

# Measuring Influence for Dependent Voters: A Generalisation of the Banzhaf Measure

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## 1. Introduction

The Banzhaf measure of voting power equals the chance of pivotality under the assumptions that (i) the voters cast their votes independently and (ii) each voter is equally likely to vote yes or no. These assumptions define the so-called Bernoulli model (Felsenthal and Machover 1998). But the Bernoulli model is of course not always an accurate representation of the real world. We will give up both assumptions of the Bernoulli model. Since it is clear how to generalize Banzhaf voting power in the absence of equiprobability (Dubey and Shapley 1979, Machover et al. 2007), we will focus our discussion on the independence assumption.

The simplest approach for generalizing the notion of standard voting power is to calculate the probability of being pivotal under the probability model at hand, which might violate the independence assumption. However, as Machover et al. (2007) show, this leads to highly counterintuitive results. We will therefore construct a new measure that quantifies the real-world influence of each voter. If the assumptions of the Bernoulli model hold then our measure simply reduces to Banzhaf voting power.

## 2. Generalized Voting Power

Suppose that with person  $i$  voting no on a proposal in the actual world, the chance of acceptance is quite low. If  $i$  were to vote yes instead, then the chance of acceptance would be much greater. Similarly with  $i$  voting yes on a proposal in the actual world, the chance of rejection is quite low. If  $i$  were to vote no instead, then the chance of rejection would be much greater. Then intuitively,  $i$  has more influence than a person  $j$  for whom these respective differences would have been notably smaller. Let us make this idea precise by constructing a measure.

We introduce the following notation. We assume that there are  $N$  voters. The  $i^{\text{th}}$  vote is modeled as a random variable  $V_i$ . We set  $V_i = 1$ , if  $i$  votes yes, and  $V_i = 0$ , if she votes no. The outcome of the vote is described by the random variable  $V$ .  $V = 1$  means that the proposal is accepted.  $V = 0$  means that the proposal is rejected.

We assume that we have a full probability model for the votes. The model provides us with probabilities for each possible voting profile (i.e. the joint probabilities of the  $V_i$  s). We assume that we can obtain conditional probabilities such as  $P(V_j = 0 \mid V_i = 1)$  from the joint probability distribution. (For an exception see Section 5 below.) Once we know the decision rule, we can calculate conditional probabilities for acceptance given some single vote or given a combination of votes, such as  $P(V = 1 \mid V_i = 1)$  or  $P(V = 1 \mid V_i = 1, V_j = 0)$ . It is not our concern in this paper how to obtain a realistic probability model from empirical data.

We will also assume that we have causal information about the votes. In our models a person's votes can be influenced by other people's votes and by one's ideological commitment. This is not inconsistent with taking voting to be an instance of free agency. How such causal information can be obtained is not our concern in this paper.

For calculating our measure of influence, we first assess the chance that a proposal is accepted given that  $i$  voted no, i.e.  $P(V = 1 | V_i = 0)$  and the chance that a proposal is rejected given that  $i$  voted yes, i.e.  $P(V = 0 | V_i = 1)$ . Subsequently, we construct counterfactual probability distributions. We ask what the chance is that a proposal would have been accepted, had  $i$  voted yes (rather than no, as  $i$  did in the actual world), i.e.  $Q_i^-(V = 1 | V_i = 1)$ . And we ask what the chance is that a proposal would have been rejected, had  $i$  voted no (rather than yes, as  $i$  did in the actual world), i.e.  $Q_i^+(V = 0 | V_i = 0)$ . Let  $D_i^-$  be the difference between the chance that the proposal would have been accepted had  $i$  voted yes and the chance that the proposal was accepted conditional on  $i$  having voted no. Let  $D_i^+$  be the difference between the chance that the proposal would have been rejected had  $i$  voted no and the chance that that the proposal was rejected conditional on  $i$  having voted yes.

$$(1) \quad D_i^- = Q_i^-(V = 1 | V_i = 1) - P(V = 1 | V_i = 0)$$

and

$$(2) \quad D_i^+ = Q_i^+(V = 0 | V_i = 0) - P(V = 0 | V_i = 1) .$$

We can now construct the measure:

$$(3) \quad D_i = D_i^- P(V_i = 0) + D_i^+ P(V_i = 1) .$$

Our definition of  $D$  need not only be motivated by an intuitive argument. There is also an ex post justification for quantifying influence in terms of  $D$ :  $D$  yields intuitively plausible results if applied to a range of examples. We now turn to such examples.

### 3. A simple example

We start with a simple three-person example featuring a Supreme Court with Scalia, Thomas and Ginsburg as voters. This example will illustrate how the  $D$ -values are calculated. We will first assume that proposals are decided on by a simple majority vote. Rather than using the terminology that is fitting for the Supreme Court, we will conduct our presentation in terms of *voters* and *proposals*.

We consider different models for the votes. Under every model, each voter votes yes with a probability of .5.

#### 3.a Opinion leader

Both Ginsburg and Scalia are influenced by and only by the details of the case. Thomas keeps a close eye on Scalia and there is .9 chance that he will vote yes, given that Scalia votes yes, and there is a .9 chance that he will vote no, given that Scalia votes no. One can thus say that Thomas's vote causally depends on Scalia's vote. The causal influences in the model are represented in Fig. 1. Let us now assess the influence of each voter by calculating her  $D$ -value.

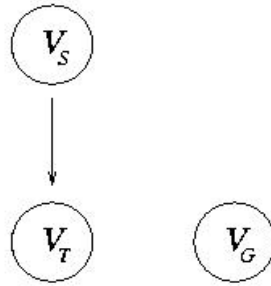


Figure 1. The causal network for the opinion leader model.

We start with the  $D$ -value of Thomas. We consider the first addend, viz.  $D_T^- P(V_T = 0)$ . Clearly,  $P(V_T = 0) = .5$ .

For calculating  $D_T^-$  we have to assume that Thomas votes no in the real world. We first turn to  $P(V = 1 | V_T = 0)$ . If Thomas votes no, then the chance that Scalia voted no is .9. Given that Thomas votes no, we only get acceptance in the real world, if both Scalia and Ginsburg vote yes. And so the conditional chance that the proposal is accepted is  $P(V = 1 | V_T = 0) = .1 * .5 = .05$ . We now turn to  $Q_T^-(V = 1 | V_T = 1) = 1 - Q_T^-(V = 0 | V_T = 1)$ . If Thomas were to vote for the proposal, this would not affect the chance that Scalia voted yes—that chance is still .9—since the causal link does not go from Thomas to Scalia. The only profile under which the proposal would be rejected, if Thomas voted yes, is a profile with Scalia and Ginsburg voting no. That chance is  $.9 * .5 = .45$ . So  $Q_T^-(V = 1 | V_T = 1) = .55$  and  $D_T^- = .55 - .05 = .5$ . The argument for  $D_T^+$  runs parallel and so  $D_T = .50$ .

Let us now calculate the  $D$ -value for Scalia. We first consider  $Q_S^-$ . If Scalia votes no, then the chance that Thomas votes no is .9. Given that Scalia votes no, we only get acceptance in the real world, if both Thomas and Ginsburg vote yes. And so the conditional chance that the proposal is accepted is  $P(V = 1 | V_S = 0) = .1 * .5 = .05$ . We now turn to  $Q_S^-(V = 1 | V_S = 1) = 1 - Q_S^-(V = 0 | V_S = 1)$ . If Scalia were to vote yes, this would affect the chance that Thomas votes no—that chance is now .1—since the causal link goes from Scalia to Thomas. The only profile under which the proposal would be rejected is a profile with Thomas and Ginsburg voting no. The chance is  $.1 * .5 = .05$ . So  $Q_S^-(V = 1 | V_T = 1) = .95$  and  $D_S^- = .95 - .05 = .9$ . The argument for  $D_S^+$  runs parallel and so  $D_S = .9$ .

Finally, we assess the influence of Ginsburg. If Ginsburg votes no, then this does not affect the chance that Thomas or Scalia vote no. Given that Ginsburg votes no, the chance that both Thomas and Scalia vote yes is  $.9 * .5 = .45$ . So  $P(V = 1 | V_G = 0) = .45$ . Suppose that Ginsburg asks herself what the chance would be that the motion had been accepted had she voted yes. The chance that both Thomas and Scalia would have voted no is  $.9 * .5 = .45$ . So  $Q_G^-(V = 1 | V_G = 1) = 1 - .9 * .5 = .55$ . Hence,  $D_G^- = .10$ . The argument for  $D_G^+$  runs parallel and so  $D_G = .10$ .

This is not unreasonable. Scalia does have more influence than Thomas, because he takes Thomas along with him as he changes votes, but not vice versa. Ginsburg has very little influence because she faces a quasi-block vote of the two other voters.

Justice	Scalia	Thomas	Ginsburg
D-value	0.9	0.5	0.1

Table 1. Results under the opinion leader model

### 3.b Common causes

Let us now change our assumptions in the following way:

1. We introduce a parameter  $\epsilon$  which permits us to vary the correlation between the votes of Scalia and Thomas. The parameter ranges from  $-1$  for full negative correlation over  $0$  for independence to  $+1$  for full positive correlation:

$$P(V_S = 1, V_T = 1) = P(V_S = 0, V_T = 0) = .25*(1 + \epsilon) .$$

and

$$P(V_S = 0, V_T = 1) = P(V_S = 1, V_T = 0) = .25*(1 - \epsilon) .$$

2. Correlations do not arise due to a direct causal influence from Scalia to Thomas or vice versa as in Subsection 3.a. Rather, they are due to a common cause (see Figure 2). Positive correlations are due to shared ideological backgrounds. Negative correlations are due to diverging ideological backgrounds. We can model this in the following way. We introduce a random variable which captures the nature of the proposal. If  $C = 0$ , then the proposal is such that both Scalia and Thomas vote no; if  $C = 1$ , then the proposal is such that Scalia votes no and Thomas votes yes; if  $C = 2$ , then the proposal is such that Scalia votes yes and Thomas votes no; if  $C = 3$ , then the proposal is such that both vote yes.  $C$  models, in Dretske's terms (Dretske 1988, pp. 42–4), a triggering cause for the voting behaviour of Thomas and Scalia. The features of the proposal trigger votes that match the ideological commitments of Thomas and Scalia. By specifying the probability values in Table 2 we can characterize the degree to which Thomas and Scalia's votes are correlated or anti-correlated.

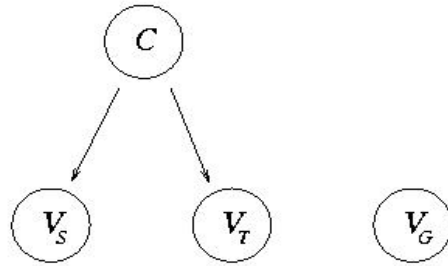


Figure 2. The causal network for the common cause model.

$C = i$	$P(V_S = 1   C = j)$	$P(V_T = 1   C = j)$	$P(C = i)$
$C = 0$	$P(V_S = 1   C = 0) = 0$	$P(V_T = 1   C = 0) = 0$	$.25*(1 + \epsilon)$
$C = 1$	$P(V_S = 1   C = 1) = 0$	$P(V_T = 1   C = 1) = 1$	$.25*(1 - \epsilon)$
$C = 2$	$P(V_S = 1   C = 2) = 1$	$P(V_T = 1   C = 2) = 0$	$.25*(1 - \epsilon)$
$C = 3$	$P(V_S = 1   C = 3) = 1$	$P(V_T = 1   C = 3) = 1$	$.25*(1 + \epsilon)$

Table 2. The probability model for the common cause model. The variable  $V_G$  is not included – it is independent from the other variables  $V_i$  and takes values of 0 and 1 with a probability of .5 each.

In the following, we will omit the details of our calculations. A general algorithm for calculating the  $Q_i$ 's and  $Q_i^+$ 's is provided in Appendix A.

Applying this algorithm, we obtain the following results:

Justice	Scalia	Thomas	Ginsburg
$D$ -value	0.5	0.5	$0.5 * (1 - \epsilon)$

Table 3. Results under the common cause model

Thus, under the common cause model, the  $D$ -values for Scalia and Thomas do not depend on the strength of the correlations. On our model, the influence of a voter is the same regardless whether there is a shared ideology with another voter or not. However, the influence of Ginsburg depends on whether and how the votes of the other justices are correlated. If Scalia and Thomas always vote the same, then Ginsburg has no influence. If, on the other hand, Scalia and Thomas always cast opposing votes, then Ginsburg has maximal influence. Clearly, if the votes of Scalia and Thomas are independent, then every voter has a  $D$ -value of .5. Note that this  $D$ -value for independent votes coincides with the

Banzhaf measure for this simple voting game. As we will show in Section 4, this is due to a more general connection between Banzhaf voting power and the  $D$ -value.

But one might object that Scalia (and Thomas) have more influence in a court in which their votes are correlated than in a court in which they vote independently. This is indeed not an unreasonable interpretation of influence. Let us call this the *block* interpretation. It is not the interpretation that is captured by the  $D$ -measure though. On the  $D$ -measure, Scalia's influence is the same with correlated or independent votes, because if he had voted differently, the chance of rejection or acceptance would have been equally affected. But we can also construct a measure  $D^*$  which is in line with the block interpretation of influence. Let  $D^*$  be the normalized measure of  $D$ , i.e. every  $D$ -value is divided by the sum of the  $D$ -values. Then  $D^*$  is monotonically increasing in  $\epsilon$  and so according to this measure the influence of Scalia and Thomas increases as we move from anticorrelated votes over independence to correlated votes.

### 3.c Dictator

So far we have assumed simple majority voting with equal weights. Let us now change the weights as follows: Scalia has a block vote of three votes, whereas Thomas and Ginsburg have only one vote each. The Supreme Court issues a yes (no) vote if and only if there are at least three yes (no) votes. Thus, Scalia is a dictator, whereas Thomas and Ginsburg are dummies.

Whether we calculate these results under the model under which Scalia is an opinion leader or under the common cause model with any value of  $\epsilon$ , we obtain the following results:

Justice	Scalia	Thomas	Ginsburg
$D$ -value	1	0	0

Table 4. Results for a voting rule in which Scalia is a dictator.

These results are very plausible. Only the dictator has influence, whereas the dummies do not have influence. This gives rise to the following observations:

1. The  $D$ -value significantly depends on the voting rule. As we change the voting rule and keep the model for the voting profile fixed, the value of  $D$  changes. This is important, since, in voting theory, we are particularly interested in how different voting rules affect the influences of the voters.
2. The results also show that the  $D$ -value does not suffer from a problem that affects other generalizations of Banzhaf voting power for probability models different from the Bernoulli model. One way of generalizing Banzhaf voting power starts from the observation that standard voting power is a linear transform of the probability of your vote coinciding with the outcome of the vote. One can then quantify influence as the probability of the coincidence of your vote and the outcome of the vote under the assumption of the probability model that we adopt for the voting profiles. As Machover et al. (2007) argue, this will not work, because, in the dictator model, every dummy vote that is perfectly correlated with the vote of the dictator will obtain the same value of the

measure as the dictator. Our  $D$ -value does not have this problem. Even if Thomas blindly follows Scalia as an opinion leader or even if Scalia and Thomas's ideologies completely overlap ( $\epsilon = 1$ ),  $D$  assigns non-identical values to the dictator Scalia and the dummy Thomas. The values of  $D$  are not dependent on the correlations between Scalia and Thomas's vote.

#### 4. The relation to Banzhaf Voting Power

We will now show that the  $D$ -measure reduces to Banzhaf voting power under conditions of independence and equiprobability. On grounds of equiprobability:

$$(4) \quad P(V_i = 0) = P(V_i = 1) = .5 .$$

To understand the role of independence, consider first dependent voters, viz. Thomas with Scalia as opinion leader. Suppose that Thomas votes no. This teaches us something about Scalia's vote in the real world. On our interpretation of counterfactuals, this knowledge has to be taken into account in calculating  $Q_i^-(V = 1 | V_t = 1)$ . Thus,  $Q_i^-(V = 1 | V_t = 1)$  does not simply equal  $P(V = 1 | V_t = 1)$ . Next consider Ginsburg under the same model. The fact that Ginsburg votes no in the real world teaches us nothing about other votes, because Ginsburg's vote is independent. So  $Q_G^-(V = 1 | V_G = 1)$  does equal  $P(V = 1 | V_G = 1)$ . More generally, if there is independence, we have

$$(5) \quad Q_i^-(V = 1 | V_i = 1) = P(V = 1 | V_i = 1) \text{ and } Q_i^+(V = 0 | V_i = 0) = P(V = 0 | V_i = 0) .$$

So by (4) and (5),

$$(6) \quad D_i = .5 (P(V = 1 | V_i = 1) - P(V = 1 | V_i = 0) + P(V = 0 | V_i = 0) - P(V = 0 | V_i = 1)) .$$

And by the probability calculus:

$$(7) \quad D_i = .5 (P(V = 1 | V_i = 1) - P(V = 1 | V_i = 0) + (1 - P(V = 1 | V_i = 0)) - (1 - P(V = 1 | V_i = 1))) .$$

Thus,

$$(8) \quad D_i = P(V = 1 | V_i = 1) - P(V = 1 | V_i = 0) .$$

Let us now consider the probability of the vote coinciding with  $i$ 's vote, i.e.  $\psi_i$ :

$$(9) \quad \psi_i = P(V = 1 | V_i = 1)P(V_i = 1) + P(V = 0 | V_i = 0)P(V_i = 0) .$$

By (4) and the probability calculus,

$$(10) \quad \psi_i = .5(P(V = 1 | V_i = 1) - P(V = 1 | V_i = 0) + 1) .$$

From (8) and (10):

$$(11) \quad \psi_i = .5(D_i + 1) .$$

So,

$$(12) \quad D_i = 2 \psi_i - 1 .$$

But we know that

$$(13) \quad \beta'_i = 2 \psi_i - 1 ,$$

Where  $\beta'_i$  denotes the Banzhaf measure of voting power for  $i$ . Hence,  $D_i$  coincides with  $\beta'_i$  in the special case of equiprobability and independence.

### 5. Machover's Example

Let us now turn to a more complex example. Suppose that we have a five person Supreme Court with simple majority voting and equal weights for each justice. The 12 profiles in which at least four voters cast the same vote each occur with probability  $1/12 - \epsilon$  for  $0 \leq \epsilon \leq 1/12$ . The 20 remaining profiles each occur with probability  $.6\epsilon$ . At  $\epsilon = 0$ , we reproduce the example that Machover et al. (2007), p. 3 specify – call it Machover's example, for short. Probability models with a finite  $\epsilon$  correspond to generalizations of Machover's example.

If set  $\epsilon$  at zero, the probability of pivotality is zero for every justice. This shows that we cannot measure influence by means of the probability of pivotality, because, as Machover et al. (2007, p. 3) write, “it would be absurd to claim that every voter here is powerless, in the sense of having no influence over the outcome of divisions.” So let us examine whether our  $D$ -measure would yield more fitting values.

For  $0 < \epsilon \leq 1/12$ , we specify a causal interpretation that is consistent with the probability model. We first calculate the conditional probabilities  $P(V_A = 1)$ ,  $P(V_B = 1 | V_A = 1)$ ,  $P(V_B = 1 | V_A = 0)$ , ...,  $P(V_E = 1 | V_A = 0, V_B = 0, V_C = 0, V_D = 0, V_E = 0)$ . We then impose the following opinion leader model. A is not influenced by any other voter. B's vote is influenced by and only by A's vote. B votes yes with probability  $P(V_B = 1 | V_A = 1)$  if A votes yes, and with probability  $P(V_B = 1 | V_A = 0)$  if A votes no. C is influenced by both A and B's vote and so on. So A is an opinion leader for voters B through E, B is an opinion leader for voters C through E, ... and E is an opinion leader for nobody. This is illustrated in Figure 3.



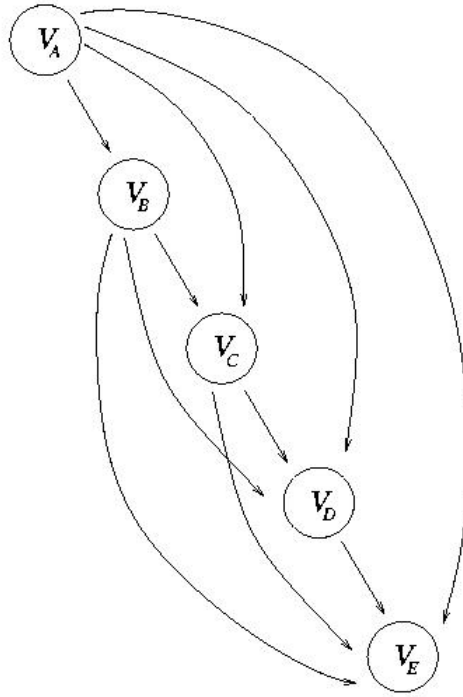


Figure 3. The causal relations that we assume for the generalization of Machover's example.

Of course there are many other causal models that are consistent with the probability model. E.g. a permutation of A, ..., E would also yield a consistent opinion leader model. Furthermore, many more common cause models could be spelled out that are consistent with the probability model. But we will assume that our particular causal model appropriately represents the influences in the real world.

We calculate the  $D$ -values for this causal model following our methodology. As an example,  $D_A = 2/3 - 9.2\varepsilon$ . Subsequently we calculate the limits as  $\varepsilon$  goes to 0 for all voters. In Table 5, we see that the  $D$ -values cascade downwards as we move from voters A to E. This squares very nicely with the fact that A is an opinion leader to more voters than B, B is an opinion leader to more voters than C etc.

$i$	$\lim_{\varepsilon \rightarrow 0} Q_i^-(V=1   V_i=1)$ $\lim_{\varepsilon \rightarrow 0} Q_i^+(V=0   V_i=0)$	$\lim_{\varepsilon \rightarrow 0} P(V=0   V_i=1)$ $\lim_{\varepsilon \rightarrow 0} P(V=1   V_i=0)$	$\lim_{\varepsilon \rightarrow 0} D_i$
A	5/6	1/6	4/6
B	2/3	1/6	3/6
C	1/2	1/6	2/6
D	5/12	1/6	3/12
E	1/6	1/6	0

Table 5. Results for the extension of Machover's example

What happens when we set  $\varepsilon$  at 0 – as is the case of Machover's example (2007, p. 3)? When  $\varepsilon = 0$  we face a problem in calculating the  $D$ -value of voter D. Suppose that  $V_D = 0$

in the real world. We ask what the chance of acceptance would be, if  $D$  were to have voted yes, i.e.  $Q_D^-(V=1 | V_D=1)$ . One profile under which  $V=1$  is the profile  $V_A=1, V_B=0, V_C=0, V_D=1, V_E=1$ . But

$$(14) Q_D^-(V_A=1, V_B=0, V_C=0, V_D=1, V_E=1 | V_D=1) = Q_D^-(V_E=1 | V_A=1, V_B=0, V_C=0, V_D=1) Q_D^-(V_A=1, V_B=0, V_C=0 | V_D=1)$$

with

$$(15) Q_D^-(V_E=1 | V_A=1, V_B=0, V_C=0, V_D=1) = P(V_E=1 | V_A=1, V_B=0, V_C=0, V_D=1)$$

(see Appendix A for details). But note that  $P(V_E=1 | V_A=1, V_B=0, V_C=0, V_D=1)$  is undefined since  $P(V_A=1, V_B=0, V_C=0, V_D=1) = 0$ . Hence we cannot calculate the  $Q_D^-(V=1 | V_D=1)$  for a probability model with extreme values. For this reason, we stipulated a non-extreme  $\varepsilon$ -model and calculated the limiting value of the  $D$ -measure.

But one might object that there are other families of models that approach Machover's example in some limit. For example, we could set the probability of one profile with three yes-votes and two no-votes at  $.4\varepsilon$  and set the probability of another such profile at  $.8\varepsilon$ . Again, we will recover Machover's example, as we set  $\varepsilon$  at zero. However, if we take the limit  $\varepsilon \rightarrow$  zero, we will obtain a slightly different limit for the  $D$ -value of  $D$ . But it can be shown that for all such families of models, the  $D$ -values for  $A, B, C$  and  $E$  are unaffected in the limit  $\varepsilon \rightarrow$  zero and the  $D$ -value for  $D$  ranges from 0 to  $1/3$ , i.e. it takes the  $D$ -values of  $C$  and  $E$  as its bounds.

## 6. Causal Learning

Our assessment of a voter's influence depends on a causal model. But how do we know whether correlations are due to opinion leaders or to ideological commitments? And if they are due to opinion leaders, how do we know whether Scalia is an opinion leader for Thomas or vice versa? The standard line in Causal Learning is that a probability model yields conditional independence structures that define a class of causal models. For this class, it may be possible to obtain bounds on the  $D$ -values of the voters. But if we want to identify a unique causal model we need additional information. One may look at the temporal structure: If Thomas always casts his vote after Scalia, then the causal direction is clear. Or one may appeal to experiment: For instance, one could toggle Scalia's vote and see whether Thomas follows suit. Once a unique causal model is specified, we can assess the  $D$ -value of each voter. But how this is done and whether this can be done is not the subject of our inquiry.

Our recourse to causal information might give rise to two objections.

The first objection is that voting theorists are interested in the influence a voter has due to the decision rule rather than the influence that the voter has due to her influence on other voters. Our reply is that, in the real world, both kinds of influences are entangled in complicated ways. They cannot just be isolated. Our  $D$ -value measures the influence that a voter has in virtue of a decision rule *and* her influence on others.

Second, one might object that is too demanding in practice to require causal information for calculating the  $D$ -measure. Our reply is that one cannot quantify the influence that a voter has in virtue of a decision rule and her influence on others without using causal

information. The uncertainty in our causal information will be reflected in the uncertainty in the  $D$ -measure.

## Appendix A. The algorithm for calculating the $Q_i^{*/}$ -values

We will follow Balke and Pearl 1994, though our notation diverges. Let us take  $Q_i^*(V = 1 | V_i = 1)$  as an example. This is the probability of the counterfactual that the proposal would have been accepted, had  $i$  voted yes, though  $i$  votes no in the actual world.  $i$ 's vote is an event that is embedded in a causal structure. It is caused by certain events and it causes certain effects. Isolate all the non-effects of  $i$ 's vote. If we learn that  $i$  votes no in the actual world, then this teaches us something about some of these non-effects of  $i$ 's vote. We determine a joint probability model for the non-effects of  $i$ 's vote, conditional on  $i$  voting no. Subsequently we set  $i$ 's vote at yes, as if this came about, in Lewis's terms, by a miracle, that is, as if some exogenous force interfered in the course of nature and changed the event from voting no to voting yes. We evaluate how the effects of  $i$ 's vote would be affected by the probability model over the non-effects of  $i$ 's vote conjoint with  $i$  voting yes.

Formally, for evaluating  $Q_i^*(V = 1 | V_i = 1)$  we consider the actual world in which  $V_i = 0$ . Let the random variables  $C_1, \dots, C_n$  be the non-effects of  $V_i$ . The  $C_j$ s may include other votes and variables representing common causes. We calculate the joint probabilities  $P(C_1 = c_1, \dots, C_n = c_n | V_i = 0)$  where the  $c_j$ s range over the possible values for  $C_j$  for each  $j$ . For each combination of the  $C_1 = c_1, \dots, C_n = c_n$ , we then multiply  $P(C_1 = c_1, \dots, C_n = c_n | V_i = 0)$  with  $P(V = 1 | C_1 = c_1, \dots, C_n = c_n, V_i = 1)$ . That is, we ask: What is the probability of acceptance, if  $V_i = 1$ , but if the non-effects of  $V_i$  are as they are in the actual world. By summing the products

$$P(C_1 = c_1, \dots, C_n = c_n | V_i = 0) * P(V = 1 | C_1 = c_1, \dots, C_n = c_n, V_i = 1)$$

for every possible combination  $C_1 = c_1, \dots, C_n = c_n$ , we obtain  $Q_i^*(V = 0 | V_i = 1)$ . To calculate  $P(V = 1 | C_1 = c_1, \dots, C_n = c_n, V_i = 1)$ , we average over all the effect variables of  $V_i$ , call them  $E_1, \dots, E_m$ :

$$P(V = 1 | C_1 = c_1, \dots, C_n = c_n, V_i = 1) = \sum_{e_1} \dots \sum_{e_m} P(V = 1 | C_1 = c_1, \dots, C_n = c_n, V_i = 1, E_1 = e_1, \dots, E_m = e_m) \\ \times P(E_1 = e_1, \dots, E_m = e_m | C_1 = c_1, \dots, C_n = c_n, V_i = 1).$$

In terms of Bayesian Networks, we can characterize our procedure as follows:

1. Construct a Bayesian Network with variables (1) for the votes of each voter, (2) for the common causes and (3) for the outcome of the vote. Insert arrows for opinion leaders as in Figure 1, for common cause ideological commitments as in Figure 2, and arrows from each voter into the outcome of the vote modeling the decision rule.
2. Read off the prior probabilities  $P(V_i = 1)$  and  $P(V_i = 0)$  from this network.
3. Set the value of the variable for voter  $i$  at no and read off the probability of acceptance, i.e.  $P(V = 1 | V_i = 0)$ .
4. Determine the joint probability distribution over the non-effect variables of  $V_i$  conditional on  $V_i = 0$ .

5. Construct a node for the combination of all non-effect variables  $C_1, \dots, C_n$  and insert this joint probability distribution as a new prior.
6. Erase the nodes for the single non-effect variables along with their incoming and outgoing arrows.
7. Insert the requisite arrows from the combined non-effect variable to the effect variables of  $V_i$ , including the node for the outcome of the vote, and put in the concomitant conditional probability distributions. Note that  $V_i$  is a root node in this new network.
8. Set the value of the variable for voter  $i$  at yes in this new network. Read off the probability of acceptance. This is  $Q_i^-(V = 1 | V_i = 1)$ .
9. A similar procedure yields  $P(V = 0 | V_i = 1)$  and  $Q_i^+(V = 0 | V_i = 0)$ .

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