The Value of Research Based on Simple Assumptions about Voters' Preferences*

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(*Draft*)

Abstract

Early research on the general topic of the probability that various paradoxical election outcomes might be observed was typically based on very simple models to describe the likelihood that voters might have different preference rankings on the available candidates. The three primary assumptions of this type are the Dual Culture Condition, the Impartial Culture Condition and the Impartial Anonymous Culture Condition. Research that is based on these assumptions continues to be considered in the literature, despite the fact that they are often criticized by empirically focused researchers for not reflecting realistic voting scenarios. The objective of this study is to clarify the intent and value of the basic research that is based on these simple assumptions. While no claim has ever been made that any of these assumptions reflect realistic scenarios, they still do add very significant value to research on the probability that various paradoxical election outcomes might be observed. In particular, they show that most extreme voting paradoxes should be expected to be rare events.

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1 Introduction

Many people have found it to be very interesting to think about strange and counterintuitive outcomes that might possibly be observed when a group of voters takes on the task of selecting a winning candidate from a set of available candidates. Books have been written to describe many of these paradoxical outcomes and to categorize them according to the types of unusual behaviors that they display. The categories of voting paradoxes that are defined by Nurmi (1999) are used in this current study. The most famous of these paradoxical voting outcomes is Condorcet's Paradox, or the Condorcet Effect, which is named after the renowned 18th century French mathematician-philosopher who formally described this phenomenon. We address this particular voting paradox at this point, so that it can be used as a basis for further discussion. Other voting paradoxes will be developed in detail later in the study.

A description of the phenomenon that is known as Condorcet's Paradox begins with the definition of a given possible combination of voters' preferences for three candidates $\{A,B,C\}$ in an election. Voters are assumed to have complete and rational preference rankings on the candidates, and the six possible preference rankings that voters might have on the three candidates are listed in Figure 1.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Figure 1. A voting situation from a three-candidate election.

The n_i terms in Figure 1 denote the number of voters who have the associated preference rankings on the candidates. That is, n_3 voters have Candidate B as most preferred, Candidate C as least preferred, and Candidate A as middle-ranked. With a total of n voters, $n = \sum_{i=1}^{6} n_i$. A *voting situation* is denoted by $\boldsymbol{n}$, and it defines a specific combination of voters' preference rankings, $(n_1, n_2, n_3, n_4, n_5, n_6)$, that could be observed.

Borda (1784) developed a procedure that extends the basic principle of majority rule in two-candidate elections to scenarios that involve three candidates, by looking at the basic majority rule relation as applied to pairs of candidates. Let AMB denote the event that Candidate *A* defeats *B* by *Pairwise Majority Rule* (*PMR*) when only Candidates *A* and *B* are considered. By ignoring the relative position of *C* in the possible preference ranking for any of the individual voter's rankings in Figure 1, we see that ***AMB*** if $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$. By using the same basic logic, we then find that ***AMC*** if $n_1 + n_2 + n_3 > n_4 + n_5 + n_6$ and ***BMC*** if $n_1 + n_3 + n_5 > n_2 + n_4 + n_6$. Candidate *A* would be the *Pairwise Majority Rule Winner* (*PMRW*) if both ***AMB*** and ***AMC***, which would make it an exceptionally good candidate for selection as the most preferred candidate according to the voters' preference rankings in the associated voting situation. The *Pairwise Majority Rule Loser* (*PMRL*) is then defined in the obvious way.

Condorcet (1785a) makes very strong arguments that the PMRW should always be chosen as the winner of an election, which has resulted in this principle being commonly referred to as the *Condorcet Criterion*. But, Condorcet then continued on with an analysis of PMR relationships to make a fascinating discovery with his famous example of a voting situation with 60 voters on three candidates, as shown in Figure 2:

$$\begin{array}{ccccc} A & B & B & C & C \\ B & A & C & A & B \\ C & C & A & B & A \\ n_1 = 23 & n_3 = 2 & n_4 = 17 & n_5 = 10 & n_6 = 8. \end{array}$$

Figure 2. A voting situation showing a PMR cycle from Condorcet (1785a)

Condorcet notes that we have a voting situation in this example that results in what he called a "*contradictory system*", and has come to be widely known as *Condorcet's Paradox*. In particular, we find that by using PMR comparisons with the voting situation in Figure 2: ***AMB*** (33-27), ***BMC*** (42-18), and ***CMA*** (35-25). We therefore have a cycle in the PMR relations on the three candidates, so that no candidate emerges as being superior to each of the remaining candidates. Given Condorcet's strong arguments that the PMRW should always be selected as the winner, we are left with a difficult question in this case. In particular, "Which candidate should be selected as the winner?" No matter which candidate we select in this example, a majority of voters would prefer some other candidate for selection.

Condorcet wrote at length about the possibility that these cyclical majorities on pairs of candidates might occur, and he made some attempts to assess the likelihood that such outcomes might happen [Condorcet (1785b, 1785c)]. A great deal of effort has been expended since Condorcet's early work to identify other voting paradoxes and to obtain probability representations for the likelihood that various voting paradoxes, including Condorcet's Paradox, might be observed in election settings. The basic motivation behind this work has been to determine if any of these possible paradoxical events might actually pose a real threat to the stability of elections. There have been significant advances in recent years in the modeling techniques that have been employed to develop these probability representations, and our objective here is to survey some of the results that have been obtained from the most elementary models that have been used. The primary goal is to show that significant results can indeed be obtained with analysis that is based on these basic models, and we focus on outcomes for three-candidate elections.

2 Calculating Probabilities for Observing Voting Paradoxes

The general procedure for calculating the probability that a voting paradox might be observed is quite direct. If we consider the example of Condorcet's Paradox from the immediately preceding section, it is sufficient to enumerate all possible voting situations for a specified *n*, and identify the subset of all possible voting situations for which a PMR cycle exists. Then, the probability of observing Condorcet's Paradox would be obtained by summing the probabilities that the voting situations in that subset will be observed. The outcome will obviously be completely driven by the specific mechanism that determines the probability with which each specific voting situation is observed.

Three probability models have formed the bulk of the traditional basis for assigning probabilities to voting situations: the Dual Culture Condition (DC), the Impartial Culture Condition (IC) and the Impartial Anonymous Culture Condition (IAC). We begin by describing each of these models. While doing this, some subtle differences between these models will be pointed out, along with the resulting impact that these differences will have on the characteristics of the voting situations that are obtained from them. Once that is complete, these models will be analyzed to determine what they can tell us about the probability that a number of different voting paradoxes might be observed.

2.1 Dual Culture Condition

Specific voting situations are not obtained directly with the DC model. Instead, DC describes the probability that specific *voter preference profiles*, or profiles, will be observed. A voter preference profile identifies the specific preference ranking that each voter has for the candidates, so that voters' preferences are not anonymous in a profile. However, once a profile has been established, it is easy to accumulate the voters' preferences according to the possible preference rankings to obtain the associated voting situation for that profile. Since the preferences of specific voters can not be identified in a voting situation, voters' preferences are anonymous in a voting situation.

The probability that any specific voter preference profile will be observed can be considered to be the result of a process that randomly generates n individual voter's preference rankings on the candidates. In this situation, we let $\boldsymbol{p}$ denote a six-dimensional vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ for the three-candidate case, where p_i denotes the probability that a randomly selected voter from the population of potential voters will have the corresponding preference ranking on candidates that is shown in Figure 3.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
p_1	p_2	p_3	p_4	p_5	p_6

Figure 3. Voter preference ranking probabilities for a three-candidate election.

That is, a randomly selected voter will have a probability p_1 of having the linear preference ranking with Candidate A being most preferred, Candidate C being least preferred and Candidate B being middle-ranked.. A very critical assumption is made at this point, in that each voter's preference ranking on candidates is assumed to be arrived at independently of the preferences of all other voters.

Following the traditional methods that are used in any analysis of this type of probability modeling, we start with an urn that contains some total number of balls, with each ball being one of six different colors. Each color corresponds to one of the six possible preference rankings on the three candidates. The proportions of the total number of balls of each color that are in the urn are equal to their associated probabilities for the specified $\boldsymbol{p}$. Then, balls are sequentially drawn at random from the urn over n different trials, with the selected ball being returned to the urn after its color is noted on each draw. The random selection of balls is being done with replacement during the experiment so that the probability of observing any particular possible preference ranking for an individual voter does not change from draw to draw. The color of the ball that is drawn during the i^{th} step of this sequential drawing is used to assign the associated preference ranking to the i^{th} voter before the ball is placed back in the urn.

As noted previously, the voting situation, $\boldsymbol{n}$, that results from any given voter preference profile with its identifiable voters can be obtained simply by determining the number of voters in the voter preference profile that have each of the six possible preference rankings. The probability that any given $\boldsymbol{n}$ will be observed from the identifiable voters in such a randomly generated voter preference profile with this urn model is then given directly by the multinomial probability $n! \prod_{i=1}^{6} \frac{p_i^{n_i}}{n_i!}$.

The DC assumption represents a special case of $\boldsymbol{p}$ vectors such that the probability that a randomly selected voter will have any preference ranking on the candidates is the same as the probability that the same voter will have the dual, or inverted, preference ranking on the candidates, with $p_1 = p_6$, $p_2 = p_5$ and $p_3 = p_4$ in Figure 3. In order to describe the context of DC, let $A \succ B$ denote the outcome that Candidate A is preferred to B in a specific voter's preference ranking on candidates.

Let $\Delta(A,B)$ denote the difference between the sum of the p_i values for preference rankings with $A \succ B$ and $B \succ A$. The same definition is extended in the obvious fashion to all pairs of candidates, so that

$$\begin{aligned}\Delta(A,B) &= p_1 + p_2 + p_4 - p_3 - p_5 - p_6 \\ \Delta(A,C) &= p_1 + p_2 + p_3 - p_4 - p_5 - p_6 \\ \Delta(B,C) &= p_1 + p_3 + p_5 - p_2 - p_4 - p_6 .\end{aligned} \tag{1}$$

When each voter's preference ranking is independent of all of the other voters' preference rankings, a randomly selected voter will be more likely to have a preference ranking with $A \succ B$ than with $B \succ A$ whenever $\Delta(A,B) > 0$.

The law of large numbers requires that a randomly generated voting situation with $n \to \infty$ must have $A\boldsymbol{M}B$ if $\Delta(A,B) > 0$ for any pair of candidates like A and B. As a result, Candidate A will be the PMRW with probability approaching one whenever both $\Delta(A,B) > 0$ and $\Delta(A,C) > 0$, B will be the PMRW with probability approaching one if both $\Delta(A,B) < 0$ and $\Delta(B,C) > 0$, and C will be the PMRW with probability approaching one if both $\Delta(A,C) < 0$ and $\Delta(B,C) < 0$. There will be a PMR cycle $A\boldsymbol{M}B\boldsymbol{M}C\boldsymbol{M}A$ with probability approaching one with $n \to \infty$ if each of $\Delta(A,B) > 0$, $\Delta(B,C) > 0$ and $\Delta(A,C) < 0$, and the reverse PMR cycle with $A\boldsymbol{M}C\boldsymbol{M}B\boldsymbol{M}A$ will exist with probability approaching one if each of $\Delta(A,C) > 0$, $\Delta(B,C) < 0$ and $\Delta(A,B) < 0$.

By selectively constructing $\boldsymbol{p}$ to define the likelihood that randomly generated voting situations are observed from the urn experiment described above, it is easy to contrive situations as $n \to \infty$ for which either a PMRW must exist with near certainty, or for which a PMR cycle must exist with near certainty. However, there is a *complete balance* over all pairs of candidates on an expected value basis for a randomly selected voter when $\Delta(A,B) = \Delta(A,C) = \Delta(B,C) = 0$. When some $\boldsymbol{p}$ has this complete balance, it is neither intentionally forcing a PMRW to exist nor intentionally forcing a PMR cycle to exist. Moreover, such a complete balance of individual voter's preferences on all pairs of candidates only exists with $\boldsymbol{p}$ vectors that meet the restriction of DC.

All of this leads to the conclusion that any results that are obtained with the assumption of DC represent a somewhat extreme case in which no candidate has any expected advantage whatsoever when the preferences on *pairs of candidates* are examined for a voter that is randomly selected from the population of voters. It is very important to emphasize that this balance of preferences applies to *individual voter's preferences on pairs of candidates* with DC. It does not preclude the possibility that some candidates might be ranked as most preferred, or least preferred, with greater likelihood than some other candidate in the preference ranking of a randomly selected voter. For example, DC applies to the particular case with $p_1 = p_6 = 1/2$ and $p_2 = p_3 = p_4 = p_5 = 0$, so that Candidates A and C must always be ranked as either most preferred, or least preferred, in a randomly selected voter's preference ranking, while *B* must always be the middle-ranked candidate.

Another observation from the analysis of the assumption of complete balance of individual voter's preferences follows from a consideration of the resulting proportions of voters with preferences on pairs of candidates in a voting situation. That is, the proportion of voters with $A \succ B$ in a random voting situation will approach one-half with certainty as $n \to \infty$ if $\Delta(A,B)=0$. The relative margins of all PMR wins and losses on pairs of candidates in voting situations will therefore be relatively small with a complete balance of preferences for individual voters. As a result, this will lead to an environment that is conducive to the occurrence of voting paradoxes that involve PMR cycles in voting situations. When the assumption of DC is being utilized, it can therefore be expected that exaggerated estimates will be obtained for the likelihood that voting paradoxes that involve PMR cycles will be observed in the resulting voting situations. But, it is important to stress that the DC assumption is neither forcing a PMRW to exist nor forcing a PMR cycle to exist as $n \to \infty$.

2.2 Impartial Culture Condition

The *Impartial Culture Condition* (*IC*) is a refinement of DC which assumes that $p_i = 1/m!$ in an *m*-candidate election, so that each possible preference ranking on the candidates is equally likely to represent the preferences of a randomly selected voter. Since IC is a special case of DC, the preferences of any given voter are assumed to be independent of all other voters' preferences, and there is a complete expected balance of preferences on pairs of candidates for a randomly selected voter. The additional restriction of IC beyond DC requires that there is also a complete balance on the expected ranking position for all candidates, so that all candidates are equally likely to be most preferred, least preferred or middle ranked for a randomly selected voter. All of these assumptions make IC the 'purest' assumption, since no candidate will have any advantage whatsoever when it is compared to any other candidates in the preference rankings of a randomly selected voter.

2.3 Impartial Anonymous Culture Condition

The *Impartial Anonymous Culture Condition* (*IAC*) is not based on the use of any particular ***p*** to generate a random voter preference profile that will then be used to obtain a random voting situation. Instead, the concept of IAC is based directly on the assumption that each possible voting situation with n voters is equally likely to be observed. IAC also produces an expected balance of preferences on pairs of candidates. However, this balance does not apply to the preferences of specific individual voter's with IAC, it applies over all possible voting situations with anonymous voters. This balance follows from partitioning the set of all possible voting situations into pairs. To form a pair of voting situations in the partition, each voting situation is matched with the unique voting situation that interchanges voter preference rankings according to: $n_1 \leftrightarrow n_6$, $n_2 \leftrightarrow n_5$, and $n_3 \leftrightarrow n_4$.

This transformation matches every voting situation with its *dual voting situation*, which effectively reverses the preference ranking on candidates for every voter. Thus, for any two candidates, *A* and *B*, the number of voters with $A \succ B$ in one of the voting situations will have the same number of voters with $B \succ A$ in the matching voting situation in the partition. Since both voting situations are equally likely to be observed under IAC, there is an expected balance between the number of voters with $A \succ B$ and with $B \succ A$ within the pair of voting situations. This observation extends to all of the pairs of voting situations in the partition, since all voting situations are equally likely to be observed with IAC. In the event that $n_1 = n_6$, $n_2 = n_5$, and $n_3 = n_4$, the interchange of rankings matches the voting situation with itself. In this case, the difference in the number of rankings with $A \succ B$ and with $B \succ A$ is not cancelled out over a pair of equally likely voting situations, but within this particular voting situation itself.

If a voting situation is selected at random from the set of all possible voting situations with IAC, it is therefore equally likely that either *A**M**B* or *B**M**A* will be observed for all possible pairs of candidates. Estimates for the likelihood that voting paradoxes that involve PMR cycles will be observed can therefore be expected to be exaggerated with IAC. However, it is important to stress that the IAC assumption is neither forcing a PMRW to exist nor forcing a PMR cycle to exist. The DC assumption also requires that it is equally likely that *AMB* or *BMA* for all possible pairs of candidates in a voting situation, since it is based on the more restrictive requirement that it is equally likely to have $A \succ B$ or $B \succ A$ for each individual voter. IAC does not directly specify anything about the preferences of any individual voter.

The IAC assumption is a very simple concept, and it can take on some other equivalent interpretations. For example, Berg (1985) shows an interesting connection between IC and IAC through a discussion of *Pólya-Eggenberger* (*P-E*) probability models. These models are best described in the context of generating random voter preference profiles by drawing colored balls from an urn, following earlier discussion. The experiment starts with balls of six different colors being placed in the urn. For each possible individual preference ranking, there are A_i balls of the particular color that corresponds to the i^{th} possible individual preference ranking, and the A_i values vary according the components of the predetermined $\boldsymbol{p}$. A ball is drawn at random and the corresponding individual preference ranking is assigned to the first voter. The ball is then replaced, just as in the original experiment, but now α additional balls of the same color are also placed into the urn. A second ball is then drawn, the corresponding ranking for its color is assigned to the second voter, and the ball is replaced along with α additional balls of the same color. The process is repeated over *n* trials to obtain an individual preference ranking for each of the *n* voters. When $\alpha > 0$, the color of the ball that is drawn for the first voter will have an increased likelihood of representing the color of the ball that is drawn for the second voter, and so on.

These P-E-based *contagion models* create an increasing degree of dependence among the voters' preferences as α increases. However, there is no dependence among voters' preferences for the particular case with $\alpha = 0$, and the special case of a P-E model in which $A_i = 1$ for $i = 1,2,3,4,5,6$ is obviously identical to the assumption of IC when $\alpha = 0$. The particularly interesting observation is that the same P-E model is equivalent to IAC when $\alpha = 1$, to indicate the IAC implies that some degree of dependence exists among voters' preferences in voting situations.

Another interesting connection between IAC and the *Uniform Culture Condition* (*UC*) is developed in Gehrlein (1981). We have described how the probability that a voting paradox will be observed can be calculated for a specified $\boldsymbol{p}$ that describes the probability that n individual voters will have the preference rankings in Figure 3. UC assumes that all such $\boldsymbol{p}$ with $\sum_{i=1}^{6} p_i = 1$ are equally likely to be observed. For any voting paradox, different probabilities will be obtained for observing that paradox with different $\boldsymbol{p}$. But, if we consider the expected value of the probability that this voting paradox will be observed over all possible $\boldsymbol{p}$ with UC, the result will be identical to the probability that is obtained for n voters with IAC.

3 Relevance of DC, IC and IAC Based Probability Models

It was mentioned previously that an extensive amount of research has been conducted to develop probability representations for the likelihood that various voting paradoxes will occur with the assumptions of DC, IC and IAC; and it is obviously of interest to discuss the relevance of the probability estimates that result from such studies. This is particularly true since a number of recent studies have clearly raised this issue after performing empirical analysis to reach the *extraordinarily unsurprising* conclusion that the distribution of voters' preferences in most election results do not correspond to anything like DC, IC or IAC. The most notable empirical studies of this type include Regenwetter et al (2006) and Tideman and Plassmann (2008). We shall see that there are in fact many very good reasons to explain why it is indeed very relevant to consider the results that are obtained with such probability models.

3.1 General Arguments

A number of general arguments that support investigations that are based the use of assumptions like DC, IC and IAC to develop probability representation are summarized in Gehrlein and Lepelley (2004), given the fact that we have already determined that they are likely to represent scenarios that exaggerate the probability that paradoxical voting events that involve PMR relationships will occur:

- They are very useful when large amounts of relevant empirical data are not available, which is typically the case when analyzing elections.
- They can show that some paradoxical events are very unlikely to be observed. That is, if we use conditions that tend to exaggerate the likelihood of observing paradoxes to

find that the probability for some paradox is small with such calculations, then this paradox is assuredly very unlikely to be observed in reality.

- They can suggest the relative impact that paradoxical events can have on different types of voting situations. For example, different voting rules can be compared on the basis of their relative likelihood of electing the PMRW.
- By using such probability models to obtain closed form representations, it is easy to observe the impact of varying specific parameters of voting situations or voter preference profiles, which is more difficult to do with other approaches.
- The probability representations that are obtained are directly reproducible and verifiable with mathematical analysis, which is not as simple to do with other approaches.
- Analysis of this type can be useful to find out if the relative probabilities of paradoxical outcomes on various voting mechanisms behave in a consistent fashion over a number of different assumptions about the likelihood that voting situations or voter preference profiles are observed.

With very few exceptions, actual elections are only conducted with one voting rule being used, and it typically is not at all easy to compare the resulting election outcome to what else might have happened if some different voting rule had been used. In fact, it is not always straightforward to determine exactly what actually did happen in an election, based only on typical election results. Fishburn (1980) considers the restrictions under which it is possible simply to determine whether or not the PMRW has been selected as the winner of an election, based only on the reported vote counts from the election. It is assumed that voters have weak ordered preferences on candidates and assumptions are established to define admissible voting behavior. The severity of these restrictions leads Brams and Fishburn (1983a, pg 95) to conclude that

> Because of the varieties of strategies that are allowed and the paucity of detail about how people voted, the likelihood of concluding that the winner is a (PMRW) …. is often small if not zero.

As a result, other factors about voting behavior must typically be assumed with some model that reconstructs the preferences of voters from the reported ballot outcomes in an election, simply to determine which candidate was the PMRW, let alone to determine what might have happened if a different voting rule had been used. The significant difficulties that can arise from making such assumptions in these models that reconstruct voters' preferences are pointed out in the conclusion of an empirical study by Regenwetter et al (2002, pg 461)

> Similarly, we conclude from the analysis of four … data sets … that even the most basic and subtle changes in modeling approaches can affect the outcome on any analysis of voting or ballot data against the Condorcet criterion.

This conclusion was reached when actual data sets were examined to determine the resulting PMR ranking on candidates with the use of a very basic and plausible model, and it was found that very different rankings could be obtained with very minor changes in a preference threshold parameter in their model.

We now proceed to develop some of the types of basic results that can be obtained by analyzing probability representations that are obtained with the simple assumptions of DC, IC and IAC.

3.2 Results from the DC Assumption

As a specific example of the some of the types of analyses that are suggested in the list of general arguments that is presented above, we consider some results that follow from probability representations that Condorcet's Paradox is observed with the assumption of DC. Let $P^S_{PMRC}(3,n,DC)$ denote the probability that a Strict PMR cycle, or an occurrence of Condorcet's Paradox, is observed in a three-candidate election with an odd number n voters for a specified $\boldsymbol{p}$ vector from the subspace of DC. A Strict PMR relationship indicates that no PMR ties exist on any of the pairs of candidates. A representation for $P^S_{PMRC}(3,n,DC)$ follows directly from related work in Gehrlein and Fishburn (1976a) as given in (2)

$$P_{PMRC}^{S}(3,n,DC) =$$

$$1 - \sum_{m_1=0}^{\frac{n-1}{2}} \sum_{m_2=0}^{\frac{n-1}{2}-m_1} \sum_{m_3=0}^{\frac{n-1}{2}-m_1} \frac{n!}{m_1!\,m_2!\,m_3!\,m_4!} \begin{Bmatrix} \left(\frac{1}{2}-p_3\right)^{n-m_2-m_3} p_3^{m_2+m_3} + \\ \left(\frac{1}{2}-p_1\right)^{n-m_2-m_3} p_1^{m_2+m_3} + \\ \left(\frac{1}{2}-p_2\right)^{n-m_2-m_3} p_2^{m_2+m_3} \end{Bmatrix}, \tag{2}$$

for odd n with $m_4 = n - m_1 - m_2 - m_3$.

The limiting case for large electorates as $n \to \infty$ is addressed in Fishburn and Gehrlein (1980) to lead to a representation for $P_{PMRC}^{S}(3,\infty,DC)$ in (3)

$$P_{PMRC}^{S}(3,\infty,DC) = \frac{1}{4} - \frac{1}{2\pi} \sum_{j=1}^{3} Sin^{-1}\left(1-4p_j\right). \tag{3}$$

Computed values of $P_{PMRC}^{S}(3,\infty,DC)$ that are obtained from (3) are listed in Table 1 for each $p_1, p_2 = .000(.025).500$. The range of values in Table 1 is truncated since it is obvious from (3) that $P_{PMRC}^{S}(3,\infty,DC)$ is invariant to permutations of p_1, p_2 and p_3.

The limiting probability values in Table 1 show that $P_{PMRC}^{S}(3,\infty,DC)$ goes to zero if any of p_1, p_2 or p_3 is equal to zero, it is also proved that $P_{PMRC}^{S}(3,\infty,DC)$ is maximized for the special case of IC, with $p_i = 1/6$ for $1 \le i \le 6$, for $\boldsymbol{p}$ that are consistent with the assumption of DC. It has already been concluded that DC can be expected to produce exaggerated estimates of the probability that paradoxical outcomes that involve PMR relationships will be observed. By adding the fact that that $P_{PMRC}^{S}(3,\infty,DC)$ is maximized with IC suggests that $P_{PMRC}^{S}(3,n,IC)$ estimates are very likely to produce significant overestimates of the likelihood with which Condorcet's Paradox can be expected to be observed.

It follows directly from (3) that

$$P_{PMRC}^{S}(3,\infty,IC) = \frac{1}{4} - \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right) \approx .088. \tag{4}$$

Table 1. Computed Values of $P_{PMRC}^{S}(3,\infty,DC)$

	p_2										
p_1	.000	.025	.050	.075	.100	.125	.150	.175	.200	.225	.250
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.025	.000	.041	.048	.051	.053	.054	.055	.055	.056	.056	.056
.050	.000	.048	.057	.062	.065	.068	.069	.070	.070	.071	.070
.075	.000	.051	.062	.069	.073	.075	.077	.078	.079	.079	.078
.100	.000	.053	.065	.073	.077	.080	.082	.083	.083	.083	.082
.125	.000	.054	.068	.075	.080	.083	.085	.086	.086	.085	.083
.150	.000	.055	.069	.077	.082	.085	.087	.088	.087	.085	.082
.175	.000	.055	.070	.078	.083	.086	.088	.088	.086	.083	.078
.200	.000	.056	.070	.079	.083	.086	.087	.086	.083	.079	.070
.225	.000	.056	.071	.079	.083	.085	.085	.083	.079	.071	.056
.250	.000	.056	.070	.078	.082	.083	.082	.078	.070	.056	.000
.275	.000	.056	.070	.077	.080	.080	.077	.070	.056	.000	
.300	.000	.055	.069	.075	.077	.075	.069	.055	.000		
.325	.000	.055	.068	.073	.073	.068	.055	.000			
.350	.000	.054	.065	.069	.065	.054	.000				
.375	.000	.053	.062	.062	.053	.000					
.400	.000	.051	.057	.051	.000						
.425	.000	.048	.048	.000							
.450	.000	.041	.000								
.475	.000	.000									
.500	.000										

This result indicates that a significant overestimate of the probability that Condorcet's Paradox will be observed in a three-candidate election is approximately nine percent. We can therefore conclude that such observations should actually be infrequent phenomena, which follows the logic of the second general argument in the list that is given above. Moreover, this outcome is completely consistent with many empirical results that indicate that while Condorcet's Paradox is not a commonly observed election outcome, it does occasionally occur.

In general, $P^S_{PMRC}(m,n,IC)$ values should be viewed as an upper bound on $P^S_{PMRC}(m,n,\boldsymbol{p})$ when $\boldsymbol{p}$ vectors are not biased either to produce a PMR cycle or to produce a PMRW. They have never been intended to produce estimates of the probability that Condorcet's Paradox would ever be observed in any actual voting scenario, but they can tell us a great deal about the likelihood of extreme cases.

The calculated $P_{PMRC}^{S}(3,\infty,DC)$ values in Table 1 indicate that there is a great deal of variability over the range of $\boldsymbol{p}$ vectors in DC, and it is natural to wonder if there is some natural underlying explanation for this variation. Many studies have been conducted to evaluate the impact that various measures of the consistency of voters' preferences in a population will have on the probability that a PMRW exists. It is intuitively appealing to speculate that paradoxical voting outcomes should become less likely to be observed as a population of voters tends to have preferences that are more mutually consistent. This degree of the consistency of voters' preferences can be defined in the context of *social homogeneity*. The preferences of a population of voters would be totally homogeneous if every member of that society had exactly the same preference ranking on the candidates. The opposite extreme is a situation that reflects a situation like IC, where the individual voters have preferences that are completely dispersed over all possible preference rankings on the candidates.

Simple measures of the amount of dispersion among the p_i terms in $\boldsymbol{p}$ vectors have been used as a gauge of the amount of social homogeneity among voters' preferences in a population. Abrams (1976) considers such a measure of homogeneity for three-candidate elections, with

$$H(\boldsymbol{p}) = \sum_{i=1}^{6} p_i^2 . \tag{5}$$

$H(\boldsymbol{p})$ is maximized when $p_i = 1$ for some ranking, so that all voters will have identical preference rankings on candidates, and it is minimized with the assumption of IC, with $p_i = 1/6$ for all $1 \le i \le 6$. Increased values of $H(\boldsymbol{p})$ will generally tend to reflect increased levels of homogeneity for a population of voters. With a large value of $H(\boldsymbol{p})$, we would expect an increased likelihood of observing random voting situations from such a population that have voters' preferences that are clustered around one, or a few, of the possible linear rankings on candidates. As $H(\boldsymbol{p})$ increases, intuition therefore suggests that $P_{PMRC}^{S}(3,n,\boldsymbol{p})$ should also be expected to decrease.

Fishburn and Gehrlein (1980) show that $P_{PMRC}^{S}(3,\infty,DC)$ decreases as $H(\boldsymbol{p})$ increases for $\boldsymbol{p}$ vectors in DC when $H(\boldsymbol{p})$ is changed by keeping one of p_1, p_2 or p_3 fixed while changing the other two. Of course, p_4, p_5 and p_6 must also change accordingly to keep $\boldsymbol{p}$ in accord with the definition of DC. An expected negative relationship is also found between $H(\boldsymbol{p})$ and $P_{PMRC}^{S}(3,n,\boldsymbol{p})$ for general $\boldsymbol{p}$ with independent voters, but this relationship does tend to deteriorate as the number of voters gets very large.

An important conclusion can be reached from these two observed relationships that exist between $H(\boldsymbol{p})$ and $P_{PMRC}^{S}(3,n,\boldsymbol{p})$ for $\boldsymbol{p}$ vectors with independent voters. In particular, the impact of any possible dependence among voters' preferences is completely eliminated as a potential component of an explanation of the source of this relationship. This provides an example application of the fourth item in the list of general arguments for developing such representations above, since it allows for an analysis of the impact of varying just one specific parameter of voting situations. That is, a relationship exists between $H(\boldsymbol{p})$ and $P_{PMRC}^{S}(3,n,\boldsymbol{p})$ when the direct impact of dependence among voters' preferences is excluded.

3.3 Results from the IAC Assumption

A representation for the probability $P_{PMRC}^{S}(3,n,IAC)$ that Condorcet's Paradox is observed for odd n with the assumption of IAC is directly obtainable from a result in Gehrlein and Fishburn (1976b), with

$$P_{PMRC}^{S}(3,n,IAC)=\frac{(n-1)(n+7)}{16(n+2)(n+4)},\ \text{for odd } n. \tag{6}$$

Two interesting observations can be made by considering the limiting result as $n\to\infty$ in (6), with $P_{PMRC}^{S}(3,\infty,IAC)=1/16$. The first of these observations goes back to a consideration of the relationship between the assumptions of IAC and UC that was discussed above. That is, if all possible $\boldsymbol{p}$ vectors are equally likely to be observed as $n\to\infty$, then the expected value of $P_{PMRC}^{S}(3,\infty,\boldsymbol{p})$ is 1/16, or about six percent. This again verifies that the IC assumption, which leads to $P^{S}_{PMRC}(3,\infty,IC)$ equal to about nine percent, gives an exaggerated estimate of the probability that Condorcet's Paradox will be observed. Since the proportion of all possible $\boldsymbol{p}$ vectors that meet the restrictions of DC is of measure zero, this observation and previous discussion jointly lead to an alternative form of this conclusion. That is, only 1/16 of all possible $\boldsymbol{p}$ vectors will result in an observation of Condorcet's Paradox as $n \to \infty$, while 15/16 of all possible $\boldsymbol{p}$ vectors will result in the existence of a PMRW.

The second general observation that stems from (6) follows from a comparison of the limiting results that are obtained for the probability that Condorcet's Paradox is observed with IC and with IAC. The limiting probability is approximately nine percent with IC, while it is reduced to approximately six percent with IAC. Both of these assumptions were found to result in an expected balance in PMR comparisons on all pairs of candidates, so it is natural to wonder what else remains to explain the difference. The earlier discussion of P-E probability models indicated that the difference between IC and IAC stems from the fact that IAC introduces a degree of dependence among voters' preferences while IC does not do so. As a result, the presence of a degree of dependence among voters' preferences can be isolated as a cause for reducing the probability that Condorcet's Paradox will be observed. This observation must be balanced with an understanding that some link must be expected to exist between social homogeneity and the degree of dependence among voters' preferences.

We have said a great deal about how these different probability models can be used to analyze the likelihood that Condorcet's Paradox might be observed, and to isolate different parameters of voting situations that will have an impact on this probability. Attention is now focused on what has been discovered much more recently by applying these same techniques in the consideration of other voting paradoxes.

4 Incompatibility Paradoxes

Incompatibility Paradoxes occur in voting situations when there are multiple definitions as to which candidate should be viewed as being the 'best' possible candidate among the set of available candidates, and where these definitions cannot be satisfied simultaneously by a voting rule. Condorcet's Paradox reflects one such incompatibility paradox. Two other incompatibility paradoxes are Borda's Paradox and Condorcet's Other Paradox, and we consider the relative likelihood that each will be observed.

4.1 Borda's Paradox

Borda (1784) presented another early example of a voting paradox. The background of Borda's Paradox is associated with the use of a Weighted Scoring Rule (WSR) to determine the winner of an election. A WSR is defined in terms of weights $(1, \lambda, 0)$, with $1 \geq \lambda \geq 0$ for a three-candidate election. Each voter assigns a score of one to their most preferred candidate, a score of zero to their least preferred candidate and a score of λ to their middle-ranked candidate. The WSR winner is the candidate that receives the most total points from all voters. The most commonly use WSR is *Plurality Rule* (*PR*) with $\lambda = 0$, such that each voter gives a score of one to their most preferred candidate. Let *A**P**B* denote the outcome that Candidate *A* beats *B* by PR. A variation of this voting rule is *Negative Plurality Rule* (*NPR*) with $\lambda = 1$, such that each voter casts a vote for their two more preferred candidates. This is equivalent to having each voter cast a vote against some candidate, where the candidate with the fewest negative votes is declared the winner. Let *A**N**B* denote the outcome that Candidate *A* beats *B* by NPR.

Borda presented an example voting situation in Figure 4 for 21 voters.

$$\begin{array}{cccc} A & A & B & C \\ B & C & C & B \\ C & B & A & A \\ n_1 = 1 & n_2 = 7 & n_5 = 7 & n_6 = 6\,. \end{array}$$

Figure 4. An example voting situation from Borda (1784)

If PR is used with the voting situation that is shown in Figure 4, we have the outcomes *A**P**B* (8-7), *A**P**C* (8-6) and *B**P**C* (7-6) to give a linear ranking by PR, with *A**P**B**P**C*. A very different and very paradoxical result is observed with the use of PMR. Here, *B**M**A* (13-8), *C**M**A* (13-8) and *C**M**B* (13-8) to give a linear PMR ranking, with *C**M**B**M**A*. With this particular voting situation, PR and PMR completely reverse the election rankings on the three candidates. This specific phenomenon is referred to as representing an occurrence of a *Strict Borda Paradox*.

Borda was particularly concerned about the fact that the PMRL could be chosen as the winner by PR, leading to his suggestion that PR should never be used. Given this primary source of concern, a less stringent *Strong Borda Paradox* is defined as a situation in which PR elects the PMRL, without necessarily having a complete reversal in PR and PMR rankings. Both forms of Borda's Paradox can obviously be observed with other voting rules than PR. Borda proposed a procedure that he referred to as "*election by order of merit*", that has come to be widely known as *Borda Rule* (*BR*), to deal with this type of situation, and BR is equivalent to a WSR with $\lambda = 1/2$ For a general voting situation, as described in Figure 1, with *n* voters and three candidates, the *Borda Score* for Candidates *A*, *B* and *C* under BR would respectively be $BS(A)$, $BS(B)$ and $BS(C)$ with:

$$\begin{aligned} BS(A) &= (n_1+n_2)+(n_3+n_4)/2 \\ BS(B) &= (n_3+n_5)+(n_1+n_6)/2 \\ BS(C) &= (n_4+n_6)+(n_2+n_5)/2\,. \end{aligned} \tag{7}$$

For the particular example in Figure 4, we obtain $BS(C)=13$, $BS(B)=10.5$, and $BS(A)=8$. If we let *A**B**B* denote the event that Candidate *A* beats *B* by BR, we get a linear ranking on the candidates, with *C**B**B**B**A*. This ranking of candidates by BR is now in the reverse order of the ranking by PR, and it is in perfect agreement with the ranking that was obtained by PMR. It has since been proved that BR can never elect the PMRL as the unique winner, so it is completely resistant to the possibility of exhibiting both a Strict Borda Paradox and a Strong Borda Paradox. However, every WSR other than BR can exhibit both of these this phenomena, and representations have been obtained for the associated limiting probabilities for each in Diss and Gehrlein (2009).

Let $P_{StBP}^{WSR(\lambda)}(3,\infty,IC*)$ denote the conditional limiting probability as $n \to \infty$ that a Strict Borda Paradox is observed with a WSR that uses weights $(1,\lambda,0)$, conditional on the existence a strict PMRW with the assumption of IC. When there are only three candidates, a requirement that a strict PMRW exists is equivalent to a requirement that a strict PMR ranking exists for odd *n*. Then, $P_{StBP}^{WSR(\lambda)}(3,\infty,IAC*)$ is defined in the same fashion. It is proved that $P_{StBP}^{WSR(\lambda)}(3,\infty,IC*)$ = $P_{StBP}^{WSR(1-\lambda)}(3,\infty,IC*)$, and the same relationship is also valid with IAC. Computed values of both $P_{StBP}^{WSR(\lambda)}(3,\infty,IC^*)$ and $P_{StBP}^{WSR(\lambda)}(3,\infty,IAC^*)$ are listed in Table 2 for each $\lambda = .00(.05).50$.

Table 2. Computed values of $P_{StBP}^{WSR(\lambda)}(3,\infty,IC^*)$ and $P_{StBP}^{WSR(\lambda)}(3,\infty,IAC^*)$ ^.

λ	$P_{StBP}^{WSR(\lambda)}(3,\infty,IC^*)$	$P_{StBP}^{WSR(\lambda)}(3,\infty,IAC^*)$
.00	.0126	.0111
.05	.0100	.0091
.10	.0077	.0073
.15	.0057	.0056
.20	.0039	.0040
.25	.0024	.0027
.30	.0013	.0016
.35	.0006	.0008
.40	.0002	.0003
.45	.0000	.0000
.50	.0000	.0000

^ From Diss and Gehrlein (2009).

The results from Table 2 indicate that $P_{StBP}^{WSR(\lambda)}(3,\infty,IC^*)$ and $P_{StBP}^{WSR(\lambda)}(3,\infty,IAC^*)$ both decrease as λ increases for the interval $0 \le \lambda \le .5$, so that the likelihood of the outcome is maximized by both PR and NPR. However, these probabilities are typically less than one percent in all cases. Given that the IC and IAC scenarios can be expected to exaggerate the probability that paradoxical events that involve PMR relationships will be observed, it can easily be concluded that actual observations of a Strict Borda Paradox should be very rare events, which is completely consistent with empirical studies. Since these probabilities are so small, no really significant differences can be observed between the cases of IC and IAC from Table 2.

The definition of a Strong Borda Paradox specifies requirements that are not as stringent as the requirements for a Strict Borda Paradox, so it is obvious that it should have a greater probability of being observed. Computed values of the limiting conditional probabilities $P_{SgBP}^{WSR(\lambda)}(3,\infty,IC^*)$ and $P_{SgBP}^{WSR(\lambda)}(3,\infty,IAC^*)$ that a Strong Borda Paradox is observed with IC and IAC respectively are listed in Table 3 for each

$\lambda = .00(.05).50$. As before, these IC* and IAC* representations are conditional on the existence of a PMRW.

Table 3. Computed values of $P_{SgBP}^{WSR(\lambda)}(3,\infty,IC*)$ and $P_{SgBP}^{WSR(\lambda)}(3,\infty,IAC*)$^.

λ	$P_{SgBP}^{WSR(\lambda)}(3,\infty,IC*)$	$P_{SgBP}^{WSR(\lambda)}(3,\infty,IAC*)$	λ	$P_{SgBP}^{WSR(\lambda)}(3,\infty,IC*)$	$P_{SgBP}^{WSR(\lambda)}(3,\infty,IAC*)$
.00	.0371	.0296	.50	.0000	.0000
.05	.0303	.0242	.55	.0001	.0002
.10	.0238	.0192	.60	.0007	.0013
.15	.0179	.0146	.65	.0021	.0033
.20	.0126	.0105	.70	.0046	.0061
.25	.0081	.0070	.75	.0081	.0096
.30	.0046	.0042	.80	.0126	.0136
.35	.0021	.0021	.85	.0179	.0178
.40	.0007	.0007	.90	.0238	.0223
.45	.0001	.0001	.95	.0303	.0269
.50	.0000	.0000	1.00	.0371	.0315

^ From Diss and Gehrlein (2009).

Similar to observations from Table 2 for a Strict Borda Paradox, it is seen that $P_{SgBP}^{WSR(\lambda)}(3,\infty,IC*) = P_{SgBP}^{WSR(1-\lambda)}(3,\infty,IC*)$, but this symmetry relationship is no longer valid for a Strong Borda Paradox with IAC. The probabilities in Table 3 are obviously greater than the associated probabilities in Table 2, and they are maximized with the use of NPR for both IC and IAC, with PR having a marginally smaller probability than NPR for IAC. However, all of these probabilities remain less than four percent in all cases. This indicates that observations of a Strong Borda Paradox should be unlikely events, which is consistent with results from empirical studies that show that they do occasionally occur. The increase in dependence among voters' preferences that is inherent to the IAC assumption reduces the already small probabilities of observing a Strong Borda Paradox with the assumption of IC for all $0 \le \lambda \le .5$. But, there are some instances in which the IAC probabilities are actually greater than the associated IC probabilities when $\lambda > .5$.

Since we have already concluded that IC and IAC based probabilities can be expected to exaggerate the likelihood of observing paradoxes that are based on PMR relationships, neither of these two forms of Borda's Paradox can be viewed as posing a significant threat to typical voting scenarios with a small number of candidates. This conclusion follows, despite the fact that neither IC nor IAC is ever expected to mirror the reality of any given election.

4.2 Condorcet's Other Paradox

Condorcet (1785d) gives the example voting situation in Figure 5 to show a phenomenon that has come to be known as *Condorcet's Other Paradox*.

$$\begin{array}{cccccc} A & A & B & C & B & C \\ B & C & A & A & C & B \\ C & B & C & B & A & A \\ n_1 = 30 & n_2 = 1 & n_3 = 29 & n_4 = 10 & n_5 = 10 & n_6 = 1. \end{array}$$

Figure 5. A voting situation from Condorcet (1785d)

Condorcet notes that $A\mathbf{M}B$ (41-40) and $A\mathbf{M}C$ (61-20) in this voting situation, so that Candidate A is the PMRW, and then goes on to compute $Score(A,\lambda)$ and $Score(B,\lambda)$ for Candidates A and B when the WSR with weights $(1,\lambda,0)$ is used, and:

$$\begin{aligned} &Score(A,\lambda) = 1*31 + \lambda*39 + 0*11 \\ &Score(B,\lambda) = 1*39 + \lambda*31 + 0*11. \end{aligned} \tag{8}$$

In order for Candidate A to be elected by this WSR, we must therefore have:

$$\begin{aligned} &Score(A,\lambda) > Score(B,\lambda) \\ &31 + 39\lambda > 39 + 31\lambda \\ &8\lambda > 8 \\ &\lambda > 1. \end{aligned} \tag{9}$$

This contradicts the basic definition of a WSR, so that no WSR, including BR, can elect the PMRW in this example voting situation, which is Condorcet's Other Paradox. This observation led Condorcet to the conclusion that no WSR should ever be used to determine the winner of an election.

It is of definite interest to obtain some estimate of the relative probability with which this paradox might be observed, since it has a highly significant impact on the relevance of using a WSR. Merlin et al (2002) obtain a limiting conditional representation as $n \to \infty$ for the probability that a similar event is observed in a three-candidate election, given that a PMRW exists. They consider the probability that a given candidate that is not the PMRW will be the winner over the range of all possible WSR's with $1 \geq \lambda \geq 0$. With the assumption of IC, this limiting probability is estimated to be .01808. Given that IC will tend to create voting situations that have a PMRW with relatively small PMR margins over other candidates, this gives an estimate for scenarios in which Condorcet's Other Paradox should be more likely to be observed. And, we find that this probability is still small for a scenario that is expected to exaggerate it.

Gehrlein and Lepelley (2009) obtain a different representation for this limiting conditional probability and find a very similar numerical result with IC. Moreover, a limiting representation is also found with IAC, and the resulting conditional probability is reduced to 19/1620 = .01173. So, the already small IC related probability is further reduced with the introduction of some degree of dependence among voters' preferences with IAC. More relaxed conditions are also introduced to consider probabilities that are more closely associated with the pure definition of Condorcet's Other Paradox, but very little change resulted in the associated probabilities that have just been given. It therefore follows that there is very little reason to expect that Condorcet's Other Paradox would ever be observed in any realistic three-candidate election.

5 Monotonicity Paradoxes

Monotonicity Paradoxes represent situations in which some reasonable definition has been established to determine which candidate should be viewed as being the 'best' available candidate, and where a voting rule has been selected and that voting rule is not monotonic. *Monotonicity* of a voting rule requires consistency of election outcomes as voters' preferences change. That is, increased support (decreased support) for a candidate in voters' preferences should not be detrimental (beneficial) to that candidate in the election outcome. The No Show Paradox is one specific type of a Monotonicity Paradox.

5.1 No Show Paradox

The *No Show Paradox* is developed in Brams and Fishburn (1983b), with an example in which some subset of voters chooses not to participate in an election, and then prefers the resulting winner to the winner that would have been selected if they had actually participated in the election. The winner of an election is determined by *Negative Plurality Elimination Rule* (*NPER*) in a three-candidate election in this example. In the first stage, voters cast votes according to NPR. The candidate that receives the fewest number of votes is then eliminated, and the ultimate winner is selected in the second stage by using PMR on the remaining two candidates.

Consider a voting situation with 21 voters and three candidates, as shown in Figure 6.

$$\begin{array}{cccccc} A & A & B & C & B & C \\ B & C & A & A & C & B \\ C & B & C & B & A & A \\ n_1 = 3 & n_2 = 5 & n_3 = 5 & n_4 = 2 & n_5 = 3 & n_6 = 3. \end{array}$$

Figure 6. An example voting situation from Brams and Fishburn (1983b).

In the first stage of NPR voting, Candidates *A*, *B*, and *C* receive 15, 14 and 13 votes respectively. Candidate *C* is therefore eliminated in the first stage and then *B**M**A* by a vote of 11-10 in the second stage, to select *B* as the overall winner.

Voters with the linear preference ranking $A \succ B \succ C$ would not get their most preferred candidate in this situation, since *B* is the ultimate election winner. But, suppose that two of these particular voters had not participated in this election for some reason. The resulting voting situation for the 19 remaining voters is shown in Figure 7.

$$\begin{array}{cccccc} A & A & B & C & B & C \\ B & C & A & A & C & B \\ C & B & C & B & A & A \\ n_1 = 1 & n_2 = 5 & n_3 = 5 & n_4 = 2 & n_5 = 3 & n_6 = 3. \end{array}$$

Figure 7. The modified example voting situation from Brams and Fishburn (1983b).

In the first stage of NPR voting with this modified voting situation, Candidates *A*, *B*, and *C* respectively receive 13, 12 and 13 votes. Candidate *B* is eliminated in the first stage and then *A**M**C* by a vote of eleven to eight in the second stage. Since the winner in this modified voting situation is *A*, the two voters with linear preferences $A \succ B \succ C$ who did not participate will now have their most preferred candidate chosen as the winner. These two voters have therefore obtained a more preferred outcome from the election with NPER as a result of not participating in the election, which violates the definition of monotonicity.

Probability representations for the limiting probability $P_{NSP}^{VR}(3,\infty,IC)$ that the No Show Paradox is observed with the assumption of IC are obtained in Lepelley and Merlin (2001) for thee voting rules (VR). The analysis includes NPER, as described above, along with Plurality Elimination Rule (PER) and Borda Elimination Rule (BER). PER and BER operate in the same fashion as NPER, by using PR and BR respectively in the initial stage to determine which candidate is eliminated in the first round of voting. Limiting representations for $P_{NSP}^{VR}(3,\infty,IAC)$ are also obtained for both PER and NPER. A representation for $P_{NSP}^{BER}(3,\infty,IAC)$ is obtained in Wilson and Pritchard (2007). All results are summarized in Table 4.

Table 4. Probability values for $P_{NSP}^{VR}(3,\infty,IC)$ and $P_{NSP}^{VR}(3,\infty,IAC)$.

VR	$P_{NSP}^{VR}(3,\infty,IC)$	$P_{NSP}^{VR}(3,\infty,IAC)$
PER	.0558	.0408
NPER	.1623	.0425
BER	.0502	.0243

Occurrences of a Monotonicity Paradox are very often associated to the presence of a PMR cycle in voting situations. Consequently, it should be expected that the introduction of some degree of homogeneity or dependence in voters' preferences will considerably reduce the vulnerability of WSR runoff systems to these paradoxes. This expectation is clearly shown to exist in Table 4, where the $P_{NSP}^{VR}(3,\infty,IC)$ probabilities are significantly greater than their associated $P_{NSP}^{VR}(3,\infty,IAC)$ probabilities, particularly for NPER. With the exception of the entry for $P_{NSP}^{NPER}(3,\infty,IC)$, all probabilities remain less likely than the probability that Condorcet's Paradox will be observed with IC and IAC. The No Show Paradox should therefore have a relatively low probability of being observed, particularly with PER and BER.

The impact of using assumptions like IC for these probability calculations can also be considered from the fact that PMR is used on the second stage of all of these elimination rules. The use of IC will tend to support the generation of voting situations for large *n* such that there will be a relatively close PMR comparison in this second stage, so there will be a good chance of either of the two candidates being selected as the winner in the second stage, resulting in an exaggerated chance that the outcome in the second stage might be changed with the removal of some subset of voters' preferences from the election.

6 Choice Set Variance Paradoxes

Choice Set Variance Paradoxes represent situations in which a series of propositions are put before voters, where each individual issue will be approved or disapproved by majority rule voting. A paradoxical result then arises when the overall final election outcome on the propositions represents a result that is somehow inconsistent with the underlying preferences of the voters. We consider two such paradoxes in the form of Ostrogorski's Paradox and the Majority Paradox.

6.1 Ostrogorski's Paradox

Suppose that there are *m* independent issues that are to be presented to *n* voters and that each individual issue will be approved or disapproved by majority rule voting. There are two parties, *R* and *L*, that have opposing positions on each of the issues. Each voter therefore has a position that is in agreement with either Party *R* or Party *L* on each individual issue, but each voter does not necessarily agree with the position of the same party on every issue. A voter is considered to be a member of Party *R* (Party *L*) if their individual position on issues is in agreement with Party *R* (Party *L*) over a majority of the issues that are being considered. The outcome of voting on each issue will be determined to be in agreement Party *R*, or Party *L*, based on the majority rule outcome of voting on that issue.

Consider the example in Figure 8 where each of five voters has preferences on three different issues.

Issue	Voter 1	2	3	4	5	Position Winner
1	*L*	*L*	*R*	*R*	*R*	*R*
2	*L*	*R*	*L*	*R*	*R*	*R*
3	*R*	*L*	*L*	*R*	*R*	*R*
Party Membership	*L*	*L*	*L*	*R*	*R*	

Figure 8. An example voting situation from Bezembinder and Van Acker (1980).

The results in Figure 8 indicate for example that Voter 1 has preferences on Issues 1 and 2 that are in agreement with Party *L*, while this voter has preferences that are in agreement with Party *R* on Issue 3. Since Voter 1 is in agreement with Party *L* on a majority of issues by a 2-1 margin, this voter is listed as having a membership affiliation with Party *L*. Using this same logic, three of the five voters have a membership affiliation with Party *L*, to make it the *Majority Party* (*MP*) by a 3-2 margin. However, given the preferences of the voters on the issues, the position of Party *R* will win by a 3-2 majority margin on all issues. So, the position of Party *R* wins on every issue, while a Party *L* is the MP.

Deb and Kelsey (1987) define this very contrary outcome as a *Strict Ostrogorski Paradox*, and it was first discussed in Ostrogorski (1902). A less restrictive outcome of a *Weak Ostrogorski Paradox* occurs when Party *R* (Party *L*) is the MP, while a majority of election outcomes on issues are in agreement with the position of Party *L* (Party *R*) .

Probability representations for the likelihood that various forms of Ostrogorski's Paradox are observed are developed in Gehrlein and Merlin (2009a) with an application of the IC assumption. That is, each possible assignment of voters' preferences on the *m* issues, according to party positions, is assumed to be equally likely to be observed. This will tend to result in voting situations in which there is a small relative margin of victory for the determination of the MP as $n \to \infty$. Such a balanced outcome will make it easier for paradoxical outcomes to be observed on majority rule votes on the issues, compared to scenarios in which most voters are expected to have the same party membership.

Representations are obtained for the limiting probability $P_{MP}^{\infty}(m,k,IC)$ as $n \to \infty$ that the majority rule outcomes on exactly *k* issues are in agreement with the MP positions in an *m*-candidate election. It follows that $P_{MP}^{\infty}(m,0,IC)$ is the probability that a Strict Ostrogorski Paradox will be observed, and that results become less paradoxical as *k* increases for a given *m*. Computed values of all possible $P_{MP}^{\infty}(m,k,IC)$ are listed in Table 5 for each $m = 2,3,4$.

Table 5. Probability values of $P_{MP}^{\infty}(m,k,IC)$.

	m		
k	2	3	4
0	.0000	.0104	.0005
1	.5000	.2187	.0594
2	.5000	.5312	.3750
3		.2396	.4406
4			.1245

Given the completely balanced nature of the IC assumption, the maximum agreement values in Table 5 occur for *k* values near $(m+2)/2$. Since we know that these probabilities are expected to produce exaggerated estimated of paradoxical outcomes, it is clear that the likelihood of observing an extreme Strict Ostrogorski Paradox is very small. Moreover, there is strong evidence to suggest that strong versions of a Weak Ostrogorski Paradox can also be expected to be relatively rare. While less stringent occurrences of a Weak Ostrogorski Paradox will have greater probabilities of being observed, it can also be pointed out that such outcomes are not very paradoxical. It is also found that creating a bias toward situations in which individual voter's preferences on issues that are more uniformly consistent with the position of either Party R or Party L, has a significant impact on the probability that Ostrogorski's Paradox will be observed.

Following the discussion above regarding the possibility of contriving situations in which Condorcet's Paradox must occur; it is also possible to do the same type of thing with Ostrogorski's Paradox. That is, models can be developed to significantly increase the probability that these paradoxes will be observed, but these models typically are based on very implausible situations.

6.2 Majority Paradox

The Majority Paradox is similar in nature to Ostrogorski's Paradox, and it was developed in Feix et al (2004). With Ostrogorski's Paradox, we were concerned about the number of majority rule outcomes on issues that were in agreement with the MP. With the Majority Paradox, we are concerned instead about the number of majority rule outcomes on issues that are in agreement with the *Overall Majority Party* (*OMP*). Party *R* (Party *L*) is the OMP if there are more *R* (*L*) entries than *L* (*R*) entries in the *mn* different party position associations for preferences of the voters over all of the issues. The example in Figure 8 shows the fifteen different preference agreements that the five voters have with the parties on the three issues, with nine Party *R* agreements and six Party *L* agreements.

The party membership of each voter in Figure 8 has no impact on the definition of the Majority Paradox; we simply note that Party *R* is the OMP in this example since it beats Party *L* by a 9-6 margin in the set of all voters' preferences on issues. The *Majority Paradox* occurs if the OMP is selected as the winner in a minority of elections on issues. There can not be a *Strict Majority Paradox*, since if any party is the winner by majority rule for every issue, then that same party must also be the OMP.

Representations are obtained in Gehrlein and Merlin (2009b) for the limiting probability $P_{OMP}^{\infty}(m,k,IC)$ as $n \to \infty$ that the majority rule outcomes on exactly *k* issues are in agreement with the OMP positions in an *m*-candidate election with the same IC assumption that was used in the discussion of representations for Ostrogorski's Paradox. Computed values of all possible $P_{OMP}^{\infty}(m,k,IC)$ are listed in Table 6 for each $m = 2,3,4$.

Table 6. Probability values of $P_{OMP}^{\infty}(m,k,IC)$.

	m		
k	2	3	4
0	.0000	.0000	.0000
1	.5000	.1623	.0417
2	.5000	.5877	.3750
3		.2500	.4583
4			.1250

The computed Majority Paradox probabilities in Table 6 are similar to the values of the Ostrogorski Paradox probabilities that were observed in Table 5, so similar conclusions can be drawn. That is, there is strong evidence to suggest that strong versions of a Majority Paradox can be expected to be relatively rare. While less stringent occurrences of a Majority Paradox will have greater probabilities of being observed, such outcomes are not highly paradoxical.

7 Conclusion

We have seen that the classic assumptions for producing probability representations for the likelihood that voting paradoxes will be observed do have valid uses for isolating the effect that different parameters can have on these probabilities. By using the fact that these classic assumptions will tend to exaggerate the probability of observing paradoxes that involve PMR relationships, we have been able to show that the probability of observing any extreme paradoxical results in an election is very small for a number of different paradoxes. It is also consistently observed that the introduction of a degree of dependence among voters' preferences will further reduce these already small probabilities. However, there were some minor aberrations in this observation for paradoxes that involve WSR's.

References

Abrams R (1976) The voter's paradox and the homogeneity of individual preference orders, Public Choice 26: 19-27.

Berg, S (1985) Paradox of voting under an urn model: The effect of homogeneity. *Public Choice* 47: 377-387.

Bezembinder T, Van Acker P (1980) The Ostrogorski Paradox and its relation to nontransitive choice. Unoublished paper.

Borda, J de (1784) A paper on elections by ballot. In Sommerlad and McLean (1989), 122-129.

Brams SJ, Fishburn PC (1983a) Approval voting. Birkhäuser Publishers, Boston.

Brams SJ, Fishburn PC (1983b) Paradoxes of preferential voting. Mathematics Magazine 56: 207-214.

Condorcet, Marquis de (1785a) An essay on the application of probability theory to plurality decision making: An election between three candidates. In Sommerlad and McLean (1989), 69-80.

Condorcet, Marquis de (1785b) An essay on the application of probability theory to plurality decision making: Elections. In Sommerlad and McLean (1989), 81-89.

Condorcet, Marquis de (1785c) An essay on the application of probability theory to plurality decision making: Part five." In Sommerlad and McLean (1989), 109-118.

Condorcet M de (1785d) An essay on the application of probability theory to plurality decision making: Hypothesis eleven. In: Sommerlad F, McLean I (1989, eds) The political theory of Condorcet. University of Oxford Working Paper, Oxford, pp 90-108.

Deb R, Kelsey D (1987) On constructing a generalized Ostrogorski paradox: necessary and sufficient conditions. Mathematical Social Sciences 14: 161-174.

Diss M, Gehrlein WV (2009) Borda's Paradox and weighted scoring rules. University of Caen, unpublished manuscript.

Feix MR, Lepelley D, Merlin VR, Rouet JL (2004) The probability of conflicts in a U. S. presidential type election. Economic Theory 23: 227-257.

Fishburn PC (1980) Deducing majority candidates from election data. Social Science Research 9: 216-224.

Fishburn PC, Gehrlein WV (1980) Social homogeneity and Condorcet's paradox. Public Choice 35: 403-420.

Gehrlein, WV (1981) The expected probability of Condorcet's paradox. *Economics Letters* 7: 33-37.

Gehrlein WV, Fishburn PC (1976a) The probability of the paradox of voting: A computable solution. Journal of Economic Theory 13: 14-25.

Gehrlein WV, Fishburn PC (1976b) Condorcet's paradox and anonymous preference profiles. Public Choice 26: 1-18.

Gehrlein WV, Lepelley D (2004) Probability calculations in voting theory: An overview of recent results. In: Wiberg M (ed) Reasoned choices: Essays in honor of Hannu Nurmi. The Finnish Political Science Association, Turku, Finland: 140-160.

Gehrlein WV, Lepelley D (2009) A note on Condorcet's Other Paradox. Economics Bulletin, Volume 29, August 18.

Gehrlein WV, Merlin V (2009a) On the probability of the Ostrogorski Paradox. Unpublished manuscript.

Gehrlein WV, Merlin V (2009b) The probability of the Majority Paradox. Unpublished manuscript.

Lepelley D, Merlin V (2001) Scoring run-off paradoxes for variable electorates. Economic Theory 17: 53-80.

Merlin V, Tataru M, Valognes F (2002) On the likelihood of Condorcet's profiles. Social Choice and Welfare 19: 193-206.

Nurmi, H (1999) Voting Paradoxes and How to Deal with Them, Springer, Berlin.

Ostrogorski M (1902). La démocratie et l'organisation des partis politiques. Calmann Levy, Paris.

Regenwetter M, Grofman B, Marley AAJ (2002) On the model dependence of majority preference relations reconstructed from ballot or survey data. Mathematical Social Sciences 43: 451-466.

Regenwetter M, Grofman B, Marley A, Tsetlin I (2006) Behavioral social choice. Cambridge University Press.

Sommerlad, F and I McLean (1989). *The Political Theory of Condorcet*, University of Oxford Working Paper.

Tideman TN, Plassmann F (2008) Evaluating voting methods by their probability of success: An empirical analysis. Virginia Polytechnic Institute and State University, unpublished manuscript.

Wilson MC, Pritchard G (2007) Probability calculations under the IAC hypothesis. Mathematical Social Sciences 54; 244-256.