a proposer being chosen. As shown in Laruelle and Valenciano (2002), *Prob* (i is decisive$|$ $i \in S$) generates for different p's (s.t., $p(S)$ depends on S's size) all semivalues[8]. In other words, (i-S)-protocols, for p's such that the probability of a coalition depends on its size, generates all $(p_i^W)_{i \in N}$ such that this vector is the normalization of a semivalue. In particular this includes two particular distinguished cases.

Shapley-Shubik index: If $p_s = \frac{\frac{1}{s}}{\sum\limits_{t=1}^{n} \frac{1}{t}} \frac{1}{\binom{n}{s}}$ (for any $S \neq \emptyset$), then the probability of player i being the proposer under (i-S)-protocol is given by the Shapley-Shubik index (1954) index of the voting rule, that is, $p_i^W = Sh_i(W)$. Note the way of choosing a coalition underlying this protocol: a coalition's size (from 1 to n) is chosen *with probability inversely proportional to the size*, then one coalition of size s is chosen at random, all of them being equally probable. Note the difference with other probabilistic models of the Shapley-Shubik index. In Laruelle and Valenciano (2005a, 2002) this 'strange' probability distribution emerges in two completely different contexts, where probability distributions over vote configurations (or coalitions) represent either a probabilistic 'voting behavior' or a random coalition in a TU coalitional game. But note that in the present context it makes sense to weight coalitions inversely to their size. It seems easier to gather the support of a few than that of many.

Banzhaf normalized: If $p_s = \frac{1}{2^n}$, then the probability of player i being the proposer under (i-S)-protocol is given by the normalized Banzhaf (1965) index of the voting rule.

(S-i)**-Protocols** (Choose first S, then choose i): Assume a given probability distribution over coalitions p, and the following protocol: Choose a coalition S according to p. Choose at random a player i in S. If $S \in W$ and $S \setminus i \notin W$, player i is the proposer, otherwise recommence until a proposer is chosen.

[8]Semivalues were introduced by Weber (1979) (see also Dubey, Neyman and Weber (1981)).

The probability of player i being the proposer after the first two steps is given by

$$\sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} \frac{1}{s} p_s = \sum_{S: i \in S} \frac{1}{s} p_s (v_W(S) - v_W(S \setminus i)), \tag{17}$$

but in general possibly no player is chosen as proposer after one single round. Nevertheless the actual probabilities of being proposer after applying an (S-i)-protocol are proportional to the probabilities given by (17), which yields in a different way the family of normalized semivalues. But in this protocol a same p gives a different semivalue than for (i-S)-protocols, or the other way round a same semivalue is associated with different probabilities over coalitions in either protocol. In particular the two best known semivalues result for the following probabilities.

Shapley-Shubik index: If $p_s = \frac{1}{n+1} \frac{1}{\binom{n}{s}}$, then the probability of player i being the proposer (S-i)-protocol is given by the Shapley-Shubik index (1954) index of the voting rule, that is, $p_i^W = Sh_i(W)$. Thus in terms of (S-i)-protocol the Shapley-Shubik index emerges for the more familiar probabilistic model of choosing a size at random, and then a coalition of that size at random.

Banzhaf normalized: If $p_s = k \frac{s}{2^n}$, where k is a constant resulting from normalization, then the probability of player i being the proposer under (S-i)-protocol is given by the normalized Banzhaf (1965) index of the voting rule. Note that in this protocol the probability of a coalition is weighted by its size[9].

But more general than the two previous types of protocol is the following one of which both can be seen as particular cases:
(i, S)**-Protocols** (Choose simultaneously i and S): As was already pointed out any $W \mapsto p_W$ such that $p_W(i, S)$ satisfies (14) and (15) abstracts a protocol that yields in the limit a particular case of (5). This includes as particular cases also some power indices less familiar than Shapley-Shubik's and Banzhaf's and outside the family of normalized semivalues, as Deegan-Packel's (1978), Holler-Packel's (1983) or even Johnston's (1978) index.

But again the Shapley-Shubik index emerges associated with a very simple selection procedure:

Shapley-Shubik's Protocol (formulation 1): (i) Choose an order in N at random, and let the players join a coalition in this order up to a winning coalition S is formed. (ii) Then the last player entering S is the proposer.

Under this protocol:

$$Prob\ (i \text{ is the proposer}) = Sh_i(W).$$

[9] Note that if $p_s = \frac{1}{2^n}$ for all S, then the probability of a player being chosen the proposer is *not* the normalized Banzhaf index of that player.

It is worth remarking the simplicity of this procedure within the family of protocols described above. First, the formation of the coalition appears in this case as a sequential process, which seems at once natural and the simplest. Alternatively and equivalently it can be described as at every step choosing at random one player to join the coalition. As to the choice of the 'swinger' as proposer one may wonder why not any of the other players decisive in S. But it does not make any difference is the second step is replaced by this: Choose at random one of the players decisive in S. It can be easily seen that the procedure is equivalent. Thus the protocol can be alternatively specified like this:

Shapley-Shubik's Protocol (formulation 2): (i) Starting from the empty coalition, choose one player at random every time from the remaining players up to a winning coalition S is formed. (ii) Then choose at random one of the players decisive in S.

Thus, *under this protocol each player as a probability of being the proposer equal to his/her Shapley-Shubik index for the current voting rule.*

In sum combining Theorem 3 with the above discussion we have the following results which are the noncooperative counterpart of Theorems 1 and 2:

Theorem 5 *Let an N-person committee with preference profile $B = (D, d)$ satisfying the conditions specified in section 2, bargaining under voting rule W. (i) For all r $(0 < r < 1)$, under any $(i$-$S)$-protocol, any $(S$-$i)$-protocol, and any (i, S)-protocol, there exists a stationary subgame perfect equilibrium (SSPE). (ii) In the limit when $r \to 1$, the SSPE payoffs tend to the weighted Nash bargaining solution of B with weights given by the probabilities of being proposer determined by W and the protocol.*

In (ii), when the protocol is either a $(S$-$i)$-protocol or a $(S$-$i)$-protocol, the weights are given by a normalized semivalue of the voting rule (i.e., of the simple TU game v_W). In particular, for the Shapley-Shubik protocol we have:

Theorem 6 *Let an N-person committee with preference profile $B = (D, d)$ satisfying the conditions specified in section 2, bargaining under voting rule W. (i) For all r $(0 < r < 1)$, under the Shapley-Shubik protocol there exists a stationary subgame perfect equilibrium (SSPE). (ii) In the limit when $r \to 1$, the SSPE payoffs tend to the weighted Nash bargaining solution of B with weights given by the Shapley-Shubik of the voting rule W. (iii) If $B = \Lambda$, for all r $(0 < r < 1)$, the SSPE payoffs are given by the Shapley-Shubik index of the voting rule W.*

Similar results, replacing *SSPE* by *ϵ-SSPE,* are obtained from Theorem 4 if the protocols in Theorems 5 and 6 are modified in the sense that the probability of breakdown is replaced by a probability of the rejecters leaving the game whenever a rejection occurs. In this case, it is necessary to specify the way of choosing the proposer (i.e., the probabilities for each of the remaining players) when any group of players has abandoned the game.

Remarks 2:

(i) In the light of this bargaining model the voters' 'power' becomes 'bargaining power' in the specific game-theoretic sense, so clarifying the old conceptual ambiguity concerning the game theoretic notion of 'value' when applied to simple games representing voting rules, and its alternative interpretation as 'decisiveness', or likelihood of playing a crucial role in a decision. In a bargaining committee[10], according to this model, *the source of power is the likelihood of being the proposer, related to the likelihood of being decisive via the protocol.*

(ii) By (iii) of Theorem 6, in the case of a committee with a TU preference profile, i.e., if $B = \Lambda$, the ex ante *SSPE* payoffs are given by the Shapley-Shubik index of the voting rule W, whatever r $(0 < r < 1)$. Thus the limit result for $r \to 1$ is trivial in this case. But observe that the noncooperative "implementation" of the Shapley-Shubik index of the voting rule W (or equivalently, of the Shapley value of the associated simple game v_W) is different from previous ones. In this model $Sh(W)$ represents an expectation in a precise sense, in which no player (unless the rule is a dictatorship) has a chance of getting the whole cake, although the proposer would benefit (the less so the bigger r) from this role. Observe also that when $r \to 0$, in the limit the proposer will have the whole cake, though the ex ante expectations are the same. In other words, for $r \to 0$, in the limit we have a reinterpretation of original Shapley's model applied to the simple game associated to the voting rule.

(iii) This is the noncooperative counterpart of the fact pointed out in Laruelle and Valenciano (2005b) that $Nash^{Sh(W)}(B)$ yields $Sh(W)$ when $B = \Lambda$. In cooperative terms it was emphasized there that this particular case is not the relevant point, but the fact that $Sh(W)$ appears 'up there' *for all* B with a new meaning: the bargaining power that the voting rule confers to the players. Now in this noncooperative model this interpretation is corroborated and clarified: this is so (in the limit for $r \to 1$) for a specific and particularly simple protocol.

5 RELATED WORK

For the two-person case the only voting rule satisfying the conditions assumed is the unanimity rule, and consequently we are back to the classical two-person bargaining problem

[10] On the contrary, as argued in Laruelle, Martínez and Valenciano (2005), in a 'take-it-or-leave-it' committee, where the relevant notion is that of success, decisiveness seems secondary.

considered by Nash (1950). In this case the protocol and result in Theorem 5 becomes very similar to Rubinstein's (1982) alternating offers model and result, and extends Binmore (1987), paralleling the way in which Theorem 1 yields Nash's characterizing result in this case. While for the particular case in which the 'preference profile' of an n-player committee is TU-like, Theorem 6 establishes that whatever the probability of breakdown the *SSPE* payoffs are given by the Shapley-Shubik index of the voting rule, paralleling the way in which Theorem 2 yields in this case Shapley-Dubey's characterization of Shapley-Shubik index.

Hart and Mas-Colell's (1996) noncooperative bargaining model in the context of NTU games yields in the limit the Maschler-Owen's (1992) consistent value, and therefore as particular cases the Nash bargaining solution and the Shapley value[11]. But their setting is different and in their model rejections are dealt with by eliminating the proposer with some probability. As the class of situations considered here is isomorphic to a subclass of NTU games (see Laruelle and Valenciano (2005b)), their bargaining procedure can be applied to our case, but it gives a different result. In fact they propose the same protocol for all NTU games. Thus when applied to the NTU game associated (equivalent in fact as it encodes exactly the same information in a different way) to a bargaining committee (B, W) their protocol, unlike ours, is independent of the voting rule W.

There are also several noncooperative models related to the Shapley value. Gul's (1989) model of bilateral meetings in a transferable utility economy yields the Shapley value as a limit case. Pérez-Castrillo and Wettstein (2001) propose a bidding mechanism that implements the Shapley value. But our setting and endeavour here is different, for we are not interested in 'implementing' formula (1) or (2) in the sense of providing a noncooperative game that 'decentralizes' a desired outcome[12].

In the political context, apart from the seminal Baron and Ferejohn (1989) mentioned in the introduction, there are several papers in its wake worth mentioning. See for instance Harrington (1990), Merlo and Wilson (1995, 1998), Banks and Duggan (2000), Eraslan and Merlo (2002), Eraslan (2002), Snyder, Ting and Ansolabehere (2004), Montero (2005).

6 CONCLUSIONS

We have provided noncooperative foundation to the results obtained in Laruelle and Valenciano (2005b) from a cooperative approach: Theorems 5 and 6 give noncooperative support to formulae (1) and (2), to which axiomatic support was previously given. Often

[11] See also Vidal-Puga (2005a, 2005b).

[12] Nevertheless, as a by-product (Remark 2 (ii)), the particularization of Theorem 6 to this case provides a very simple implementation of the Shapley value of the TU game associated to a voting rule.

the noncooperative approach is vindicated as superior to the cooperative one because it incorporates strategic considerations in richer models and provides what can be seen as positive predictions. While cooperative 'solutions', often interpreted in normative terms, are seen with suspicion by many. But we see no conflict here, but rather two different approaches in search of a same goal (Binmore, Osborne and Rubinstein (1992), Binmore (2005)). This view is consistent with the results in the case under consideration, where both approaches become complementary, and provide two alternative and mutually supporting and consistent points of view for modeling a specific situation. In this case the complexity of the situation, a committee bargaining in search of consensus or unanimous agreement 'in the shadow of a voting rule,' makes any 'prediction' problematic and dependent on the bargaining protocol. We have studied a class of specially simple protocols determined by the voting rule used by the committee. But it would be naive to interpret the results obtained as crisp positive 'predictions'. In other words, no credible prediction can be expected without adding some more relevant detail in the model. Of course the results can be stated as a 'conditional prediction': If (i) players are rational in the sense of von Neumann-Morgenstern model; (ii) the whole situation, including the players preferences, is transparent to all players (i.e., information is complete); (iii) the protocol is respected; and (iv) a *SSPE* is considered a good guide for action, then Theorems 5 and 6 predict something. But the more 'if's', i.e., the more honest and precise the statement, the more dubious sounds its predicting power. Nevertheless the conclusions based on these models and results, as shown in the previous section, are consistent with the most important bargaining models (Rubinstein (1982), Binmore (1987)), and basically consistent also with the intuition gained from the observation of real world situations. In sum, if Theorems 1 and 2 from Laruelle and Valenciano (2005b) support (5) and (7) axiomatically, yielding as particular cases Nash bargaining solution (when W is is the unanimity rule) and the Shapley-Shubik index (when $B = \Lambda$, but also for any B in formula (7)), now, in exact consistency, Theorems 5 and 6 support non cooperatively (5) and (7), and yield also as particular limit cases Nash bargaining solution and the Shapley-Shubik index.

The variety of protocols compatible with (5) corroborate the 'indeterminacy' of the problem from a strictly positive point of view, as argued in Laruelle and Valenciano (2005b). But this enhances the simplicity of the protocol that yields the Shapley-Shubik index in (7) as a limit case, conferring it a 'focal appeal' as a reference term. Nevertheless, despite our calling it "Shapley-Shubik's protocol" this is surely not the unique simple protocol that yields this result. It has been shown how different protocols may yield the same outcome in the limit, and this one in particular.

It is also worth stressing the clarification of the old conceptual ambiguity concerning the game theoretic notion of 'value' when applied to simple games representing voting rules, and its alternative interpretation as the likelihood of playing a crucial role in a decision, since Shapley-Shubik (1954). In the bargaining model provided here: (i) The voters' 'power' becomes 'bargaining power' in the precise game-theoretic sense of the term; (ii) The source of such power is the likelihood of being the proposer, related to the likelihood of being decisive via the protocol; and (iii) When the preference profile is TU, these bargaining powers coincide with the payoff vector, that is, the 'value' of the game.

In particular these results provide additional support (to the one obtained from the cooperative approach) to the normative assessment of the 'bargaining power' that a voting rule confers to its users in a bargaining committee as given by formulae (5) and (7), an issue of consequence when the choice of the voting rule is at stake[13].

References

[1] Banks, J. S., and J. Duggan (2000): "A bargaining model of Collective Choice," *American Political Science Review* **94**, 73-88.

[2] Banzhaf, J. F. (1965): "Weighted Voting doesn't Work: A Mathematical Analysis," *Rutgers Law Review* **19**, 317-343.

[3] Baron, D.P., and J. A. Ferejohn (1989): "Bargaining in Legislatures," *American Political Science Review* **83**, 1181-1206.

[4] Bergantiños, G, B. Casas-Méndez, M.G. Fiestras-Janeiro, and J. J. Vidal-Puga (2005): "A Focal-Point Solution for Bargaining Problems with Coalition Structure," Mimeo.

[5] Binmore, K. (1987): "Perfect Equilibria in Bargaining Models" in *The Economics of Bargaining,* edited by K. Binmore and P. Dasgupta, Oxford: Blackwel, 77-105.

[6] Binmore, K. (1998): *Game Theory and the Social Contract II, Just Playing.* Cambridge: MIT Press.

[7] Binmore, K. (2005): *Playing for Real* (to be published by Oxford University Press).

[8] Binmore, K., M. J. Osborne and A. Rubinstein (1992): Noncooperative models of bargaining, inIn: *Handbook of Game Theory with Economic Applications* (Vol. 1), Aumann, R. J., and S. Hart, eds, Elsevier Science Publishers, B. V.,179-225.

[13] In Laruelle and Valenciano (2004) we use the results obtained from the cooperative approach to address this issue.

[9] Deegan, J., and E. W. Packel (1978): "A New Index of Power for Simple n-Person Games," *International Journal of Game Theory* **7**, 113-123.

[10] Dubey, P. (1975): "On the Uniqueness of the Shapley Value," *International Journal of Game Theory* **4**, 131-139.

[11] Dubey, P., A. Neyman, and R. J. Weber (1981): "Value Theory without Efficiency," *Mathematics of Operations Research* **6**, 122-128.

[12] Eraslan, H. (2002): "Uniqueness of Stationary Equilibrium Payoffs in the Baron-Ferejohn Model," *Journal of Economic Theory* **103***, 11-30.*

[13] Eraslan, H., and A. Merlo (2002): "Majority rule in a Stochastic Model of Bargaining," *Journal of Economic Theory* **103**, 31-48.

[14] Gul, F. (1989): "Bargaining Foundations of the Shapley Value," *Econometrica* **57**, 81-95.

[15] Hart, S. and A. Mas-Colell (1996): "Bargaining and Value," *Econometrica* **64**, 357-380.

[16] Holler, M. J., and E. W. Packel (1983): "Power, Luck and the Right Index," *Journal of Economics* **43**, 21-29.

[17] Johnston, R. J. (1978): "On the Measurement of Power: Some Reactions to Laver," *Environment and Planning A* **10**, 907-914.

[18] Kalai, E. (1977): "Nonsymmetric Nash Solutions and Replications of 2-person Bargaining," *International Journal of Game Theory* **6**, 129-133.

[19] Laruelle, A., R. Martínez, and F. Valenciano (2005): "Success versus decisivenss in voting situations," forthcoming in *Journal of Theoretical Politics.*

[20] Laruelle, A., and F. Valenciano (2001): "Shapley-Shubik and Banzhaf Indices Revisited," *Mathematics of Operations Research* **26**, 89-104.

[21] Laruelle, A., and F. Valenciano (2002): "Potential, Value and Probability," *Discussion Paper* **27/2002**, Departamento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.

[22] Laruelle, A., and F. Valenciano (2004): "Bargaining, in committees of representatives: the optimal voting rule," Discussion Paper 45/2004, Departamento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.

[23] Laruelle, A., and F. Valenciano (2005a): "Assessing success and decisiveness in voting situations," *Social Choice and Welfare*, **24**, 171-197.

[24] Laruelle, A., and F. Valenciano (2005b): "Bargaining in committees as an extension of Nash's bargaining theory," forthcoming in *Journal of Economic Theory.*

[25] Laruelle, A., and F. Valenciano (2005c): "Cooperative bargaining foundations of the Shapley-Shubik index," mimeo.

[26] Maschler, M., and G. Owen (1992): "The consistent Shapley value for games without side payments," In: *Rational Interaction*, R. Selten, ed., New York: Springer-Verlag, 5-12.

[27] Merlo, A., and C. Wilson (1995): "A stochastic model of sequential bargaining with complete information," *Econometrica* **63**, 371-399.

[28] Merlo, A., and C. Wilson (1998): "Efficient delays in a stochastic model of bargaining," *Economic Theory* **11**, 39-55.

[29] Montero, M. (2005): "Noncooperative bargaining foundations of the nucleolus in majority games," forthcoming in *Games and Economic Behavior.*

[30] Nash, J. F. (1950): "The Bargaining Problem," *Econometrica* **18**, 155-162.

[31] Pérez-Castrillo, D. and D. Wettstein (2001): "Bidding for the Surplus: A Non-cooperative Approach to the Shapley Value," *Journal of Economic Theory* 100, 274-294.

[32] Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica* **50**, 97-109.

[33] Shapley, L. S. (1953): "A Value for N-person Games," *Annals of Mathematical Studies* **28**, 307-317.

[34] Shapley, L. S., and M. Shubik (1954): "A Method for Evaluating the Distribution of Power in a Committee System," *American Political Science Review* **48**, 787-792.

[35] Snyder, J. M., M. M Ting, and S. Ansolabehere (2004): "Legislative bargaining under weighted voting", mimeo.

[36] Vidal-Puga, J. (2005a): "A bargaining approach to the Owen value and the Nash solution with coalition structure," *Economic Theory* **25**(3), 679-701.

[37] Vidal-Puga, J. (2005b): "The Harsanyi paradox and the 'right to talk' in bargaining among coalitions," mimeo.

[38] von Neumann, J. and O. Morgenstern (1944): 1947, 1953, *Theory of Games and Economic Behavior.* Princeton: Princeton University Press.

[39] Weber, R. J. (1979): "Subjectivity in the Valuation of Games." In: *Game Theory and Related Topics*, Moeschlin, O. and D. Pallaschke, eds, North Holland Publishing Co, Amsterdam, 129-136.