On the Nucleolus as a Power Index

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Abstract

This paper argues that the nucleolus can compete with the Shapley value as a measure of P-power. It currently has more solid non-cooperative foundations for majority games. It also identifies a set of attractive coalitions that are expected to form (unlike the Shapley value, which is based on the values of all coalitions) and does better than the Shapley value at some postulates of voting power. On the negative side, it may give a payoff of zero to players that are not dummies, though this possibility is excluded for constant-sum weighted majority games.

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1 Introduction

Consider the classical problem of dividing a dollar by majority rule. There are n members in the voting body and a voting rule characterized by a set of winning coalitions. How powerful is each member of this voting body? A measure of power is the expected share of the budget for each player. This concept of power is what Felsenthal and Machover (1998) call P-power. They point out that the outcome of the bargaining process will not generally be deterministic[1], and the index of P-power will be an average of the possible outcomes, weighted by their probability.

In this context, the most widely accepted measure of power is the Shapley value, introduced by Shapley in 1953 as a measure of the expected payoff from playing a general cooperative game. The Shapley value can be seen as a weighted average of several possible outcomes (the simplest possibility is Shapley's original story of players entering randomly into a room and receiving their marginal contribution to the value of the existing group). Roth (1977a, 1977b) and Laruelle and Valenciano (2004) give further axiomatic support to the Shapley value as a measure of power in divide-the-dollar games.

In contrast, the nucleolus (introduced by Schmeidler in 1969, see also Maschler 1992) answers a different type of question: what is the most stable way of dividing the dollar between the n players? Unlike the Shapley value, the nucleolus seems to presuppose the formation of the grand coalition and therefore seems inappropriate as a measure of P-power. The nucleolus is the most stable outcome given that the grand coalition forms, not the average of

[1] Consider the simple majority game with three players. If there was a unique deterministic outcome, symmetry points to the grand coalition with every player receiving $\frac{1}{3}$. However, this outcome seems too fragile. If we accept that a two-player coalition will eventually form, symmetry dictates that each of the three possible coalitions will be equally likely.

several possible outcomes which may not involve the grand coalition at all. Accordingly the nucleolus is not even mentioned by Felsenthal and Machover (1998), though it appears in Pajala's (2002) literature list.

Nevertheless, this paper will discuss an alternative interpretation of the nucleolus and argue that it can compete with the Shapley value as a measure of P-power. The most solid reason lies in the field of noncooperative foundations, but the paper will also discuss some general properties of the Shapley value and the nucleolus.

The nucleolus will find support in the Baron-Ferejohn (1989) model. This is not a model introduced in order to provide noncooperative foundations to any particular solution concept, and it is fairly popular with political scientists.[2] The Shapley value finds little or no support in this model. The paper will also discuss why the existing literature on noncooperative foundations of the Shapley value is either not applicable or not fully convincing for majority games.

The question arises of whether the nucleolus is a good power index if we abstract from the bargaining process. A possible (and solid) reason why the nucleolus has been ignored as a power index is that it may assign zero to players who are not dummies; this seems counterintuitive and indeed Felsenthal and Machover (1998) include "vanishing only for dummies" as one of the postulates any power index must obey. On the other hand, vanishing only for dummies is incompatible with core selection. Moreover, this property of the nucleolus can be justified if we interpret the nucleolus as a system of competitive prices. Finally, the nucleolus does better than the Shapley value with respect to the added blocker postulate.

The remainder of the paper is organized as follows. Section 2 contains some preliminaries on majority games and the nucleolus. Section 3 provides

[2] A search in the Social Sciences Citation Index reveals that the paper has been cited 161 times; about half of those citations appeared in political science journals.

an alternative interpretation of the nucleolus as a system of competitive prices for the players. Section 4 illustrates how those competitive prices can arise as the equilibrium of a natural modification of the Baron-Ferejohn model. Section 5 discusses some properties of the nucleolus as a power index, and section 6 concludes.

2 Preliminaries

2.1 Majority games

Let $N = \{1, ..., n\}$ be the set of players. $S \subseteq N$ ($S \neq \varnothing$) represents a generic coalition of players, and $v : 2^n \to \mathbb{R}$ with $v(\varnothing) = 0$ is a function that assigns to each coalition the total payoff its members can divide among themselves. The function v is called *characteristic function.* In the context of *majority* (also called *simple*) games, $v(S) \in \{0, 1\}$ for all $S \subseteq N$, $v(\varnothing) = 0$, and $v(N) = 1$. A coalition S is called *winning* iff $v(S) = 1$ and *losing* iff $v(S) = 0$. It is called *minimal winning* iff $v(S) = 1$ and $v(T) = 0$ for all T such that $T \subset S$. The set of winning coalitions is denoted by W; this set contains the same information as the function v. The set of minimal winning coalitions is denoted by W^m. A player i such that $v(S \cup i) = v(S)$ for all S is called a *dummy player.* A player who belongs to all winning coalitions is called a *veto player* or a *blocker.*

A simple game is *proper* iff $v(S) = 1$ implies $v(T) = 0$ for all $T \subset N \backslash S$. It is *constant-sum* iff $v(S) + v(N \backslash S) = 1$ for all $S \subseteq N$. It is a *weighted majority game* iff there exist n nonnegative numbers (weights) $w_1, ..., w_n$ and a nonnegative number q such that $v(S) = 1$ if and only if $\sum_{i \in S} w_i := w(S) \geq q$. We will denote a weighted majority game by $[q; w_1, ..., w_n]$. The pair $[q; w]$ is called a *representation* of the game v. A weighted majority game has many possible representations, but not all of them are equally convenient.

A representation w is called *normalized* iff $\sum_{i \in N} w_i = 1$; it is *homogeneous* iff $\sum_{i \in S} w_i = q$ for all $S \in W^m$. Not all weighted majority games admit a homogeneous representation. A weighted majority game admitting a homogeneous representation is called a *homogeneous game*.[3]

2.2 The nucleolus

Let (N, v) be a majority game and $x = (x_1, ..., x_n)$ be an *imputation*, that is, a payoff vector with $x_i \geq v(i)$ and $x(N) = v(N)$. For any coalition S the value $e(S, x) = v(S) - x(S)$ is called the *excess* of S at x.

For any imputation x let $S_1, ..., S_{2^{|N|}-1}$ be an ordering of the coalitions for which $e(S_l, x) \geq e(S_{l+1}, x)$ for all $l = 1, ..., 2^{|N|} - 1$ and let $E(x)$ be the vector of excesses defined by $E_l(x) = e(S_l, x)$ for all $l = 1, ..., 2^{|N|} - 1$. We say that $E(x)$ is *lexicographically less* than $E(y)$ if $E_l(x) < E_l(y)$ for the smallest l for which $E_l(x) \neq E_l(y)$. The *nucleolus* is the set of imputations x for which the vector $E(x)$ is lexicographically minimal. Schmeidler (1969) shows that the nucleolus consists of a unique imputation. It is contained in the classical bargaining set (Davis and Maschler, 1967) and in the kernel (Davis and Maschler, 1965).

The excess of coalition S at x is a measure of how dissatisfied coalition S is with imputation x. We can think of the excess as a measure of how likely S would be to depart from the grand coalition. The nucleolus minimizes the maximum excess, and thus is one of the (possibly many) solutions of the

[3]Homogeneous representations are preferable because they give a more accurate description of the situation. For example, games $[5; 4, 3, 2]$ and $[2; 1, 1, 1]$ are identical in terms of the characteristic function, but only the second representation is homogeneous. This representation reflects the fact that all players are in a symmetric position.

following linear programming problem[4]

$$\begin{aligned} & \min \ e \\ \text{s.t. } x(S)+e &\geq 1 \text{ for all } S \in \mathbf{W} \\ x(N) &= 1 \\ x_i &\geq 0 \text{ for all } i \in N. \end{aligned}$$

An important property of the nucleolus follows from the fact that it is a solution to this linear programming problem. To present this property, we need some definitions.

For every majority game v and every payoff vector x, let $b_1(x,v)$ be the set of those $S \subseteq N$ for which $\max\{v(S)-x(S) : S \subseteq N\}$ is attained and $b_0(x) = \{\{i\} : x_i = 0\}$.

Let $\mathcal{C}$ be a collection of nonempty subsets of N. We say that the collection is *balanced* iff there exist strictly positive numbers $(\lambda_S)_{S\in\mathcal{C}}$ such that, for each $i \in N$, $\sum_{S\ni i} \lambda_S = 1$. The numbers $(\lambda_S)_{S\in\mathcal{C}}$ receive the name of *balancing weights.*

Proposition 1 *(Kohlberg, 1971) Let v be a majority game. If x is the nucleolus of v, then $b_0(x) \cup b_1(x,v)$ is balanced.*

This property will play an important role in the next two sections.

3 The nucleolus as a competitive system of prices

The important property of the balancing weights $(\lambda_S)_{S\in b_1(x.v)}$ is not that they add up to 1 for each player, but that they add up to the same constant

[4]This will be the case even if there is a player i with $v(i) = 1$.

for each player who gets a positive payoff. We can change this constant by rescaling the weights to obtain another set of weights λ'_S. In particular, suppose we rescale the weights in such a way that $\sum_{S\in b_1(x,v)} \lambda'_S = 1$. The weights $(\lambda'_S)_{S\in b_1(x,v)}$ can be interpreted as the probabilities of coalition S forming (cf. Albers 1974 p. 5). Then each player with $x_i > 0$ will be in the final coalition with the same probability (which turns out to be precisely the total payoff x assigns to a coalition of maximum excess), and a player with $x_i = 0$ appears in the final coalition no more often than one with $x_i > 0$. This interpretation of the balancing weights doesn't seem widespread - indeed Albers dismisses it in the related context of balanced aspirations.

Why interpret the weights $(\lambda'_S)_{S\in b_1(x,v)}$ as the probabilities of each coalition forming? Suppose the imputation $x := (x_1, ..., x_n)$ is related to a system of prices players charge for their participation in a coalition. Which sets of prices are stable? We can make two assumptions with respect to what happens when a coalition forms:

1. A privileged player i (the proposer) selects a coalition S, pays each player $j \in S\backslash\{i\}$ the price x_j and pockets the residual, which will generally be higher than x_i. In this case player i will choose S to solve the following problem

$$\max_{\substack{S\in W\\ S\ni i}} 1 - \sum_{j\in S\backslash\{i\}} x_j$$

   This problem is equivalent to $\max_{S\in W, S\ni i} 1 - \sum_{j\in S} x_j$. In other words, given a price vector x player i always proposes a coalition of maximum excess containing him. Because he only pays the others x_j and keeps the whole excess, he wants to maximize that excess. The set $b_1(x, v)$ becomes prominent (though in general not all players will belong to one of the coalitions in $b_1(x, v)$, if x is the nucleolus all players with

$x_i > 0$ do). If x is the nucleolus it is reasonable to assume that only coalitions in $b_1(x,v)$ will form.

2. Alternatively, we can assume that no player is privileged and that if a coalition S forms, the players in S will divide the payoff proportionally to x. Again, coalitions in $b_1(x,v)$ emerge as the most profitable and it is reasonable to assume that they will form. This is because both the surplus above $\sum_{j \in S} x_j$ and the share of the surplus a player receives are maximized for coalitions of maximum excess. If x is the nucleolus, *ex post* payoff division will correspond to a balanced aspiration (see Cross (1967)).

In any of the two cases we will have identified the coalitions that may form given the system of prices. Each of these coalitions will form with a certain probability. If, *for any probabilities* we assign to those coalitions, there is a group of players that belongs to the final coalition more often than another group of players, the price system is not stable. There is an "excess demand" for some players and their price should rise at the expense of some players who belong to the final coalition less often. The only exception is the case in which the players who are less often in the final coalition are already getting 0. On the other hand, if we can assign probabilities to the coalitions so that all players who get a positive payoff are in the final coalition with equal probability, we have a competitive price system.

Let the system of prices coincide with the nucleolus. As we have seen, in any of the two cases above the coalitions that will form given these prices are the ones with maximum excess. In either case, since the set of coalitions of maximum excess is balanced, we can assign probabilities to them such that all players with positive payoff are in the coalition with the same probability, and players who are getting 0 are no more often than other players in the final coalition.

Example 1 $[5; 2, 2, 2, 1, 1, 1]$. *The nucleolus is* $(\frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$. *Given this, a large player will be indifferent between proposing a coalition of type* $[221]$ *and* $[2111]$. *A small player will also be indifferent between them. There are nine coalitions of the first type and three of the second type (twelve in total). A large player is in seven coalitions, and a small player is in six. Thus, if all coalitions were equally likely the large players would be in excess demand. However, we can assign probabilities to coalitions such that all players are in the final coalition with probability* $\frac{5}{9}$ *(the total nucleolus payoff of a coalition of maximum excess). If we assign* $\frac{2}{27}$ *to type* $[221]$ *and* $\frac{1}{9}$ *to* $[2111]$ *this will be the case.*

The Shapley value of this game is $(\frac{7}{30}, \frac{7}{30}, \frac{7}{30}, \frac{3}{30}, \frac{3}{30}\ \frac{3}{30})$. *Given these prices, all players will propose a coalition of type* $[2111]$. *A large player is in only one of those, whereas a small player is in three of those. Thus, regardless of what probabilities we assign to the coalitions each small player will be in the final coalition with probability* 1*; each large player will be (on average) with probability* $\frac{1}{3}$. *There is a sense in which the small players are more in demand and should raise their price.*

Example 2 $[5; 3, 2, 2, 1]$. *The nucleolus is* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. *Coalitions* $[32]$, $[221]$ *and others in which we add player* $[1]$ *to a minimal winning coalition would be optimal. We can find weights* ($\frac{1}{3}$ *for each minimal winning coalition) such that all players who get a positive payoff are in the final coalition with the same probability* ($\frac{2}{3}$), *and the player who gets a zero payoff is in the final coalition with a smaller probability* ($\frac{1}{3}$). *Even though player* 4 *is "providing something for nothing" when he enters a coalition, he is not in excess demand.*

The Shapley value is $(\frac{5}{12}, \frac{3}{12}, \frac{3}{12}, \frac{1}{12})$. *Given these prices we can give two interpretations. According to the first approach both types of coalitions* $[32]$ *and* $[221]$ *are possible (and then we cannot find probabilities such that every player is in the final coalition with equal probability) or, according to the* *second approach, only* [221] *is possible (and the same can be concluded). In neither case is the Shapley value a competitive price system.*

4 Noncooperative foundations

4.1 The Baron-Ferejohn model

Baron and Ferejohn's (1989) influential paper introduced a legislative bargaining game based on Rubinstein (1982) and Binmore (1987). In their paper n symmetric players must divide a budget by simple majority. Each player has an equal chance of being recognized to be the proposer; once a proposer is recognized he proposes a division of the budget. The rest of players then vote "yes" or "no"; if a majority of the players supports the proposal then it is implemented and the game ends; otherwise we come back to the previous situation in which nature chooses a proposer, each player being chosen with equal probability. Baron and Ferejohn focus on stationary subgame perfect equilibria. In a stationary equilibrium, strategies do not depend on any elements of the history of the game other than the current proposal, if any. It is important to emphasize that Baron and Ferejohn's model appeared in a political science journal; nothing seems to connect their paper with the field of noncooperative foundations.

In extending the model to general voting games we must choose whether to keep the recognition probabilities identical for all players, or to have asymmetric probabilities. If the game is a weighted majority game, we may want to select each player with a probability proportional to his number of votes (this is done by Baron and Ferejohn in one of their examples). This extension has a straightforward interpretation if players are parties, different number of votes correspond to different number of representatives, and each representative is selected to be the proposer with equal probability.

In Montero (2001), I extend the Baron-Ferejohn model to any proper simple game, and show that the nucleolus can always be obtained as the unique equilibrium expected shares in the Baron-Ferejohn game, provided that the recognition probabilities coincide with the nucleolus. Since the recognition probabilities are itself a measure of bargaining power (an input of the game, which in principle need not be related to the voting rule), the nucleolus is a sort of self-confirming power index in this noncooperative game. As for other recognition probabilities, the nucleolus seems more likely to emerge as an equilibrium than the Shapley value.

Example 3 *Consider the game* $[5; 3, 2, 2, 1, 1]$. *The nucleolus of this game is* $(\frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9})$. *In the Baron-Ferejohn bargaining procedure with recognition probabilities* $\theta = (\frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9})$, *the only stationary equilibrium expected payoff is precisely* $(\frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9})$.

The idea of the proof of this result is as follows. Expected equilibrium payoffs act as reservation prices: if a proposal is rejected, nature starts the game all over again and, since strategies are stationary, each player receives his equilibrium payoff. It is then a best response for a player to accept any offer that gives him at least his equilibrium payoff.

Given this vector of prices, it is optimal for the proposer to propose a coalition of maximum excess. In this example all minimal winning coalitions are of maximum excess: $\{1,2\}$, $\{1,3\}$, $\{1,4,5\}$, $\{2,3,4\}$ and $\{2,3,5\}$. This collection is balanced; a set of balancing weights is $\lambda_{\{1,2\}} = \lambda_{\{1,3\}} = \frac{1}{5}$, $\lambda_{\{1,4,5\}} = \frac{3}{5}$, $\lambda_{\{2,3,4\}} = \lambda_{\{2,3,5\}} = \frac{2}{5}$. Consider the following strategy for the proposer: if he belongs to S, he proposes S with probability λ_S, and offers each other player in S their price. Because $\sum_{S \ni i} \lambda_S = 1$, the proposer's strategy is completely determined. Moreover, if all players follow these strategies, expected payoffs indeed coincide with $(\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9})$.

Consider for example player 1. With probability $\frac{1}{3}$ he is selected to be the proposer; he then proposes $\{1,2\}$ or $\{1,3\}$ - offering the other player his price of $\frac{2}{9}$ and thus obtaining $1-\frac{2}{9}=\frac{7}{9}$ - or alternatively $\{1,4,5\}$ - offering each of the other two players $\frac{1}{9}$ and thus also obtaining $\frac{7}{9}$ -. With probability $\frac{2}{9}$ player 2 is selected to be the proposer; player 2 proposes $\{1,2\}$ with probability $\lambda_{\{1,2\}}=\frac{1}{5}$, and pays player 1 his price, $\frac{1}{3}$; the same applies to player 3. Each of players 4 and 5 is selected with probability $\frac{1}{9}$, proposes $\{1,4,5\}$ with probability $\lambda_{\{1,4,5\}}=\frac{3}{5}$ and offers $\frac{1}{3}$ to player 1. Player 1's expected payoff is then

$$\frac{1}{3}\left[1-\frac{2}{9}\right]+\left[\frac{4}{9}\frac{1}{5}+\frac{2}{9}\frac{3}{5}\right]\frac{1}{3}=\frac{1}{3}.$$

These strategies have the property that the probabilities of each coalition forming are proportional to the balancing weights, vindicating the interpretation of balancing weights as related to the probability of each coalition forming. Notice also that each player is in the final coalition with the same probability, in this case $\frac{5}{9}$. Thus Montero (2001) contains a justification of the arguments in the previous section in a strategic model of coalition formation in which players are free to propose any coalition with any payoff division.

If we consider arbitrary (but symmetric in the sense that players of the same type are treated equally) recognition probabilities, the nucleolus seems to be more likely to arise as an equilibrium than the Shapley value. This is because, as a price vector, the nucleolus makes the proposer indifferent between several attractive coalitions, whereas the Shapley value usually induces strict preferences over coalitions. The following example illustrates this point.

Example 4 *Consider the game* $[3;2,1,1,1]$ *and a protocol that selects player* 1 *with probability* θ_1 *and each other player with probability* $\frac{1-\theta_1}{3}$. *The nucleolus of this game is* $(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$ *and the Shapley value is* $(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$. *The* *nucleolus can be obtained for any* $\theta_1 \le \frac{1}{2}$*; the Shapley value is only obtained for* $\theta_1 = \frac{3}{5}$.

Given the price vector $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$, each of the players with 1 vote is indifferent between proposing to player 1 and proposing to the other two players: in both cases the proposer pays a total of $\frac{2}{5}$. We can construct an equilibrium in which player 1 proposes to each other player with probability $\frac{1}{3}$, and each other player proposes to player 1 with probability λ, where λ can be found from player 1's expected payoff equation $\frac{2}{5} = \theta_1 \left[1 - \frac{1}{5}\right] + (1 - \theta_1)\lambda\frac{2}{5}$. The solution to this equation, $\lambda = \frac{1-2\theta_1}{1-\theta_1}$, is between 0 and 1 for $\theta_1 \le \frac{1}{2}$.

In contrast, given the price vector $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ player 1 is overpriced and receives no proposals, and the expected payoff equation becomes $\frac{1}{2} = \theta_1 \left[1 - \frac{1}{6}\right]$, which has only one solution.

Even if the Shapley value makes players indifferent between the relevant coalitions, it may be impossible to obtain it as an equilibrium as the following example of a game with a nonempty core illustrates (for a discussion of the relationship between the Baron-Ferejohn model and the core see Banks and Duggan, 2000).

Example 5 *Consider the game* $[3; 2, 1, 1]$. *The Shapley value of this game is* $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. *There is no protocol* θ *for which expected equilibrium payoffs coincide with the Shapley value. In contrast, the nucleolus is obtained for any protocol such that* $\theta_1 > 0$.

Let y_1 be the expected equilibrium payoff for player 1 and y_2 the expected equilibrium payoff for 2 and 3. Then expected payoff for player 1 is given by $y_1 = \theta_1 [1 - y_2] + (1 - \theta_1)y_1$. For $\theta_1 > 0$, the solution of this equation together with $y_1 = 1 - 2y_2$ is $y_1 = 1$. For $\theta_1 = 0$ we have $y_1 = 0$. Thus *no* protocol implements the Shapley value. The same applies to all power indices based on marginal contributions or that give positive values to any player who is at least in one minimal winning coalition, like the Johnston (1978) and Deegan-Packel (1978) indices.

The Baron-Ferejohn model has been criticized because of the disproportionate advantage it gives to the proposer (see e.g. Harrington (1990)). However, it can be easily modified to eliminate this advantage, as Montero (2003) shows.

Even if the core is empty ,the nucleolus may give a payoff of 0 to players that are not dummies. For example, in the game $[5; 3, 2, 2, 1]$ the nucleolus is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. This payoff can be obtained as an expected equilibrium outcome in the Baron-Ferejohn model, but only if player 4 is never selected to be proposer. However, no power index is generally supported by natural protocols like the egalitarian or the proportional protocol.[5]

4.2 Discussion of other bargaining models

An alternative noncooperative foundation for the nucleolus in majority games can be found in Young (1978). He shows that the nucleolus can be obtained as an equilibrium payoff in an asymmetric lobbying game where two lobbyists with different resources compete in order to buy the players' votes. Unfortunately, the game becomes very difficult to solve if the two lobbyists have equal resources.

The Shapley value has some noncooperative foundations of its own. The most natural model, that of Gul (1989), is not applicable to simple games because it requires that any two players benefit (in terms of the Shapley value) from forming a bloc. Winter's (1994) demand commitment model only applies to convex games and thus it is not applicable to simple games (interestingly, Morelli's (1999) demand commitment results for simple games

[5]This includes the modified nucleolus of Sudhölter (1996), which is a representation of all weighted majority games.

are closer to supporting the nucleolus than they are to supporting the Shapley value). Other noncooperative foundations of the Shapley value are formally applicable to simple games, but they give a special role to the grand coalition, which seems contradictory with the idea of a majority game: Hart and Mas-Colell (1996) and Pérez-Castrillo and Wettstein (2001) require *all* players to agree with a proposal. Vidal-Puga (2004) allows only one coalition (not necessarily the grand coalition), and players must choose between joining it or become singletons; they are not allowed to wait, even though they may actually prefer to do so. Instead in the Baron-Ferejohn model players always prefer to be proposers rather than wait.

It seems paradoxical that, while the Shapley value is usually interpreted as an expected payoff of playing the game which unlike the nucleolus does not presuppose the grand coalition, the opposite happens in the corresponding implementations: Hart and Mas-Colell require the grand coalition to form in order to obtain the Shapley value while the nucleolus can be obtained in the Baron-Ferejohn model as a "value" without giving the grand coalition a prominent role (indeed the grand coalition is never formed if the game is constant sum unless one of the players is a dictator).

5 Some properties of the Shapley value and nucleolus

The nucleolus satisfies the following property: suppose, as in our previous discussion, that only coalitions of maximum excess given a price vector x form, and payoff division inside a coalition is proportional to this price vector. Let $\mathbf{S}_i$ be the set of coalitions of maximum excess to which i belongs. If $\mathbf{S}_i \subseteq \mathbf{S}_j$ and $\mathbf{S}_j \nsubseteq \mathbf{S}_i$, one can say that i depends on j, but j does not depend on i. In this case it seems reasonable that i reduces his payoff in favor of j, unless $x_i = 0$. This property is called the *partnership condition* by Bennett (1983), and a very similar condition is postulated by Napel and Widgrén (2001) as a desirable property of a power index. The fact that the nucleolus has this property is clear from Kohlberg's result, as the following claim shows.

Claim 1 *Let (N, v) be a simple game, x the nucleolus of v, and $\mathbf{S}_i = \{S \in b_1(x, v) : i \in S\}$. Then for any two players i and j, $\mathbf{S}_i \subseteq \mathbf{S}_j$ and $\mathbf{S}_j \nsubseteq \mathbf{S}_i$ implies $x_i = 0$.*

Proof. Suppose $\mathbf{S}_i \subseteq \mathbf{S}_j$ and $\mathbf{S}_j \nsubseteq \mathbf{S}_i$, but $x_i > 0$. Because x is the nucleolus, the set $b_0(x) \cup b_1(x, v)$ must be balanced. Let $(\lambda_S)_{S \in b_0(x) \cup b_1(x,v)}$ be a set of balancing weights. Because $x_i > 0$, $\sum_{S \in b_1(x,v), S \ni i} \lambda_S = 1$. But $\sum_{S \in b_1(x,v), S \ni i} \lambda_S < \sum_{S \in b_1(x,v), S \ni j} \lambda_S = 1$, a contradiction. ■

The Shapley value does not seem to have an analogous property: if we consider the majority game with a veto player $[3; 2, 1, 1]$, players 2 and 3 clearly depend on 1 but still have a positive Shapley value.

A property enjoyed by the Shapley value but not by the nucleolus is the *symmetric gain/loss* property (see Laruelle and Valenciano (2001)). This property states that, if we compare a simple game v with the game v_S^* that results after deleting a minimal winning coalition $S \neq N$ from v, then the change in the Shapley value is the same for all players in S and for all players in $N \backslash S$. The following example illustrates this property:

Example 6 *Consider the game $(5; 3, 2, 2, 1, 1)$. This game has the following minimal winning coalitions: $\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 4, 5\}$. Players 1, 2 and 3 belong to three of those, whereas players 4 and 5 belong only to two of those.*

Now consider the modified game that has the same characteristic function except that $v(1, 4, 5) = 0$. The Shapley value of the original game is $(\frac{24}{60}, \frac{14}{60}, \frac{14}{60}, \frac{4}{60}, \frac{4}{60})$; after deleting coalition $\{1,4,5\}$ from the set of winning coalitions the Shapley value changes to $(\frac{22}{60}, \frac{17}{60}, \frac{17}{60}, \frac{2}{60}, \frac{2}{60})$. Thus, each of players 1, 4 and 5 have lost $\frac{2}{60}$. However, one may argue that coalition $\{1,4,5\}$ was crucial for players 4 and 5 but not for player 1. After the deletion of $\{1,4,5\}$, player 1 can form a coalition with either 2 or 3, whereas players 4 and 5 are now dependent on players 2 and 3. Indeed the nucleolus changes from $(\frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9})$ to the (somewhat extreme but consistent with the partnership condition) $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$. Players 4 and 5 are seriously affected, but not so player 1.

Because the nucleolus is symmetric and always belongs to the core, it divides the total payoff equally between the veto players whenever they exist. It is not surprising that the nucleolus does better than the Shapley value at the postulates related to veto players or blockers.

Felsenthal and Machover (1998) introduce the added blocker postulate (ABP) for measures of P-power. Let v be a simple game, and w another simple game that is obtained by adding an extra player with veto power to v. An index ξ satisfies ABP if whenever a and b are two nondummy players in V we have

$$\frac{\xi_a[w]}{\xi_b[w]} = \frac{\xi_a[v]}{\xi_b[v]}.$$

A flagrant violation of the postulate occurs when $\xi_a[w] > \xi_b[w]$ and $\xi_a[v] < \xi_b[v]$, or the reverse.

Because the nucleolus is in the core, it must give 0 to all players who are not veto players in game w. Thus, the nucleolus violates ABP but not flagrantly. As for the Shapley value, Felsenthal and Machover show that it flagrantly violates ABP.

Another postulate of Felsenthal and Machover is the *blocker share postulate*. This postulate says that, if i is a veto player and S a winning coalition, a P-power index must assign to i at least $\frac{1}{|S|}$. The nucleolus clearly satisfies this postulate, since it divides the payoff equally between all veto players and leaves nothing to outsiders. Felsenthal and Machover show that the Shapley value satisfies this postulate, whereas the Banzhaf, Deegan-Packel and Johnston indices may violate it.

Felsenthal and Machover also point out that a player may lose from becoming a blocker according to the Shapley value. They consider the games $[6; 5, 3, 1, 1, 1]$ and $[8; 5, 3, 1, 1, 1]$. The second game is obtained from the first by raising the quota; as a result of this player 1 becomes a veto player. The Shapley value assigns respectively $(\frac{3}{5}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$ and $(\frac{11}{20}, \frac{3}{10}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20})$ to these games; the corresponding values for the nucleolus are $(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$ and $(1, 0, 0, 0, 0)$. Clearly, if a player is *the only one* to become a blocker he cannot lose because the nucleolus gives him 1; he may lose if other players become blockers as well but this doesn't seem too paradoxical. For example consider the game $[7; 6, 3, 2, 1, 1]$, whose nucleolus is again $(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$; if the quota is raised to 12 there are three veto players and the new nucleolus is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$. Player 1 loses but this is not surprising because he has become symmetric to players 2 and 3, while originally he was more powerful.

Because the nucleolus may give a payoff of 0 to players that are not dummies, it violates one of Felsenthal and Machover (1998) postulates for a power index. This undesirable property of the nucleolus does not occur in constant-sum weighted majority games. Peleg (1968) shows that the nucleolus is always a representation of this type of games, and thus must assign a positive weight to any player who is not a dummy. If the game is homogeneous but not constant-sum the nucleolus may still be a representation (Peleg and Rosenmüller (1992) provide conditions under which this is the case).

Straffin (1988) points out that the Banzhaf index and the Shapley value may rank players differently in the game $[2; 1111] \otimes [3; 2111]$, where the notation $\otimes$ means that a majority must be obtained in both voting bodies. This example also shows that the Banzhaf index and the nucleolus may also rank players differently; it also shows that the nucleolus may appear counterintuitive as a measure of power: it assigns 0 to all players in the first game, and $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ to the players in the second game. In contrast, the Banzhaf index assigns more to players of the first type than to players of the third type. The nucleolus must assign 0 value to the players of the first type because otherwise the set of coalitions of maximum excess would not be balanced. Thus, whether we take the Shapley value or the nucleolus as a measure of P-power, a player can be more powerful than another under office-seeking behavior, but less powerful under policy-seeking behavior. Felsenthal and Machover refer to this possibility as "somewhat paradoxical".

6 Concluding remarks

This paper makes a case for the nucleolus as a power index in divide-the-dollar games, especially if the nucleolus is a representation of the game. The nucleolus can be interpreted as a competitive price system and has relatively solid noncooperative foundations. At a more fundamental level, the nucleolus identifies a set of attractive coalitions, whereas the Shapley value is determined by all coalitions.

It is common wisdom in the power indices literature that "the very idea behind voting power is that the weight of a voter is not a good measure of power" (Pajala 2002). If we adopt the nucleolus as a power index, weights will be power for some games (including all constant-sum weighted majority games), provided that we choose the right weights to represent the game. Interestingly, homogeneity is neither necessary nor sufficient for weights to be a measure of power.

The Deegan-Packel index assume that only minimal winning coalitions will form, each of them with equal probability, and players will divide the payoff equally. Clearly, the nucleolus does not assume that coalitions divide the payoff equally. It does not assume either that all minimal winning coalitions form (some minimal winning coalitions may not be of maximum excess; this is the case for a coalition of type [222] in example 1). It does not assume equiprobability of coalitions, but it does imply equiprobability of players (at least, of the players that get a positive payoff), which is an appealing property.

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