# CRITICALITY IN GAMES WITH MULTIPLE LEVELS OF APPROVAL

SREEJITH DAS

Abstract. In this paper criticality within a voting game is rigorously defined and examined. Criticality forms the basis of the traditional voting power measures frequently employed to analyse voting games; therefore understanding criticality is a pre-requisite to understanding any such analysis. The concept of criticality is extended to encompass games in which players are allowed to express multiple levels of approval. This seemingly innocuous extension raises some important questions, forcing us to re-evaluate exactly what it means to be critical. These issues have been largely side-stepped by the main body of research as they focus almost exclusively on "yes/no" voting games, the so called single level approval voting games. The generalisation to multilevel approval voting games is much more than just a theoretical extension, as any single level approval game in which a player can abstain is in effect a multilevel approval voting game.

## 1. Introduction

Weighted voting games have been studied widely in the literature with real world situations such as the EU Council of Ministers (Leech, 2002a, 2003b; Felsenthal and Machover, 1998, 2000, 2001), the US presidential electoral college (Banzhaf, 1965, 1968; Mann and Shapley, 1960; Gelman et al., 2003a,b; Saari, 2001), and shareholder voting (Leech, 2003a,c). Perhaps the most common analysis carried out is to measure the influence of individual players on the outcome of the game. To accomplish this a number of 'bespoke' influence or power metrics have been proposed and adopted. The most widely accepted voting power metrics rely on the concept of 'criticality' of a player within a coalition. Almost without exception, the work carried out in this area has concentrated on single level approval voting games, games in which players are only allowed to vote 'yes' or 'no'. This paper explores the concept of criticality when players are allowed to express multiple levels of approval.

Multiple approval levels are more than just a theoretical extension. In almost every real life voting game there is the potential for a player

This paper is to be presented at the Voting Power in Practice Workshop at the University of Warwick, 14-16 July 2009, sponsored by The Leverhulme Trust (Grant F/07-004/AJ).

to abstain, either deliberately, in an 'attempt' to express indifference[1], or passively, by simply not bothering to vote. As demonstrated in Das (2008), games with abstentions can be modelled as games with multiple levels of approval. A proposition also supported by the work of Freixas and Zwicker (2003). Ergo, this work is relevant to anyone that analyses real life voting games, as opposed to the purely theoretical ones in which every player casts a vote.

This paper is intentionally mathematically light, avoiding topics such as measure theory, in order to keep the target audience as wide as possible. As such, a more diagrammatic approach is taken, concentrating on the key ideas and concepts that form the basis for this kind of analysis. Accordingly, this paper will refer to the 'counting' of events instead of the technically correct measuring of events, thereby avoiding any unnecessary distractions from the key principles being discussed.

## 2. Power Measures - Probability Assumptions

It has been argued in Felsenthal and Machover (1998); Linder (2008); Paterson (2005) and Straffin (1977, 1978) that the common voting power measures based on the concept of criticality can be modelled as follows:

$$\text{Power Measure} = f(g(\textit{criticality}), h(PDF)),$$

where $g(\textit{criticality})$ is a function of criticality and $h(PDF)$ is a function of the probability density function of the underlying coalitions.

For instance, in a game where $N$ represents the set of players, it is often suggested that power measures can be calculated as follows. Identify a coalition $C$ that is critical for a player $i \in N$. For all such $C$, to calculate the Banzhaf measure add $\dfrac{1}{2^{|N|-1}}$ to the running count for player $i$. Or to calculate the Shapley-Shubik index add $\dfrac{(|C|-1)!(|N|-|C|)!}{|N|!}$ instead. In both cases, the $g(\textit{criticality})$ function is the operation that identifies the critical coalition and the $h(PDF)$ function is $\dfrac{1}{2^{|N|-1}}$ or $\dfrac{(|C|-1)!(|N|-|C|)!}{|N|!}$.

Using this representation the $g(\textit{criticality})$ function is independent of the $h(PDF)$ function. And furthermore, the same $g(\textit{criticality})$ function is applied to find critical coalitions in both the Banzhaf measure and the Shapley-Shubik index.

Despite initial appearances the $g(\textit{criticality})$ function is non-trivial. In fact, understanding it is of greater importance than understanding the $h(PDF)$ function. After all, without a rigorous understanding of criticality is it possible to truly understand any voting power measure?

[1]Abstaining almost never expresses indifference, see Das (2008) for details.

2.1. **Special Note.** This paper focuses on the nature of criticality and measuring the critical coalitions. During the discussions reference will be made to certain conditions imposed on the probability density function. For example, criticality might be measured over all coalitions in which player $i$ votes no. It is important to note that this can considered independently of the actual choice of probability density function. Just to reiterate this point, all the work in this paper, and any conditions imposed upon the probability density function, is independent of the actual choice of probability density function.

## 3. Criticality

The traditional voting power measures depend on the concept of criticality in a coalition. The idea is simple, if a player $i$ can change the outcome of a coalition, either by leaving a winning coalition or by joining a losing coalition, then player $i$ is said to be **critical**. A player $i$ that is critical more often must have more influence, or power, within the weighted voting game than a player that is critical less often. A simple, logical and intuitively satisfying idea, but, as with most things in life, the devil is in the detail.

3.1. **Criticality Diagrams. Criticality diagrams** are a tool that aid in the visualisation of criticality. In a criticality diagram, there is a **source** coalition that undergoes a transition, or set operation, involving player $i$ to become the **sink** coalition. Importantly, the transition changes the state of the coalition from losing to winning or vice versa. A few examples will help illustrate these ideas.

In the following the electorate is represented by the set $N$. The set of all possible coalitions is given by the powerset $\mathcal{P}^N$ of $N$, where $|\mathcal{P}^N| = 2^{|N|}$. However, for future convenience, $\mathcal{P}^N$ will be denoted as $\Omega^N$ instead.

Let's start by taking a source coalition $\omega^{N\backslash\{i\}} \in \Omega^{N\backslash\{i\}}$ that is losing, and a sink coalition $\omega^{N\backslash\{i\}} \cup \{i\}$ that is winning.

$$\begin{array}{ccc} \text{Source} & & \text{Sink} \\ \text{(Losing)} & & \text{(Winning)} \\ \omega & \xrightarrow{\cup\{i\}} & \omega \cup \{i\} \end{array}$$

Given that the source coalition is losing and the sink coalition is winning, the operation of adding player $i$ has made the coalition change result, therefore $i$ is critical in this situation. But which coalition is $i$ critical in, $\omega$ or $\omega \cup \{i\}$? Since coalition $\omega$ undergoes the transformation operation $(\cup\{i\})$ to change its state, it is appropriate to say $i$ is critical to $\omega$. Furthermore, as player $i$ has increased its support (by joining the coalition), $i$ is said to be **Increasingly Critical** to coalition $\omega$. The set of all increasingly critical coalitions for player $i$ is denoted by $\mathbf{IC_i}$.

Adopting this convention requires the creation of a second criticality diagram, showing the transformation operation acting in the reverse direction.

$$\begin{array}{ccc} \text{Source} & & \text{Sink} \\ \text{(Winning)} & & \text{(Losing)} \\ \omega \cup \{i\} & \xrightarrow{\setminus\{i\}} & \omega \end{array}$$

In this situation the source coalition is winning and the sink coalition is losing. The operation of removing player $i$ has made the coalition change result, therefore $i$ is critical here too. As per the adopted convention, $i$ is critical to coalition $\omega \cup \{i\}$, because this is the coalition that undergoes the transformation operation ($\setminus\{i\}$) to change its state. Furthermore, as player $i$ has decreased its support (by leaving the coalition), $i$ is said to be **Decreasingly Critical** to coalition $\omega \cup \{i\}$. The set of all decreasingly critical coalitions for player $i$ is denoted by $\mathbf{DC_i}$.

The combined set, $IC_i \cup DC_i$, is called the **Total Critical** set and is represented by $\mathbf{TC_i}$. Clearly the sets $IC_i$ and $DC_i$ are mutually exclusive, as an event can not be simultaneously winning and losing.

In general, a player $i$ is critical to the source coalition(s), as these are the coalitions in which the player $i$ can effect change. Think of the source coalitions as being the arena of action or **zone of influence**[2] in which player $i$ can make a change to the outcome.

3.2. **Single Level Symmetry.** Single level approval criticality diagrams always occur in matched pairs. For every increasingly critical diagram a decreasingly critical diagram can be constructed by inverting the transition, and vice versa. This creates a bijection between the sets $IC_i$ and $DC_i$. This symmetry, in the single level approval game, masks the material differences between the many different notions of criticality to be discussed in this paper. Only when multilevel approval is examined in Section 6 will this symmetry be broken, and the need for the different notions of criticality will become clear.

## 4. Conditional and Unconditional Measures

In the following the event ($x_i = \{i_{max}\}$), or simply $\{i_{max}\}$, indicates player $i$ has voted in favour and the event ($x_i = \{i_{min}\}$), or simply $\{i_{min}\}$, indicates that player $i$ has voted against. $\Pr(X)$ is taken to represent the probability of event $X$ occurring.

Counting the elements of the set $IC_i$ creates a measure of the number of losing coalitions, that do not include player $i$, that can be made winning when $i$ decides to join them. As eluded to in Section 2, during the counting process it is possible to multiply by the probability of

[2]Thanks to Dr Bowler for coining this phrase.

such a coalition occurring, turning the simple count into a probability measure.

But what does the probability of such a coalition occurring actually mean? For example, with increasing criticality, player $i$ cannot be part of the source coalition; so should the probability be the conditional probability of the source occurring given that player $i$ is not a part of it? Or should it simply be the unconditional probability of the source occurring? In other words, should it be $\Pr(IC_i|(x_i = \{i_{min}\}))$ or $\Pr(IC_i)$? What about decreasing criticality, should it be $\Pr(DC_i|(x_i \neq \{i_{min}\}))$ or $\Pr(DC_i)$? Or maybe $\Pr(TC_i)$ is best? Rather than speculate, let's examine the original voting power papers to see what their authors originally envisaged.

4.1. **Penrose.** Penrose (1946) defines his power measure as "half the likelihood of a situation in which an individual vote can be decisive." An individual voter can be decisive by joining a losing coalition to make it winning, or by leaving a winning coalition to make it losing. Therefore half the likelihood of a voter being decisive is given by,

$$\text{Penrose}_i = \frac{\Pr(IC_i|\{i_{min}\}) + \Pr(DC_i|\{i_{max}\})}{2}.$$

The symmetry between the set $IC_i$ and $DC_i$ means the following is also true,

$$\text{Penrose}_i = \Pr(IC_i|\{i_{min}\}) = \Pr(DC_i|\{i_{max}\}).$$

In fact, because $\Pr(\{i_{min}\}) = \Pr(\{i_{max}\}) = \frac{1}{2}$, with the probability density function used by Penrose, the following is also true,

$$\text{Penrose}_i = \Pr(TC_i).$$

4.2. **Banzhaf.** Banzhaf (1965) doesn't provide an explicit definition of his power measure, instead it is inferred from the examples in his paper. It seems that Banzhaf constructs his measure as:

$$\text{Banzhaf}_i = \frac{|IC_i| + |DC_i|}{2^n}.$$

This is of course $\Pr(TC_i)$, when $\Pr(\{i_{min}\}) = \Pr(\{i_{max}\}) = \frac{1}{2}$ for all $i$. (Both Penrose and Banzhaf assume this to be the case).

4.3. **Shapley Shubik.** Shapley and Shubik (1954) state that their definition of voting power is based upon "the chance he has of being critical to the success of a winning coalition." They go on to mention an alternative index of "blocking power" which is constructed as a negative version of their original index. Before finally stating that the "blocking power" index is "is exactly equal to the index of our original definition."

From their description we interpret the Shapley-Shubik index to be the following:

$$\text{ShapleyShubik}_i = \Pr(IC_i|\{i_{min}\}) = \Pr(DC_i|\{i_{max}\}).$$

4.4. **Straffin.** Straffin (1977) constructs a voting power measure identical to the Shapley-Shubik index, based on what he terms the "homogeneity" assumption, as $\Pr(TC_i)$. He also constructs a Penrose-Banzhaf type measure under the "independence" assumption, and once again it is clearly constructed as $\Pr(TC_i)$.

4.5. **Summary.** There are different criticality sets and different probability assumptions that can be used when creating a voting power measure. Different notions of criticality can give rise to the same voting power measures because of the symmetry that exists in the single level approval game. So, if everything is the same, why does it matter what is being measured? Well, it transpires that as soon as you allow more than one level of positive approval to be expressed, i.e. any game with abstentions, things are no longer as straight forward.

Before going on to examine multilevel criticality in greater detail, the next section presents a few key definitions.

## 5. Definitions

In keeping with the concept of a mathematically light paper, a number of key definitions are given in this section using mathematically light syntax. The mathematically rigorous versions of these definitions, along with a number of other useful definitions, can be found in Appendix A.

A **generalised single choice weighted voting game** (GSCWVG) is a weighted voting game in which there are only two possible outcomes (think of them as 'winning' and 'losing'). The game is made up of a set of $N$ players. The approval level of a player $i$, denoted $\alpha_i$, is the fraction of its weight that it chooses to support the decision at hand with. If, after adding up all the support of the players, a predefined quota $q$ of the game is achieved the outcome is considered winning and the decision is approved. Otherwise the outcome is considered losing and the decision is rejected.

It may be possible for a player to express the same approval level in more than one way, for example, in some games, a player might express $\alpha_i = 0$ by voting 'no' or by abstaining. In order to cater for this eventuality, rather than consider the approval level directly, we will often consider that a player expresses a voting action $x_i \in \mathcal{X}_i$, where each $x_i$ is unique, but different $x_i$ can map onto the same $\alpha_i$.

A **discrete multilevel approval weighted voting game** (DSCWVG) is a generalised single choice weighted voting game in which the approval levels available to the players are drawn from a finite set. For example, a weighted voting game which allows the players to vote 'yes', 'no'' or 'abstain' can be considered as a discrete multilevel approval weighted voting game.

## 6. Multilevel Approval Criticality Diagrams

Despite having a name like multilevel approval criticality diagrams, which immediately conjure up an association with the ability to weight the game, they can, in fact, be used to analyse any type of generalised voting game, weighted or otherwise.

To help illustrate multilevel approval criticality diagrams we take an example $GSCWVG$ with three levels of approval, each player votes with an approval level $\alpha_i \in \{0, a, 1\}$. The actual numerical value of $a$ is unimportant for this analysis.

In the following an $\omega^{N\setminus\{i\}} \in \Omega^{N\setminus\{i\}}$ can be thought of as the multilevel equivalent of a coalition that excludes player $i$. In the single level case, notation such as $\omega \cup \{i\}$ was employed to represent a sink or source coalition. In the multilevel case, it is more appropriate to use the notation $\omega \times x_i$, where $x_i \in \mathcal{X}_i$, the set of possible voting actions player $i$ can take.

Source (Losing) Sink (Winning)

$$\begin{tikzcd}
 & \omega \times \{x_i : \alpha_i = a\} \\
\omega \times \{x_i : \alpha_i = 0\} \arrow[ur, "\alpha_i : 0 \to a"] \arrow[dr, "\alpha_i : 0 \to 1"'] & \\
 & \omega \times \{x_i : \alpha_i = 1\}
\end{tikzcd}$$

The previous diagram starts out with player $i$ expressing an approval level $\alpha_i = 0$ in the losing source event. This is transformed into two winning sink events when player $i$ expresses an $\alpha_i \neq 0$. This results in one losing event mapping onto two winning events.

Let's take a different $\omega^{N\setminus\{i\}} \in \Omega^{N\setminus\{i\}}$ that gives rise to the following diagram, mapping two losing events onto one winning event.

Source (Winning) Sink (Losing)

$$\begin{tikzcd}
 & \omega' \times \{x_i : \alpha_i = 0\} \\
\omega' \times \{x_i : \alpha_i = 1\} \arrow[ur, "\alpha_i : 1 \to 0"] \arrow[dr, "\alpha_i : 1 \to a"'] & \\
 & \omega' \times \{x_i : \alpha_i = a\}
\end{tikzcd}$$

Within single level approval, critical coalitions occurred in matched pairs, but with the addition of extra approval levels this is no longer true. This raises questions about which events should be 'counted' to create the power measure. But before those questions can be addressed, the very concept of criticality requires further attention.

## 7. Criticality Assumptions

In single level approval, criticality is taken to be the ability to change the outcome of an election by changing a player's vote from no to yes, or vice versa. With multilevel approval, when a player $i$ votes with $x_i = \{i_{min}\}$ they are saying no and when they vote with $x_i = \{i_{max}\}$ they are saying yes, but what about the levels in between? Just when exactly does it change from being a no vote to a yes vote? The answer to this question will depend upon the nature of the voting game, and will be represented by one of the following criticality assumptions.

(In the following discussions, instead of the more general voting concepts of $x_i = \{i_{min}\}$, the easier to understand concepts of $\alpha_i = 0$ are used. Also reference is made to the outcome $O$ of a game, which for ease of understanding can be considered as the outcome being winning).

7.1. **Criticality 0 - (No Means No).** In this assumption $\alpha_i = 0$ is a vote no and $\alpha_i \neq 0$ is a vote yes, with varying degrees of strength. There are sound reasons for believing this is a plausible assumption to make, especially whenever there is some element of cost involved in expressing approval. For example, most decision making bodies usually vote upon changing the status quo. As such, there is an element of personal risk involved in supporting the change. Should the change be implemented, and turn out to be less favourable, everyone who voted in favour, regardless of their approval level, will be held to account. The only ones to escape recrimination are those that voted with an approval level $\alpha_i = 0$. In other words, only the players that vote in favour of change take a risk. From this standpoint, the single level approval concept of criticality is extended from the change in vote $\alpha_i = 0$ to $\alpha_i = 1$, to the multilevel equivalent of $\alpha_i = 0$ to $\alpha_i \neq 0$ (and vice versa).

**Definition 7.1.** ***Criticality 0*** *- In a criticality* 0 *voting power measure one of the two events that define player $i$ as being critical must have player $i$ voting with its lowest possible approval. The set of criticality 0 increasing critical events for player $i$, with respect to an outcome $O$, is denoted by $O\_IC_i^0$, and the set of criticality 0 decreasing critical events for player $i$, with respect to an outcome $O$, is denoted by $O\_DC_i^0$.*

7.2. **Criticality $\delta$ - (I Just Don't No).** There is, of course, another way to look at things. For instance, pick an arbitrary point in the approval range and decide that anything less than this is a vote no, and anything greater is a vote yes. This raises questions about how you pick the point, what it means if you express an approval level neither above nor below the point and so on. All these questions are avoided by carefully choosing the definition of criticality under this assumption. Instead of looking at what happens to the outcome of the decision when a voter changes from voting no to yes, simply look at what happens when a voter either increases their approval level or decreases it.

**Definition 7.2.** ***Criticality*** $\delta$ *- In a criticality* $\delta$ *voting power measure there is no restriction on either of the events that define player* $i$ *as being critical. The set of criticality* $\delta$ *increasing critical events for player* $i$*, with respect to an outcome* $O$*, is denoted by* $O\_IC_i^\delta$*, and the set of criticality* $\delta$ *decreasing critical events for player* $i$*, with respect to an event* $O$ *is denoted by* $O\_DC_i^\delta$*.*

7.3. **Criticality 1 - (Yes Doesn't Mean No).** This is the mirror image of Criticality 0. In this assumption $\alpha_i = 1$ is a vote yes and everything else is a vote no. This assumption is presented here for completeness only and no power measures will be based upon it for two reasons. Firstly, no sound realistic reason can be thought of for this assumption to hold, as it would mean that the norm of the election body is to vote for change. Secondly, as this is the mirror image of Criticality 0, any Criticality 1 analysis can be carried out as Criticality 0. For example, a Criticality 1 analysis of the vote on "Should the airport be expanded?" is equivalent to a Criticality 0 analysis of the vote on "Should the airport not be expanded?"

## 8. Worked Example

In this section an example using multilevel criticality diagrams is given to emphasise the different notions of criticality.

A player $i$ in the game can express one of 10 possible approval levels. As in our previous examples $\omega^{N\setminus\{i\}} \in \Omega^{N\setminus\{i\}}$ denotes the multilevel equivalent of a coalition that excludes player $i$.

The table below lists the different variations of $\omega \times \{x_i\}$, in which $\alpha_i$ takes all of the 10 possible values. The table is arranged to show the events that are losing and those that are winning for this particular choice of $\omega$.

| Losing | Winning |
|---|---|
| $\omega \times \{x_i : \alpha_i = 0\}$ | $\omega \times \{x_i : \alpha_i = a_6\}$ |
| $\omega \times \{x_i : \alpha_i = a_1\}$ | $\omega \times \{x_i : \alpha_i = a_7\}$ |
| $\omega \times \{x_i : \alpha_i = a_2\}$ | $\omega \times \{x_i : \alpha_i = a_8\}$ |
| $\omega \times \{x_i : \alpha_i = a_3\}$ | $\omega \times \{x_i : \alpha_i = 1\}$ |
| $\omega \times \{x_i : \alpha_i = a_4\}$ | |
| $\omega \times \{x_i : \alpha_i = a_5\}$ | |

We now construct the criticality diagrams for this particular occurrence of $\omega$. In order to fully identify any of the criticality sets, and therefore create a voting power measure, we need to examine the diagrams for all possible $\omega^{N\setminus\{i\}} \in \Omega^{N\setminus\{i\}}$ and not just a single $\omega^{N\setminus\{i\}}$ as we do here.

8.1. **Criticality 0.** Criticality is defined as what happens when we go from expressing $\alpha_i = 0$ to $\alpha_i \neq 0$ or vice versa.

8.1.1. $IC_i^0$. This involves identifying all the losing critical events in which player $i$ has expressed an approval level $\alpha_i = 0$. There is only one such event $\omega \times \{x_i : \alpha_i = 0\}$, and below is the criticality diagram for it.

Source (Losing) — Sink (Winning)

$$\omega \times \{x_i : \alpha_i = 0\} \xrightarrow{\alpha_i : 0 \to a_6} \omega \times \{x_i : \alpha_i = a_6\}$$
$$\omega \times \{x_i : \alpha_i = 0\} \xrightarrow{\alpha_i : 0 \to a_7} \omega \times \{x_i : \alpha_i = a_7\}$$
$$\omega \times \{x_i : \alpha_i = 0\} \xrightarrow{\alpha_i : 0 \to a_8} \omega \times \{x_i : \alpha_i = a_8\}$$
$$\omega \times \{x_i : \alpha_i = 0\} \xrightarrow{\alpha_i : 0 \to 1} \omega \times \{x_i : \alpha_i = 1\}$$

Despite there being a number of transitions in this diagram, the losing critical event $\omega \times \{x_i : \alpha_i = 0\}$ occurs only once, therefore the contribution to the increasing criticality 'count' from this particular diagram is only 1.

8.1.2. $DC_i^0$. This involves identifying all the winning critical events in which player $i$ has expressed an approval level $\alpha_i \neq 0$, which become losing when $\alpha_i = 0$.

Source (Winning) — Sink (Losing)

$$\omega \times \{x_i : \alpha_i = a_6\} \xrightarrow{\alpha_i : a_6 \to 0} \omega \times \{x_i : \alpha_i = 0\}$$
$$\omega \times \{x_i : \alpha_i = a_7\} \xrightarrow{\alpha_i : a_7 \to 0} \omega \times \{x_i : \alpha_i = 0\}$$
$$\omega \times \{x_i : \alpha_i = a_8\} \xrightarrow{\alpha_i : a_8 \to 0} \omega \times \{x_i : \alpha_i = 0\}$$
$$\omega \times \{x_i : \alpha_i = 1\} \xrightarrow{\alpha_i : 1 \to 0} \omega \times \{x_i : \alpha_i = 0\}$$

This collection of diagrams contribute a total of 4 to the decreasing criticality 'count', as there are clearly four different source events, giving player $i$ the opportunity to influence the outcome in four different situations.

8.2. **Criticality $\delta$.** Under this assumption there is no restriction on the source events and the sink events. Criticality occurs when there is a change in outcome due to player $i$ increasing, or decreasing, its approval level.

8.2.1. $IC_i^{\delta}$. This involves identifying all the losing critical events in which player $i$ has expressed an approval level $\alpha_i = a_p$, and the event in which player $i$ expresses an approval level $\alpha_i = a_q$, where $a_p < a_q$, is winning. Below are the six criticality diagrams for this situation.

$$\begin{tikzcd}
\text{Source (Losing)} & \text{Sink (Winning)} \\
 & \omega \times \{x_i : \alpha_i = a_6\} \\
 & \omega \times \{x_i : \alpha_i = a_7\} \\
\omega \times \{x_i : \alpha_i = 0\} \arrow[uur, "\alpha_i : 0 \to a_6"] \arrow[ur, "\alpha_i : 0 \to a_7"] \arrow[dr, "\alpha_i : 0 \to a_8"'] \arrow[ddr, "\alpha_i : 0 \to 1"'] & \\
 & \omega \times \{x_i : \alpha_i = a_8\} \\
 & \omega \times \{x_i : \alpha_i = 1\}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Source (Losing)} & \text{Sink (Winning)} \\
 & \omega \times \{x_i : \alpha_i = a_6\} \\
 & \omega \times \{x_i : \alpha_i = a_7\} \\
\omega \times \{x_i : \alpha_i = a_1\} \arrow[uur, "\alpha_i : a_1 \to a_6"] \arrow[ur, "\alpha_i : a_1 \to a_7"] \arrow[dr, "\alpha_i : a_1 \to a_8"'] \arrow[ddr, "\alpha_i : a_1 \to 1"'] & \\
 & \omega \times \{x_i : \alpha_i = a_8\} \\
 & \omega \times \{x_i : \alpha_i = 1\}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Source (Losing)} & \text{Sink (Winning)} \\
 & \omega \times \{x_i : \alpha_i = a_6\} \\
 & \omega \times \{x_i : \alpha_i = a_7\} \\
\omega \times \{x_i : \alpha_i = a_2\} \arrow[uur, "\alpha_i : a_2 \to a_6"] \arrow[ur, "\alpha_i : a_2 \to a_7"] \arrow[dr, "\alpha_i : a_2 \to a_8"'] \arrow[ddr, "\alpha_i : a_2 \to 1"'] & \\
 & \omega \times \{x_i : \alpha_i = a_8\} \\
 & \omega \times \{x_i : \alpha_i = 1\}
\end{tikzcd}$$

| Source (Losing) | Sink (Winning) |
| --- | --- |

$$\omega \times \{x_i : \alpha_i = a_3\} \xrightarrow{\alpha_i : a_3 \to a_6} \omega \times \{x_i : \alpha_i = a_6\}$$
$$\omega \times \{x_i : \alpha_i = a_3\} \xrightarrow{\alpha_i : a_3 \to a_7} \omega \times \{x_i : \alpha_i = a_7\}$$
$$\omega \times \{x_i : \alpha_i = a_3\} \xrightarrow{\alpha_i : a_3 \to a_8} \omega \times \{x_i : \alpha_i = a_8\}$$
$$\omega \times \{x_i : \alpha_i = a_3\} \xrightarrow{\alpha_i : a_3 \to 1} \omega \times \{x_i : \alpha_i = 1\}$$

| Source (Losing) | Sink (Winning) |
| --- | --- |

$$\omega \times \{x_i : \alpha_i = a_4\} \xrightarrow{\alpha_i : a_4 \to a_6} \omega \times \{x_i : \alpha_i = a_6\}$$
$$\omega \times \{x_i : \alpha_i = a_4\} \xrightarrow{\alpha_i : a_4 \to a_7} \omega \times \{x_i : \alpha_i = a_7\}$$
$$\omega \times \{x_i : \alpha_i = a_4\} \xrightarrow{\alpha_i : a_4 \to a_8} \omega \times \{x_i : \alpha_i = a_8\}$$
$$\omega \times \{x_i : \alpha_i = a_4\} \xrightarrow{\alpha_i : a_4 \to 1} \omega \times \{x_i : \alpha_i = 1\}$$

| Source (Losing) | Sink (Winning) |
| --- | --- |

$$\omega \times \{x_i : \alpha_i = a_5\} \xrightarrow{\alpha_i : a_5 \to a_6} \omega \times \{x_i : \alpha_i = a_6\}$$
$$\omega \times \{x_i : \alpha_i = a_5\} \xrightarrow{\alpha_i : a_5 \to a_7} \omega \times \{x_i : \alpha_i = a_7\}$$
$$\omega \times \{x_i : \alpha_i = a_5\} \xrightarrow{\alpha_i : a_5 \to a_8} \omega \times \{x_i : \alpha_i = a_8\}$$
$$\omega \times \{x_i : \alpha_i = a_5\} \xrightarrow{\alpha_i : a_5 \to 1} \omega \times \{x_i : \alpha_i = 1\}$$

Again, despite there being a myriad of transitions, this collection of diagrams will only contribute 6 to the increasing criticality 'count'. There are clearly six different source events, giving player $i$ the opportunity to influence the outcome in six different situations.

8.2.2. $DC_i^\delta$. This involves identifying all the winning critical events in which player $i$ has expressed approval level $\alpha_i = a_q$, and the event in which player $i$ expresses an approval level $\alpha_i = a_p$, where $a_p < a_q$, is losing. Below are the four criticality diagrams for this situation.

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{c}\text{Source}\\ \text{(Winning)}\end{array} & \begin{array}{c}\text{Sink}\\ \text{(Losing)}\end{array} \\
& \omega \times \{x_i : \alpha_i = 0\} \\
& \omega \times \{x_i : \alpha_i = a_1\} \\
& \omega \times \{x_i : \alpha_i = a_2\} \\
\omega \times \{x_i : \alpha_i = a_6\} \arrow[ruuu, "\alpha_i : a_6 \to 0"] \arrow[ruu, "\alpha_i : a_6 \to a_1"] \arrow[ru, "\alpha_i : a_6 \to a_2"] \arrow[rd, "\alpha_i : a_6 \to a_3"'] \arrow[rdd, "\alpha_i : a_6 \to a_4"'] \arrow[rddd, "\alpha_i : a_6 \to a_5"'] & \\
& \omega \times \{x_i : \alpha_i = a_3\} \\
& \omega \times \{x_i : \alpha_i = a_4\} \\
& \omega \times \{x_i : \alpha_i = a_5\}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{c}\text{Source}\\ \text{(Winning)}\end{array} & \begin{array}{c}\text{Sink}\\ \text{(Losing)}\end{array} \\
& \omega \times \{x_i : \alpha_i = 0\} \\
& \omega \times \{x_i : \alpha_i = a_1\} \\
& \omega \times \{x_i : \alpha_i = a_2\} \\
\omega \times \{x_i : \alpha_i = a_7\} \arrow[ruuu, "\alpha_i : a_7 \to 0"] \arrow[ruu, "\alpha_i : a_7 \to a_1"] \arrow[ru, "\alpha_i : a_7 \to a_2"] \arrow[rd, "\alpha_i : a_7 \to a_3"'] \arrow[rdd, "\alpha_i : a_7 \to a_4"'] \arrow[rddd, "\alpha_i : a_7 \to a_5"'] & \\
& \omega \times \{x_i : \alpha_i = a_3\} \\
& \omega \times \{x_i : \alpha_i = a_4\} \\
& \omega \times \{x_i : \alpha_i = a_5\}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{c}\text{Source}\\ \text{(Winning)}\end{array} & \begin{array}{c}\text{Sink}\\ \text{(Losing)}\end{array} \\
& \omega \times \{x_i : \alpha_i = 0\} \\
& \omega \times \{x_i : \alpha_i = a_1\} \\
& \omega \times \{x_i : \alpha_i = a_2\} \\
\omega \times \{x_i : \alpha_i = a_8\} \arrow[ruuu, "\alpha_i : a_8 \to 0"] \arrow[ruu, "\alpha_i : a_8 \to a_1"] \arrow[ru, "\alpha_i : a_8 \to a_2"] \arrow[rd, "\alpha_i : a_8 \to a_3"'] \arrow[rdd, "\alpha_i : a_8 \to a_4"'] \arrow[rddd, "\alpha_i : a_8 \to a_5"'] & \\
& \omega \times \{x_i : \alpha_i = a_3\} \\
& \omega \times \{x_i : \alpha_i = a_4\} \\
& \omega \times \{x_i : \alpha_i = a_5\}
\end{tikzcd}
$$

Source (Winning) Sink (Losing)

$$\omega \times \{x_i : \alpha_i = 1\} \xrightarrow{\alpha_i : 1 \to 0} \omega \times \{x_i : \alpha_i = 0\}$$
$$\omega \times \{x_i : \alpha_i = 1\} \xrightarrow{\alpha_i : 1 \to a_1} \omega \times \{x_i : \alpha_i = a_1\}$$
$$\omega \times \{x_i : \alpha_i = 1\} \xrightarrow{\alpha_i : 1 \to a_2} \omega \times \{x_i : \alpha_i = a_2\}$$
$$\omega \times \{x_i : \alpha_i = 1\} \xrightarrow[\alpha_i : 1 \to a_3]{} \omega \times \{x_i : \alpha_i = a_3\}$$
$$\omega \times \{x_i : \alpha_i = 1\} \xrightarrow[\alpha_i : 1 \to a_4]{} \omega \times \{x_i : \alpha_i = a_4\}$$
$$\omega \times \{x_i : \alpha_i = 1\} \xrightarrow[\alpha_i : 1 \to a_5]{} \omega \times \{x_i : \alpha_i = a_5\}$$

Once again there are a large number of transitions, but this collection of diagrams will only contribute 4 to the decreasing criticality 'count'. There are only four different source events, giving player $i$ the opportunity to influence the outcome in only four different situations.

8.3. **Summary.** The number of transitions that occur within a set of criticality diagrams is unimportant to the resultant criticality measure ('count'). Only the number of situations in which it is possible for player $i$ to make a change to the outcome (the so called zone of influence) is important.

## 9. United Nations Security Council

In order to give some idea of the numerical differences that arise between the various notions of criticality, the example given in Felsenthal and Machover (1998) of voting in the United Nations Security Council (U.N.S.C.) is reworked below. It is important to stress that this example incorporates only one extra level of approval (abstention). The numerical differences between the different notions of criticality increase as the number of approval levels increase. Thus, in many ways, the U.N.S.C. example is at the lower end in terms of the numerical differences that can potentially arise.

In examining the voting powers of the members of the U.N.S.C. a uniform probability density function is assumed. In keeping with the approach taken in Freixas and Zwicker (2003), the decision process is modelled as a $DSCWVG$ in which a permanent member has weight 7 and a non-permanent member has weight 1. There are five permanent members and ten non-permanent members. The quota of the game is 39. The permanent member can express an approval level $\alpha_i \in \{0, \frac{6}{7}, 1\}$ and a non-permanent member can express an approval level $\alpha_i \in \{0, 0, 1\}$. (The approval level of 0 is repeated to highlight that a non-permanent member can abstain even though it has the same affect as voting no).

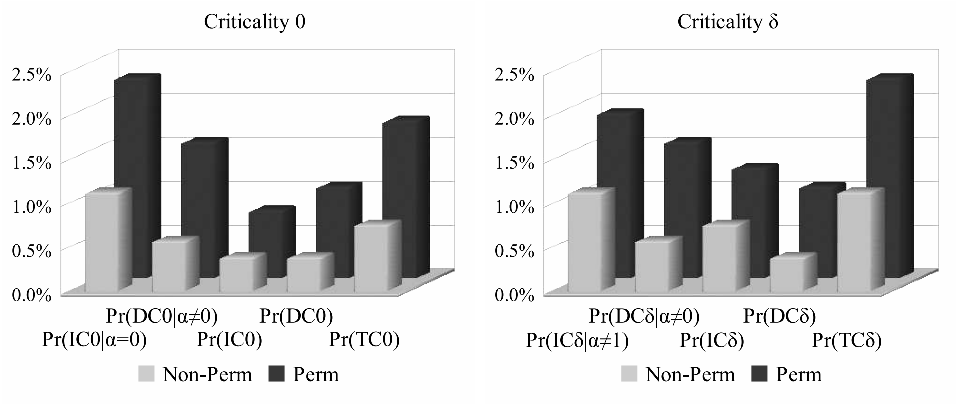


Figure 1. Voting Powers in the U.N.S.C. Game

(The data for the graphs can be found in Appendix B).

As you can see, the different criticality sets and assumptions give rise to a large number of different voting power measures, with significant differences in their results. Some of the measures vary by as much as a factor of six! No further comment is made on these results, other than to reinforce the point that different notions of criticality can not be ignored, or brushed under the carpet for calculative convenience.

## 10. Hazards of Ignoring the Different Notions of Criticality

It is difficult to consider a voting power measure in multilevel approval voting games without discussing the different criticality sets and criticality assumptions. After all, unlike single level approval, the different notions of criticality now give different results.

In order to conduct a criticality based voting power analysis of a voting game, first and foremost, an appropriate criticality assumption must be selected, then a decision must be made about which criticality set should be measured, before finally selecting an appropriate probability density function. Unfortunately, the majority of research in this area has concentrated almost exclusively on debating which probability density function is most appropriate (Banzhaf, 1965; Johnston, 1978; Laver, 1978; Paterson, 2004; Felsenthal and Machover, 1998; Leech, 2002b; Straffin, 1977). Arguably the least important factor of the voting power measure, as the probability density function is a property of the game to be analysed, and therefore not a "free choice".

Without a thorough understanding of the different notions of criticality there is a temptation to simply extrapolate the techniques used in single level approval voting to multilevel approval voting. This, however, is not without risks. For example in Felsenthal and Machover

(1998), by not defining strictly what they were calculating, it appears that they present $\Pr(IC_i^0)$ as the Banzhaf measure. The interpretation of the Banzhaf measure presented in this paper implies that it should be calculated as either $\Pr(TC_i^0)$ or $\Pr(TC_i^\delta)$. There may even be some merit in arguing that it can be calculated as the conditional probability $\Pr(IC_i^0|\{i_{min}\})$; but there is no evidence at all that the Banzhaf measure can be sensibly interpreted as the unconditional $\Pr(IC_i^0)$. To understand why, imagine increasing the number of possible approval levels such that, for all intents and purposes, it becomes a selection from a continuous range. Under the Banzhaf probability density function, this would result in $\Pr(\{i_{min}\}) \to 0$ and therefore send $\Pr(IC_i^0) \to 0$ as well, irrelevant of how influential a player is. (See Das (2008) for more details).

In Linder (2008), the author seems to be aware that multilevel approval critical events no longer occur in matched pairs. However, Linder seems to be unaware of the different notions of criticality discussed in this paper. Instead, Linder concentrates on calculating "common sense" extensions of single level power measures, without strong theoretical justification for doing so. At least Linder, in one instance, appears to be calculating the more appropriate conditional measure $\Pr(IC_i^0|\{i_{min}\})$. But why $\Pr(IC_i^0|\{i_{min}\})$? What about the other potential measures discussed in this paper?

And herein lies the crux of the problem, any kind of voting power analysis must make use of all the different notions of criticality, or give sound scientific justification for not doing so. Otherwise there is a risk the analysis will be discredited.

## 11. Which Criticality Measure Should Be Used?

So which notions of criticality should be adopted to analyse voting games? To quote the 1985 feature film, Brewster's Millions, "None of the above." Influence in a game should be measured using class conditional probabilities. As demonstrated in Das (2008), any measure based on criticality can be reduced to an expression involving class conditional probabilities. For example, the results below apply to almost all real life generalised single choice voting games. (Readers interested in the proofs should consult Das (2008) or one of the forthcoming, mathematically rigorous, publications).

$$\begin{aligned}
\Pr(IC_i^0|\{i_{min}\}) &= \Pr(\text{Winning}|\{i_{max}\}) - \Pr(\text{Winning}|\{i_{min}\}), \\
\Pr(DC_i^0) &= \Pr(\text{Winning}) - \Pr(\text{Winning}|\{i_{min}\}).
\end{aligned}$$

$$\begin{aligned}
\Pr(DC_i^\delta) &= \Pr(\text{Winning}) - \Pr(\text{Winning}|\{i_{min}\}), \\
\Pr(IC_i^\delta) &= \Pr(\text{Winning}|\{i_{max}\}) - \Pr(\text{Winning}).
\end{aligned}$$

With Pr(Winning), Pr(Winning$|\{i_{max}\}$) and Pr(Winning$|\{i_{min}\}$), almost all the criticality based voting power measures can be constructed. So why not analyse voting games using just these three probabilities instead? Anyone that wishes to construct a specific measure, based on their own choice of criticality set and assumption, can do so easily using the aforementioned probabilities.

## 12. Conclusion

In this paper an examination of criticality in voting games was carried out. A number of different criticality sets and criticality assumptions were introduced, all of which must be taken into account before a criticality based voting power analysis can be carried out.

It is of vital importance that we, as voting power researchers, must all strive to understand the different notions of criticality and their application. After all, if we can not rigorously define, and justify, our voting power measures, what hope do we have of convincing election designers to take our ideas seriously?

## Appendix A. Definitions

Despite promising a mathematically light paper, it would be remiss not to give rigorous definitions of voting games and the concepts of criticality. There is no disadvantage to the reader should they choose to skip the definitions and rely on the summary text found interlaced between them instead. In fact, it is positively encouraged, as this will ease the understanding of the main concepts and ideas presented in this paper without the risk of becoming bogged down in the mathematical theory.

### A.1. Generalised Voting Games.

**Definition A.1.** *A **player** is a probability space $(\mathcal{X}_i, \mathcal{A}_i, \mathbb{P}_i)$, where $\mathcal{X}_i$ is a set, $\mathcal{A}_i$ is a sigma-field of subsets of $\mathcal{X}_i$, and $\mathbb{P}_i$ is a countably additive, nonnegative measure with $\mathbb{P}_i(\mathcal{X}_i) = 1$. Given a set of $N$ players, where $|N| = n$, the set of all ordered $n$-tuples $(x_1, \dots, x_n)$, with $x_j \in \mathcal{X}_j$ for each $j \in 1, \dots n$ is denoted as $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ and abbreviated to $\Omega^N$. Given a player $i$, the set of all ordered $(n-1)$-tuples $(x_1, \dots, x_{i-1}, x_{i+1}, x_n)$, with $x_j \in \mathcal{X}_j$ for each $j \in 1, \dots, i-1, i+1, \dots n$ is denoted as $\mathcal{X}_1 \times \cdots \times \mathcal{X}_{i-1} \times \mathcal{X}_{i+1} \times \cdots \times \mathcal{X}_n$ and abbreviated to $\Omega^{N\setminus\{i\}}$. The action of creating a single $(n-1)$-tuple, denoted as $\omega^{N\setminus\{i\}}$, from a single $n$-tuple $\omega^N$ by removing the element $x_i$ is represented as $\omega^N \setminus x_i$. The action of creating a single $n$-tuple, denoted as $\omega^N$, from a single $(n-1)$-tuple $\omega^{N\setminus\{i\}}$ by adding an element $x_i \in \mathcal{X}_i$ is represented as $\omega^{N\setminus\{i\}} \times x_i$.*

The key concepts from the previous definition are; a player $i$ can vote by expressing one of $x_i \in \mathcal{X}_i$, $\omega^N$ represents a voting configuration (an event) with $|N|$ players, $\omega^N \setminus x_i$ represents a voting configuration with player $i$ removed (when it was previously expressing $x_i$), and $\omega^{N\setminus\{i\}} \times x_i$ represents a voting configuration in which player $i$ has joined by expressing $x_i$.

**Definition A.2.** *Given a set of $N$ players, where $|N| = n$, a set of the form $A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n : x_i \in A_i$ for each $i\}$, with $A_i \in \mathcal{A}_i$ for each $i$ is called a measurable rectangle. The product sigma field $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is defined to be the sigma field generated by all measurable rectangles. Let the product space $(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)$ be denoted as $(\Omega, \mathcal{F})$.*

**Definition A.3.** *A **generalised voting game** is a quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ such that $(\Omega, \mathcal{F}, \mathbb{P})$ is the product space generated by a set of $N$ players, $\mathbb{P}$ is the product measure and $\mathcal{W}$ is a measurable function on the subsets of $\Omega$. Such a game is denoted as a* **GVG**$(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$.

A generalised voting game is a catch all definition that encompasses almost all the different voting games that you can possibly think of. With this definition in place it is possible to define mathematically the concepts of increasing, decreasing and total criticality introduced in Section 3.

A.2. **Criticality Sets.**

**Definition A.4.** *For a $GVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ with a set of possible outcomes $\mathcal{Y}$, a player $i$, and an $\omega^N \in \Omega^N$. A player $i$ is **increasingly critical** with respect to an outcome $O \in \mathcal{Y}$ in an event $\omega^N$ if, and only if, $\mathcal{W}(\omega^N) \neq O$ and there exists an $x_i \in \mathcal{X}_i$ such that $\mathcal{W}(\omega^{N\setminus\{i\}} \times x_i) = O$. Let the set $O\_IC_i$ represent the set of increasingly critical events for a player $i$ with respect to an outcome $O$.*

Increasing criticality is a measure of a player's ability to get the outcome they want.

**Definition A.5.** *For a $GVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ with a set of possible outcomes $\mathcal{Y}$, a player $i$, and an $\omega^N \in \Omega^N$. A player $i$ is **decreasingly critical** with respect to an outcome $O \in \mathcal{Y}$ in an event $\omega^N$ if, and only if, $\mathcal{W}(\omega^N) = O$ and there exists an $x_i \in \mathcal{X}_i$ such that $\mathcal{W}(\omega^{N\setminus\{i\}} \times x_i) \neq O$. Let the set $O\_DC_i$ represent the set of decreasingly critical events for a player $i$ with respect to an outcome $O$.*

Decreasing criticality is a measure of a player's ability $i$ to prevent an outcome they don't want.

**Definition A.6.** *For a $GVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ with a set of possible outcomes $\mathcal{Y}$, a player $i$, and an $\omega^N \in \Omega^N$. A player $i$ is **totally critical** with respect to an outcome $O \in \mathcal{Y}$ in an event $\omega^N$ if it is either increasingly critical or decreasing critical with respect to the aforementioned outcome and event. Let the set $O\_TC_i$ represent the set of total critical* *events for a player $i$ with respect to an outcome $O$. In an event $\omega^N$, it is not possible to be both simultaneously increasingly and decreasing critical with respect to an outcome $O$, therefore $(O\_IC_i \cap O\_DC_i) = \emptyset$.*

Total criticality is a measure of the players zone of influence with respect to a specified outcome.

(Whenever there is no possibility of confusion the sets $O\_IC_i$, $O\_DC_i$ and $O\_TC_i$ will be denoted as $IC_i$, $DC_i$ and $TC_i$ respectively).

### A.3. **Voting Power Measures.**

**Definition A.7.** *For a $GVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ with an outcome $O$ and a player $i$, a* ***voting power measure*** *is defined as,*

$$\int g_i(\omega^N) \;\; \mu(d\omega^N).$$

*where $g_i$ is a $\mathcal{F} \setminus \mathcal{B}(\overline{\mathbb{R}})$ measurable function, $\mathcal{B}(\overline{\mathbb{R}})$ is the Borel sigma field on the set of reals along with the two singletons $\{-\infty\}$ and $\{\infty\}$, and $\mu$ is the sigma finite or sigma finite marginal measure over the applicable events. If the integration is carried out over all $\omega^N \in \Omega^N$ then it is common to take $\mu = \mathbb{P}$.*

A voting power measure is defined as the integration of a integrable function over the events in a voting game. Different voting power measures are created by integrating different functions.

**Definition A.8.** *A* ***criticality based voting power measure*** *is a voting power measure in which $g_i(\omega^N)$ is a simple function; indicating the conditional classification of an event as outcome $O$, conditional upon the behaviour of a player $i$.*

A.3.1. *Comment.* In Section 2, techniques were given for calculating power measures in simple 'yes/no' voting games. The process essentially involves identifying a critical coalition and then adding something to a 'running count'. If $\dfrac{1}{2^{|N|-1}}$ and $\dfrac{(|C|-1)!(|N|-|C|)!}{|N|!}$ are interpreted as $\mu(w)$ for the Banzhaf/Penrose measure and Shapley-Shubik index respectively, then it is clear that they are both criticality based voting power measures.

Furthermore, since it was suggested that the same $g(\mathit{criticality})$ function can be used to identify critical coalitions in both measures, it follows that they are the same criticality based voting power measure, albeit assuming a different probability distribution in the underlying events.

### A.4. **Single Choice Voting Games.**

**Definition A.9.** *A **generalised single choice voting game**, denoted as* $\mathbf{GSCVG}(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$*, is a* $GVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ *in which* $\mathcal{W} : \Omega \to \{\{0\}, \{1\}\}$*. An event* $\omega \in \Omega$ *is labelled a **winning event** if and only if* $\mathcal{W}(\omega) = 1$*, it is labelled a **losing event** otherwise.*

A generalised single choice voting game is a game in which there are only two possible outcomes. Think of them as being winning and losing.

**Definition A.10.** *A **generalised single choice monotone voting game**, denoted as* $\mathbf{GSCMVG}(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$*, is a* $GSCVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ *in which* $\mathcal{W}$ *is a monotonically increasing measurable function on the subsets of* $\Omega$*.*

A generalised single choice monotone voting game is monotonic voting game in which there are only two possible outcomes. The monotonicity of the $GSCMVG$ refers to the monotone nature of the measurable function $\mathcal{W}$. It means that if a subset of the players have formed a winning event it will remain winning if any of the players decide to increase their support.

**Definition A.11.** *The Heaviside step function* $\mathrm{H} : \mathbb{R} \to \{0, 1\}$ *with*

$$\mathrm{H}(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

**Definition A.12.** *A **generalised single choice weighted voting game**, denoted as* $\mathbf{GSCWVG}(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$*, is a* $GSCMVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ *in which every player* $(\mathcal{X}_i, \mathcal{A}_i, \mathbb{P}_i)$ *has an associated measure space* $(\mathcal{Y}_i, \mathcal{B}_i, \lambda_i)$*, with* $\mathcal{Y}_i \subset \mathbb{R}^+$*, and a* $\mathcal{A}_i \setminus \mathcal{B}_i$ *measurable function* $\mathrm{M}_i$*. A minimal element of* $\mathcal{Y}_i$ *is* $0$ *and a maximal element of* $\mathcal{Y}_i$ *is denoted as* $w_i$*, the **weight** of player* $i$*. Given an* $x_i \in \mathcal{X}_i$*, the expression* $\dfrac{\mathrm{M}_i(x_i)}{w_i}$ *is called the **approval level** of player* $i$ *and is denoted by* $\alpha_i$*. Furthermore, the measurable function* $\mathcal{W}$ *can be expressed using the Heaviside step function as follows,*

$$\mathcal{W}(\omega^N) = \mathrm{H}\left(\sum_{i \in N} (\alpha_i w_i) - q\right).$$

*where* $q$ *is called the **quota** of the game.*

The previous definition is an overly complex one, but put simply it can be expressed thus. There is a set of $n$ players. The approval level of a player $i$ is the fraction of its weight that it chooses to support the decision at hand with. If, after adding up all the support of the players, a predefined quota $q$ of the game is achieved the outcome is considered winning and the decision is approved. Otherwise the outcome is considered losing and the decision is rejected.

**Definition A.13.** *A **discrete multilevel approval weighted voting game** is a $GSCWVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ in which the approval levels available to all players are countable. Such a game is denoted as a* $\mathbf{DSCWVG}(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$.

Think of a discrete multilevel approval weighted voting game as a game in which the approval levels available to the players are drawn from a finite set. For example, a weighted voting game which allows the players to vote 'yes', 'no'' or 'abstain' can be considered as a discrete multilevel approval weighted voting game.

**Definition A.14.** *A **continuous multilevel approval weighted voting game** is a $GSCWVG(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$ in which the approval levels available to all players are uncountable. Such a game is denoted as a* $\mathbf{CSCWVG}(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W})$.

For example, a weighted voting game in which the players are allowed to express approval levels in the continuous range $[0, 1]$ can be considered as a continuous multilevel approval weighted voting game.

Even though abstentions have not been mentioned explicitly, in Freixas and Zwicker (2003) it was argued that abstention could be represented as a level of approval. In Das (2008) the same result was independently produced along with a method for calculating the actual approval level expressed when a player abstains.

A.4.1. *Comment.* The approval level of a player $i$ ($\alpha_i$) is the fraction of its weight that it chooses to support the decision at hand with. By Definition A.12, the approval level must be in the range $[0, 1]$. A game which allows negative approval levels can still be modelled as a $GSCWVG$, providing the weights and the quota are adjusted appropriately. For example, to represent a game with quota 0, in which a player can express support in the range $[-5, 5]$, the $GSCWVG$ will set the weight of the player to 10 and raise the quota by an appropriate amount to compensate (i.e. if there is only one player in the game, the quota will be 5).

There is however one slight caveat. It is a subtle but crucial point to bear in mind. If the voters believe that they have the ability to express a negative approval, then an approval level $\alpha_i$ close to zero should be thought of as a relatively strong vote no, instead of a relatively weak vote yes. This subtle difference in interpretation makes no difference to either the outcomes or analysis of the games in question, but it will have an affect on which of the criticality sets and assumptions are best suited to create power measures for them. This idea was explored in Section 7.

## Appendix B. United Nations Security Council Results

The tables below present the results of the analysis of the U.N.S.C. game.

| Voter | $\Pr(IC_i^0 \mid (\alpha_i = 0))$ | $\Pr(DC_i^0 \mid (\alpha_i \neq 0))$ | $\Pr(IC_i^0)$ | $\Pr(DC_i^0)$ | $\Pr(TC_i^0)$ |
|---|---|---|---|---|---|
| Perm | 2.27% | 1.53% | 0.76% | 1.02% | 1.78% |
| Non-Perm | 1.11% | 0.56% | 0.37% | 0.37% | 0.74% |

Table 1. Criticality 0 measures for U.N.S.C.

| Voter | $\Pr(IC_i^\delta \mid (\alpha_i \neq 1))$ | $\Pr(DC_i^\delta \mid (\alpha_i \neq 0))$ | $\Pr(IC_i^\delta)$ | $\Pr(DC_i^\delta)$ | $\Pr(TC_i^\delta)$ |
|---|---|---|---|---|---|
| Perm | 1.86% | 1.53% | 1.24% | 1.02% | 2.27% |
| Non-Perm | 1.11% | 0.56% | 0.74% | 0.37% | 1.11% |

Table 2. Criticality $\delta$ measures for U.N.S.C.

Firstly a few of comments about the numbers. Despite common practice, the results have not been normalised. This is because normalising, or producing an index as some prefer to call it, achieves nothing more than the destruction of valuable information.

Secondly, the numbers in the tables are fairly small. This is simply because notions of criticality are inextricably linked to the unconditional probability of a event being winning. In fact, a criticality diagram analysis tells us that $|DC_i| \leq |\text{Winning}|$ and $|IC_i| \leq |\text{Winning}| \times (|\mathcal{X}_i| - 1)$. In the U.N.S.C. game, $\Pr(\text{Winning}) = 1.02\%$, which explains why the numbers are so small.

And finally, even though the unconditional probabilities of $\Pr(IC_i)$ and $\Pr(DC_i)$ are presented, it is not envisaged anyone would use them as a sensible measure of power for obvious reasons.

## References

Banzhaf, J. (1965). Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–342.

Banzhaf, J. (1968). One man, 3.312 votes: A mathematical analysis of the electoral college. *Villanova Law Review*, 13:304–332.

Das, S. (2008). *Class Conditional Voting Probabilities*. PhD thesis, Birkbeck College, University of London.

Felsenthal, D. and Machover, M. (1998). *The Measurement of Voting Power*. Edward Elgar.

Felsenthal, D. and Machover, M. (2000). Enlargement of the EU and weighted voting in its Council of Ministers. Technical report, London School of Economics.

Felsenthal, D. and Machover, M. (2001). The Treaty of Nice and Qualified Majority Voting. Technical report, London School of Economics.

Freixas, J. and Zwicker, W. (2003). Weighted voting, abstention and multiple levels of approval. *Social Choice and Welfare*, 21:399–431.

Gelman, A., Katz, J., and Bafumi, J. (2003a). Standard voting power indexes don't work: An empirical analysis. Social science working paper, California Institute of Technology.

Gelman, A., Katz, J., and Tuerlinckx, F. (2003b). The mathematics and statistics of voting power. Social Science Working Paper 1141, California Institute of Technology.

Johnston, R. (1978). On the measurement of power: Some reactions to Laver. *Environment and Planning A*, 10:907–914.

Laver, M. (1978). On Johnston on Laver on Johnston. *Environment and Planning A*, 10:959–960.

Leech, D. (2002a). Designing the voting system for the EU Council of Ministers. *Public Choice*, 113:437–464.

Leech, D. (2002b). Shareholder voting power and corporate governance: A study of large British companies. *Homo Oeconomicus*, XIX(3):345–373.

Leech, D. (2003a). *Competition, Monopoly and Corporate Governance, Essays in Honour of Keith Cowling*, chapter Incentives to Corporate Governance Activism. Edward Elgar.

Leech, D. (2003b). Computing power indices for large voting games. *Management Science*, 49:831–838.

Leech, D. (2003c). Power indices as an aid to institutional design: the generalised apportionment problem. *Yearbook on New Political Economy*.

Linder, I. (2008). A special case of Penrose's limit theorem when abstention is allowed. *Theory and Decision*.

Mann, I. and Shapley, L. (1960). Values of larges games IV: Evaluating the Electoral College by Montecarlo Techniques. Technical Report RM-2651, The Rand Corporation, Santa Monica.

Paterson, I. (2004). Voting rules anchored but voting power adrift - logical probability to the rescue in Europe. In *Workshop on Voting and Collective Decision Making*. Institute for Advanced Studies, Vienna.

Paterson, I. (2005). A lesser known probabilistic approach to the Shapley-Shubik index and useful related voting measure. In *EPCS 2005*.

Penrose, L. (1946). The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, 109:53–57.

Saari, D. (2001). *Decisions and Elections*. Cambridge Univeristy Press.

Shapley, L. and Shubik, M. (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48:787–792.

Straffin, P. (1977). Homogeneity, independence and power indices. *Public Choice*, 30:107–118.

Straffin, P. (1978). *Game Theory and Political Science*, chapter Probability models for power indices, pages 477–510. New York University Press.