The impact of group coherence on the likelihood of voting paradoxes[1]

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Abstract.
The authors have recently investigated the following question: does the expected relationship between increased levels of mutual coherence in voter's preferences and the probability that paradoxes are observed hold up? This investigation analyzes various voting paradoxes (among which the Condorcet paradox, the Condorcet winner paradox, the Borda paradox…), considers a number of voting rules (plurality voting, plurality with a runoff, Borda count, Coombs' method…) and introduces a number of different and original parameters in order to measure the degree of mutual coherence in voter's preferences. The purpose of the presentation is to give an overview of the methodology on which this analysis is based and to summarize some of the most salient results obtained by the authors. Some of these results are not yet published and are forthcoming in a book entitled "Voting Paradoxes and Group Coherence" (Springer, 2010).

Extended Abstract.

1. Methodology

Let n be the number of voters. In an election with three candidates *A, B,* and *C*, there are six possible complete preference orders that voters may have, as shown in Figure 1.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 1. The six possible linear preference rankings on three candidates.

Here, n_i denotes the number of voters that have the associated complete preference ranking on the candidates. Any given combination of $n_i's$ such that $\sum_{i=1}^{6} n_i = n$ is referred to as a voting situation.

1.1. Weak Measures of Group Coherence.

The first measure of group coherence we consider is directly inspired from Black's single-peakedness. We introduce a *Parameter b*, that measures the minimum number of times that some candidate is bottom ranked, or is least preferred, in the preferences of the n voters in a voting situation, to serve as a simple measure of the proximity of a voting situation to representing perfectly single-peaked preferences in a three-candidate election, where

[1] Abstract of the presentation for the Leverhulme Trust sponsored 2010 Voting Power in Practice workshop held at Chateau du Baffy, Normandy, from 30 July to 2 August 2010.
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$$b = Min\{n_1 + n_3, n_2 + n_4, n_5 + n_6\}. \qquad (1)$$

If b is equal to zero for a voting situation with three candidates, some candidate is never ranked as least preferred, so the voting situation represents the condition in which voters have perfectly single-peaked preferences. This would happen, for example if $n_1 + n_3 = 0$, where the definitions from Fig 1 indicate that this requires that Candidate C is never the least preferred candidate for any voter in the associated voting situation. When b is maximized at $n/3$, a voting situation reflects very disperse preferences of voters over candidates to reflect a situation that is very far removed from perfect single-peakedness.
As Parameter b increases in voting situations, the preferences of voters in a voting situation become more removed from the condition of perfect single-peakedness. Another perspective on this issue is that a voting situation with a small Parameter b reflects a situation in which there is some candidate that very few voters think is the worst of the three candidates. The electorate would be somewhat united by their *weak* support of, or lack of complete opposition to, the election of such a candidate. In that sense, this candidate can be viewed as a *Weak Positively Unifying Candidate* that voters would not generally think of as reflecting the worst possible outcome if that candidate were to be elected.

Following the development of Parameter b above, *Parameter t* measures the proximity of a voting situation to meeting the condition of perfectly single-troughed preferences (see Vickery, 1960), with

$$t = Min\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \qquad (2)$$

The definition of $n_i's$ in Fig. 1 are used to define Parameter t as the minimum number of times that some candidate is top-ranked as the most preferred candidate in the voters' preference rankings, so that a voting situation is perfectly single-troughed if $t = 0$, and the value of t then reflects the relative proximity of a voting situation to the condition of perfect single-troughedness. Any candidate that very few voters rank as the most preferred candidate in a voting situation can be viewed as a *Weak Negatively Unifying Candidate* since none of the voters would generally think of the election of this candidate as reflecting the best possible outcome. The electorate would be weakly unified by their opposition to, or lack in complete support of, the election of such a candidate.

According to Ward (1965), a candidate is said to be *perfectly polarizing* if this candidate is never middle ranked, or ranked at the center, of any voter's preference ranking. That is, every voter will either consider this candidate to be either the most preferred or the least preferred. The definition of $n_i's$ in Fig. 1 are used to define *Parameter c* to reflect the proximity of a voting situation to the condition of perfect polarization, with

$$c = Min\{n_3 + n_4, n_1 + n_6, n_2 + n_5\}. \qquad (3)$$

If $c = 0$, some candidate is perfectly polarizing, since all voters will rank that candidate as either least preferred or most preferred, and the value of c measures the proximity of a voting situation to the condition of perfect polarization. Any candidate that very few voters rank in the middle of their preference ranking can generally be viewed as a *Weak Polarizing Candidate*.

1.2 Strong Measures of Group Coherence

Stronger measures of group coherence are developed in Gehrlein (2009), and each of these measures is a more restrictive variation of Parameters *b*, *t*, and *c*. A Weak Positively Unifying Candidate was defined as some candidate that is ranked as least preferred by a small proportion of voters in a voting situation, and the proximity of a voting situation to having a perfect Weak Positively Unifying Candidate is measure by Parameter *b*. A candidate would more strongly reflect the notion of being a positively unifying candidate by being ranked as most preferred by a large proportion of the voters in a voting situation. *Parameter t** is defined accordingly from the definition of the $n_i's$ in Fig. 1, with

$$t^* = Max\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \quad (4)$$

If $t^* = n$, the same candidate is ranked as most preferred by all voters, making it a perfect *Strong Positively Unifying Candidate*, and Parameter *t** is used as a measure of the proximity of a voting situation to this condition.

The same basic logic can be used to strengthen the definition the proximity of a voting situation to having perfect Weak Negatively Unifying Candidate, as measured by Parameter *t*. *Parameter b** is defined accordingly by

$$b^* = Max\{n_5 + n_6, n_2 + n_4, n_1 + n_3\}. \quad (6)$$

If $b^* = n$, the same candidate is ranked as least preferred by all voters, making it a perfect *Strong Negatively Unifying Candidate*, and Parameter *b** is used as a measure of the proximity of a voting situation to this condition.

Parameter *c* measured the proximity of a voting situation to the condition of perfect weak polarization. The strong measure that is associated with this parameter is *Parameter c**, with

$$c^* = Max\{n_3 + n_4, n_1 + n_6, n_2 + n_5\}. \quad (7)$$

If $c^* = n$, the same candidate is middle-ranked in the preferences of all voters, so that this candidate is neither extremely liked nor extremely disliked by any voter, making it a perfect *Strong Centrist Candidate*, and Parameter *c** is used as a measure of the proximity of a voting situation to this condition.

1.3 Obtaining Probability Representations

In order to determine the impact that these measures of group coherence have on the probability that a voting event occurs, attention is focused to the development of representations for the conditional probability that event, given that voting situations have specified values of these measures. These probability representations are based on a direct extension of the assumption of IAC (Impartial Anonymous Culture). For any particular $X \in \{b, t, c, b^*, t^*, c^*\}$, the *Conditional Impartial Anonymous Culture Condition* is used to develop probability representations for election outcomes, conditional on the assumption that only voting situations for which Parameter *X* has a specified value can be observed, and that each of these possible voting situations is equally likely to be observed.

Using efficient algorithms recently developed in the literature (both by the authors and other scientists[3]), we are able to obtain representations for the desired probabilities as polynomial relations giving the probabilities as a function of *n* and *X*. To illustrate, we obtain that the

[3] See Gehrlein (2005), Lepelley *et al* (2008).

representation for the probability of having a Pairwise Majority Rule Winner (or Condorcet Winner) given a specified value of parameter b for odd $n \geq 7$ is given by

$$P_{PMRW}(n,b)= \quad (8)$$

$$\frac{-b(17-21b-11b^2)+(5-26b-4b^2)n+3(2-b)n^2+n^3}{(n-3b)\left[(n+1)(n+5)-3b(2+b)\right]}, \quad \text{for } 0 \leq b \leq (n-1)/4$$

$$\frac{3(3-2b-6b^3)+(11+18b^2)n+3(1-2b)n^2+n^3}{2(\mathrm{b}+1)\left[(n+1)(n+5)-3b(2+b)\right]}, \quad \text{for } (n+1)/4 \leq b \leq (n-1)/3$$

$$\frac{3}{4}, \quad \text{for b} = n/3.$$

In contrast with relation (8), the representations we obtain are often of a very complicated nature, which makes them of limited value. For this reason, in a number of cases, we only focus on the limiting representations as n tends to infinity. We proceed as follows. Let $\alpha_b=b/n$ (we still consider the above illustration); thus, α_b denotes the minimal proportion of voters that rank a candidate in last position. Replacing b by $\alpha_b n$ in (8) and making n tend to infinity, we obtain the following representation:

$$P_{PMRW}(\alpha_b)=$$

$$\frac{11\alpha_b^3-4\alpha_b^2-3\alpha_b+1}{(3\alpha_b-1)(3\alpha_b^2-1)}, \quad \text{for } 0 \leq \alpha_b \leq 1/4$$

$$\frac{-18\alpha_b^3+18\alpha_b^2-6\alpha_b+1}{2\alpha_b(3\alpha_b^2-1)}, \quad \text{for } 1/4 \leq \alpha_b \leq 1/3$$

$$\frac{3}{4}, \quad \text{for } \alpha_b = 1/3.$$

This simple representation makes easy the analysis of the impact of a tendency toward single-peakedness on the probability that a PMRW exists.

2. Summary of Results on Borda's Paradox

A strong Borda paradox occurs when a voting rule ranks the candidates in a way that reverse the ranking obtained according to Pairwise Majority Rule (PMR). A strict Borda paradox occurs when a voting rule selects the Pairwise Majority Rule Loser '(or Condorcet Loser). The most salient results that we have obtained on these paradoxes can be summarized as follows.

1) In some specific circumstances that we have identified, the Borda's Paradox can occur with a probability that is far from being negligible: when voting situations are more and more removed from perfect single-troughedness, the strict version of the Paradox has a probability of occurrence that can be higher than 15% when plurality rule is used to rank the candidates, and the strong version can occur with a more than 30% probability when voters' preferences are far from perfect single-peakedness and when the voting rule is negative plurality rule.
2) It remains however that the overall probability that the Borda's Paradox will be observed is rather low (about 1% for the strict version and about 3% for the strong version under the IAC assumption) and significantly lower than the likelihood of Condorcet's Paradox (6,25% under the same assumption). As a consequence, Borda's

Paradox could be considered as less problematic than Condorcet's Paradox in real election settings.

3) This assertion should be balanced by the following observation, which certainly constitutes our main finding. We know from previous studies that, when voters' preferences become more internally consistent, the Condorcet's Paradox probability is reduced and tends to 0, in accordance with our intuition. The results we have obtained show that the impact of an increasing degree of mutual coherence among voters' preferences on the likelihood of Borda's Paradox is much more subtle and more difficult to analyze: it turns out that this impact depends both on the measure of mutual coherence that is considered and on the voting rule that is used. In some circumstances, the probability that the Borda's Paradox will occur increases when voters' preferences become more internally consistent.

3. Summary of Results on Condorcet Ranking Efficiency

We consider here the effectiveness of some scoring rules and scoring elimination rules at matching the complete PMR ranking on candidates. We thus define the Condorcet Ranking Efficiency of a voting rule as the conditional probability that candidate rankings are identical for both PMR and that voting rule, given that a strict PMR ranking exists.

It turns out that the results that are observed for Condorcet Ranking Efficiency are very similar to the results that were observed previously for Condorcet Efficiency when the objective was to select a single winner. The Condorcet Ranking Efficiency of Borda Rule (BR) remains somewhat stable across the complete range of all measures of group mutual coherence. BR dominates both Plurality Rule (PR) and Negative Plurality Rule (NPR) for all weak and strong measures of group mutual coherence, particularly for Parameter c and Parameter c*. While Plurality Elimination Rule (PER) does display superior performance to BR over a small range of some parameters, it very frequently exhibits extremely poor performance on the basis of Condorcet Ranking Efficiency and it is not a viable option for consideration. The efficiency of Negative Plurality Elimination Rule (NPER) is very often superior to that of BR, but there are ranges in which NPER performs very poorly for both Parameter b and Parameter b*, while BR does not do so. Since we cannot somehow exclude the possibility that voters are obtaining preference rankings with some model that will fall into the ranges in which NPER performs very poorly, the Borda Compromise suggested by our results on Condorcet Efficiencies still has a good foundation for Condorcet Ranking Efficiencies.

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