AN ANALYSIS OF THE INTRINSIC DIFFERENCES BETWEEN THE COMMONLY APPLIED VOTING POWER TECHNIQUES

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ABSTRACT. Voting power science is a field of co-operative game theory concerned with calculating the influence a voter can exert on the outcome of a voting game. The techniques used to calculate voting power have names like the Shapley-Shubik index, and the Banzhaf measure. They are invaluable when used to design democratically fair voting games, however, there is currently no consensus over which technique is best.

Ignoring the well known differences in probability models, this paper will focus upon the less well known differences in underlying measures. With the analysis showing that the Shapley-Shubik index is afflicted with a fundamental flaw, restricting its use in many real world voting games, it soon becomes apparent that the dissimilarities between the techniques extend far beyond their methods of counting voting coalitions.

1. INTRODUCTION

Voting power is a field of co-operative game theory that has seen a recent resurgence, due, in no small part, to the work of Felsenthal and Machover and their seminal book (Felsenthal and Machover, 1998). Despite the importance of the field, it is a subject that is not studied widely enough, and is poorly understood outside of the voting power community.

The concept behind voting power is simple enough. The idea is to measure the ability of an individual voter to affect the outcome of a voting game. This kind of analysis is invaluable when it comes to designing fair, and democratic, institutions. For instance, most people would agree that it is desirable to design voting within the European Union such that a country with twice the population should have twice the influence, compared with a country half the size. But the question remains, how do you go about measuring voting power?

In the literature, there have been a number of techniques proposed to measure voting power, such as Shapley and Shubik (1954); Banzhaf (1965); Coleman (1971); Deegan and Packel (1978); Johnston (1978); Straffin (1977). The two most widely used techniques are by Banzhaf,

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and Shapley and Shubik. It has previously been proposed, by Straffin (1977, 1978), that the differences between these two techniques rest solely in the underlying probability models. However, this paper will show that the differences are much more fundamental. No doubt, making such a claim will raise some interesting questions in the reader's mind. Which is why the appendix will list some anticipated questions, and associated answers.

This paper attempts to present measure theoretic ideas to as wide an audience as possible. On occasion, this will result is some mathematical notation being simplified for ease of comprehension. Any readers familiar with measure theory are asked to forgive these unavoidable simplifications.

2. Some Basic Terminology

Contrary to common practice, this paper makes a distinction between a voting power measure, and the techniques used to calculate them.

Voting Power Measure - A measurable function.

Voting Power Technique - A method used to calculate the value of a measurable function.[1]

3. Voting Power Techniques

There is no need to review each individual voting power technique, as they all work in a similar fashion. The general concept is simple; take a set of possible voting scenarios, and test each one in turn to see if the outcome is sensitive to a change in the vote of a given player i. If the outcome of the game changes, a running count of "criticality" for player i is increased (because player i is *critical* to the outcome of the given voting scenario). The result of this "criticality count", after all voting scenarios have been examined, gives the voting power of the player.

Voting power techniques are differentiated by the amount they add to the criticality count. For example, the Banzhaf technique adds $\frac{1}{2^{|N|}}$, and the Shapley-Shubik technique adds $\frac{(|C|-1)!(|N|-|C|)!}{|N|!}$; where $|N|$ is the number of players, and $|C|$ is a function of the voting scenario.

While this clearly isn't a exhaustive exposition of the many different techniques that exist, it is sufficient for the purposes of this paper.

4. Counting Blocks

Now for a short digression from voting power theory.

Imagine that you work for a Danish toy manufacturer of interlocking children's building blocks, and they have just started a recycling scheme. The amount of money they are willing to pay for a batch of

[1]The Banzhaf and Shapley-Shubik indices are examples of techniques.

blocks is dependent upon the percentage of blue blocks in the shipment (for some reason, the blue blocks are more valuable).

Now imagine you've just been handed a large pile of blocks, which we will call Ω. It's your job to calculate the percentage of blue blocks in the batch. Being an industrious type, you decide to build a machine to do this for you.

The first stage in your plan is to count how many blocks there are in total. Let's call this block counting machine $\mathbb{P}$. After one run through, we'll know how many we have in Ω, we'll call this number $\mathbb{P}(\Omega)$. Your new machine will output something like $\mathbb{P}(\Omega) = 1034$, or $\mathbb{P}(\Omega) = 32$, depending on how many blocks there are in the batch.

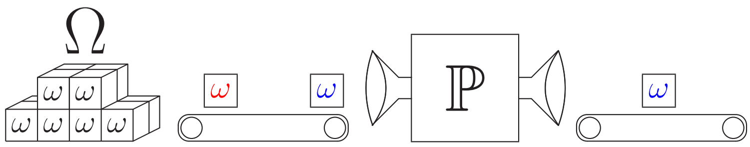

Figure 1. A Block Counting Machine

With amazing forethought you realise that, as you need to calculate a percentage, it will be more useful to have $\mathbb{P}(\Omega) = 100\%$ after all the blocks have been counted. So, you adjust the machine such that instead of adding 1 every time a little ω goes past it will add $\frac{1}{|\Omega|}$.[2] Now, after all the blocks have passed through, the machine will read $\mathbb{P}(\Omega) = 1$ (which is, of course, equivalent to $\mathbb{P}(\Omega) = 100\%$).

The second stage in your plan is to add a "magic eye" machine that can "see" if a blue block has gone past, we'll call this the $\mathbb{I}$ machine. The $\mathbb{I}$ machine is very basic, it simply outputs $\mathbb{I}(\omega) = 1$ if it sees a blue block, and $\mathbb{I}(\omega) = 0$ otherwise.

The final stage in your plan is to link the "magic eye" machine with the block counting machine, to create a super-counter machine. You connect the output of the $\mathbb{I}$ machine, with the "on/off" switch of the $\mathbb{P}$ machine. This means, whenever a blue block goes by, the $\mathbb{I}$ machine will turn on the $\mathbb{P}$ machine, allowing it to count. But if a non-blue box should pass, the $\mathbb{I}$ machine will turn off the $\mathbb{P}$ machine, preventing it from counting.

And that's it! The combined $\mathbb{I}$ and $\mathbb{P}$ machines work together to calculate the percentage of blue blocks. After all the blocks have gone through the super-counter machine, the output of the $\mathbb{P}$ machine will be the percentage of blocks that are blue.

The operation of the super-counter can be described as follows:-

(1) Start with a pile of blocks called Ω.

[2]For the purposes of this example, we shall ignore how you can come to know $\frac{1}{|\Omega|}$ before all the blocks have been counted!

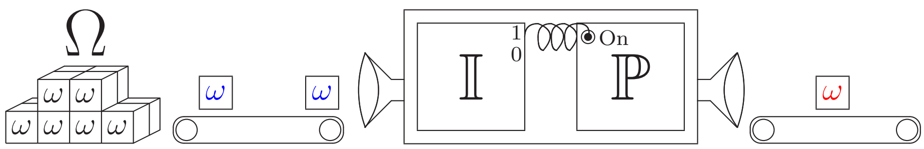

FIGURE 2. A Super-Counter Machine

(2) Take each little block ω in turn, and send it through the super-counter, one by one.
(3) If ω is blue, the $\mathbb{I}$ machine will turn on the $\mathbb{P}$ machine.
(4) If ω is not blue, the $\mathbb{I}$ machine will turn off the $\mathbb{P}$ machine.
(5) The result is given by reading the output of the $\mathbb{P}$ machine after all the blocks have passed through the super-counter.

4.1. **The Maths.** As a bit of a mathematician, you want to write down the operation of the super-counter using mathematical notation. Let's start by writing down what happens when a single block passes through the machine. We can mimic the action of the $\mathbb{I}$ machine turning the $\mathbb{P}$ machine on and off by multiplying $\mathbb{I}$ and $\mathbb{P}$ together (remember that the $\mathbb{I}$ machine outputs 1 if it is blue, and 0 otherwise).

$$\mathbb{I}(\omega) \times \mathbb{P}(\omega).$$

Next we have to represent every little block ω moving through the machine, with the result added to a running count. We could use the $\sum$ notation for this, but, for our purposes, the integral notation would be better.

$$\int_{\omega \in \Omega} \mathbb{I}(\omega) \times \mathbb{P}(\omega).$$

We're almost done, just a few more tweaks. First, let's rename the $\mathbb{I}$ function to $\mathbb{I}^{Blue}$, because the $\mathbb{I}$ machine is looking for blue blocks. Second, we get rid of the redundant $\times$ sign between $\mathbb{I}$ and $\mathbb{P}$. And third, in keeping with standard notation, we change the final ω to $d\omega$.[3]

$$\int_{\omega \in \Omega} \mathbb{I}^{Blue}(\omega)\ \mathbb{P}(d\omega).$$

We finish off our mathematical expression of the super-counter by writing down what this super-counter was designed to do. Which, in this case, is to calculate the probability of a block being blue.

$$\Pr(Blue) = \int_{\omega \in \Omega} \mathbb{I}^{Blue}(\omega)\ \mathbb{P}(d\omega).$$

[3]We normally to write $\int_x y(x)\ dx$ instead of $\int_x y(x)\ x$.

5. Non-Uniform Blocks

Satisfied in your new super-counter machine, you patiently wait for your first batch of Ω blocks to arrive. When they finally do, you receive an unwelcome surprise. Instead of a nice neat pile of individual blocks, you are given a huge mess of blocks stuck together in clumps of different sizes. The blocks have come from a school maths department where they were using them to illustrate factorials. The blocks have arrived in clumps of size $1!, 2!, 3!, 4!$, and so on. As luck would have it, each clump is made up of one colour only. Despite this, before you can use your machine, you'll have to break up these clumps into their individual little blocks. If only there was some way to modify the super-counter to cope with these clumps automatically? Fortunately, there is. And it's all to do with the $\mathbb{P}$ machine.

Instead of using the $\mathbb{P}$ machine to count blocks as they go past, the $\mathbb{P}$ machine can weigh them instead. This simple change means that even if a clump of x blocks were to go through the machine, it would still know how many went past, because they would weigh x times as much as an individual block.

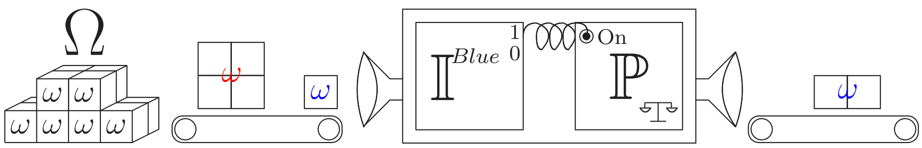

Figure 3. A Super-Measurer

As we're not really counting anymore, the machine should be renamed. It could be called a super-weigher, but calling it a super-measurer would be even better. This new super-measurer works as follows. (In our previous example, we used ω to represent an individual block, this time we can use it to represent a clump).

(1) Start with a pile of blocks called Ω.
(2) Take each clump ω in turn, and send it through the super-measurer.
(3) If the clump is blue, use the $\mathbb{I}$ machine to turn on the weighing machine $\mathbb{P}$.
(4) If the clump is not blue, use the $\mathbb{I}$ machine to turn off the weighing machine $\mathbb{P}$.
(5) The result is given by reading the total weight measured by $\mathbb{P}$ after all clumps have passed through the super-measurer.

Expressing the operation of the super-measurer mathematically gives,

$$\Pr(Blue) = \int_{\omega \in \Omega} \mathbb{I}^{Blue}(\omega)\, \mathbb{P}(d\omega). \tag{1}$$

You'll note that this is the same mathematical representation as the super-counter machine. This is because the weighing of the boxes is incorporated into the $\mathbb{P}$ function. Mathematically, we say that $\mathbb{P}$ is a measure on the subsets of Ω, and $\mathbb{I}$ is a measurable function.

5.1. **Discussion.** Let's examine the super-measurer in greater detail. It calculates the probability of a block being blue, no matter what the size of the clumps are. They could all be a uniform 1 block in size, or they could be some weird number like $(|C|-1)!(|N|-|C|)!$ in size. The super-measurer doesn't even care in which order the clumps pass through, it will still calculate $\Pr(Blue)$ in the end.

In other words, changing the distribution of the blocks doesn't change the statistic being calculated. When the blocks are distributed uniformly we are calculating $\Pr(Blue)$, and when the blocks are distributed non-uniformly we are still calculating $\Pr(Blue)$.

Looking back to Equation (1), it is clear that the $\mathbb{I}$ function defines the statistic being calculated. We call this function, the characteristic, or indicator, function of the statistic.

6. Voting Power Measures

It should be pretty clear that the block counting example was a thinly veiled analogy for the techniques used to calculate voting power. All of the commonly used voting power techniques can be recast into a block measuring scenario, with the appropriate selection of $\mathbb{P}$ and Ω. Hence we define a voting power statistic as,

$$\int_{\omega\in\Omega} \mathbb{I}(\omega)\ \mathbb{P}(d\omega).$$

Just like our block counting example, this representation is insensitive to changes in the distribution of voting scenarios. Therefore, the voting power measure is wholly defined by the measurable function $\mathbb{I}(\omega)$. Which, in the context of voting power is called a criticality indicator function.

7. Criticality Indicator Functions

Broadly speaking there are three categories of criticality functions.

Increasing Criticality - This is a measure of a player's ability to change the outcome of the voting game by increasing their support.

Decreasing Criticality - This is a measure of a player's ability to change the outcome of the voting game by decreasing their support.

Total Criticality - This is a measure of a player's ability to change the outcome of the voting game by either increasing, or decreasing, their support.

A more in-depth discussion of criticality, and the motivation for the different types can be found in Das (2011).

7.1. **Criticality Assumptions.** Almost every voting game allows a player to abstain (even if it were mandatory to vote, the player could still abstain, albeit at huge personal cost). Once we accept that a player can do more than just vote "yes" or "no", we have to augment our notions of criticality to handle this.

Criticality 0 - This criticality assumption requires that the voter must either start by initially voting "no", or that it must change its mind to end up ultimately voting "no".

Criticality δ - This criticality assumption places no restriction upon how the voter initially votes, or how it ultimately votes.

Once again, the reader is referred to Das (2011) for a greater exposition of the criticality assumptions.

7.2. **Summary.** We've introduced three different types of criticality, along with two different criticality assumptions, taken together this gives up to six different notions of criticality. They are listed in the following table, along with their abbreviated notation.

	Increasing	Decreasing	Total
Criticality 0	IC_i^0	DC_i^0	TC_i^0
Criticality δ	IC_i^δ	DC_i^δ	TC_i^δ

Table 1. The Different Notions of Criticality.

While this table is not exhaustive, it certainly covers the different criticalities measured by the commonly used voting power techniques.

8. Voting Power Techniques

With the criticality functions defined, let's look at the more common techniques and interpret them accordingly.

8.1. **Banzhaf.** Banzhaf (1965) states that the power in a legislative sense is the ability to affect outcomes. He says specifically the power of a legislator is given by the number of possible voting combinations of the entire legislature in which the legislator can alter the outcome by changing their vote. Banzhaf talks about the outcome changing when the legislator changes their vote, he does not restrict this change to only an increase in support, or a decrease in support. Hence it is a measure of Total Criticality. Nor does Banzhaf place any restrictions upon how the legislator is initially voting, or ultimately voting. Hence, it must be a Criticality δ measure. Making this a Total Criticality δ measure, i.e. Banzhaf is given by $\Pr(TC_i^\delta)$.

8.2. **Shapley Shubik.** Shapley and Shubik (1954) state that the power of an individual member of a legislative body depends on the chance they have of being critical to the success of a winning coalition. They explain that a voter can be "pivotal" when they can turn a possible defeat into a success. And they construct their index as follows.

(1) There are a group of individuals all willing to vote for some bill.
(2) They vote in order.
(3) As soon as a majority has voted for it, it is declared passed.
(4) The (pivotal) member who voted last is given credit for passing the bill.

For a moment, let's examine their term pivotal. It requires a losing voting scenario in which the voter expresses zero support towards the bill to become winning when they increase their support. Rather than call the voter pivotal, let's call it critical instead. Furthermore, as the voter becomes critical by increasing its support, let's call it increasingly critical. Finally, as the pivotal voter always starts off by expressing zero support for the bill, it must be a Criticality 0 measure. Making this an Increasing Criticality 0 measure, i.e. Shapley-Shubik is given by $\Pr(IC_i^0)$.

8.3. **Straffin.** Straffin defines his measure as the probability that player i's vote will make a difference in the outcome. Making it, like Banzhaf, a measure of Total Criticality. And, as there is no requirement for player i to be initially voting one way or another, it is a measure of Total Criticality δ, i.e. Straffin is given by $\Pr(TC_i^\delta)$.

8.4. **Other Techniques.** A detailed discussion of other techniques, and their criticality indicator functions can be found in Das and Rezek (2011). But briefly,

Coleman initiate action is $\Pr(IC^\delta | \overline{\text{Winning}})$.
Coleman prevent action is $\Pr(DC^\delta | \text{Winning})$.
Deegan-Packel is $\Pr(DC_i^0)$.
Holler-Packel is $\Pr(DC_i^0)$.
Johnston is $\Pr(DC_i^0)$.

9. Measuring Criticality

Building upon the work we did counting coloured blocks, let's build a super-measurer to measure criticality. We start with Decreasing Criticality δ.

9.1. DC_i^δ**.** Let's examine this machine in greater detail, we already know how the $\mathbb{P}$ machine works, so let's focus upon the $\mathbb{I}^{DC_i^\delta}$ machine. By definition, for a little ω to be Decreasing Criticality δ we require two conditions to hold true.

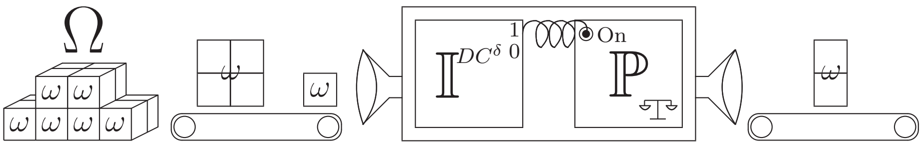

FIGURE 4. Decreasing Criticality δ

Condition 1 - The ω being measured must be winning. We'll use a function called $\mathcal{W}(\omega)$ to tell us if the box is winning, returning 1 if it is and 0 otherwise.

Condition 2 - A modified version of ω must be losing. The modified ω is the same as the original ω, except that player i has changed its vote, and is now voting "no". We'll call this new voting scenario ω'. Once again, we can use the $\mathcal{W}$ function to tell us if this condition holds true. Hence, we can build an indicator function for Decreasing Criticality δ using just ω, ω', and $\mathcal{W}$ as follows.

$$\mathbb{I}^{DC_i^\delta}(\omega) = \mathcal{W}(\omega) - \mathcal{W}(\omega').$$

The proof of this is easily given by the following truth table.

$\mathcal{W}(\omega)$	$\mathcal{W}(\omega')$	$\mathcal{W}(\omega) - \mathcal{W}(\omega')$
0	0	0
1	1	0
1	0	1

(It should be noted that the construction of ω' ensures that it is not possible for ω' to be winning, while ω is losing).

9.1.1. *The Maths.* Just as we did in the block counting example, we integrate the indicator function to calculate our statistic.

$$\Pr(DC_i^\delta) = \int_{\omega\in\Omega} \mathbb{I}^{DC_i^\delta}(\omega)\ \mathbb{P}(d\omega) = \int_{\omega\in\Omega} \mathcal{W}(\omega) - \mathcal{W}(\omega')\ \mathbb{P}(d\omega).$$

We can split the above integral to give,

$$\Pr(DC_i^\delta) = \int_{\omega\in\Omega} \mathcal{W}(\omega)\ \mathbb{P}(d\omega) - \int_{\omega\in\Omega} \mathcal{W}(\omega')\ \mathbb{P}(d\omega).$$

Look at the first integral $\int_{\omega\in\Omega} \mathcal{W}(\omega)\ \mathbb{P}(d\omega)$, this looks a lot like the expression we created for calculating the probability of a blue block, except that now we are looking for winning voting scenarios. By the same logic, $\int_{\omega\in\Omega} \mathcal{W}(\omega)\ \mathbb{P}(d\omega)$ must be the probability Pr(Winning). So,

$$\Pr(DC_i^\delta) = \Pr(\text{Winning}) - \int_{\omega\in\Omega} \mathcal{W}(\omega')\ \mathbb{P}(d\omega).$$

Now let's examine the second integral $\int_{\omega\in\Omega} \mathcal{W}(\omega')\ \mathbb{P}(d\omega)$. Solving this will require a little more effort. Recall that the function $\mathbb{P}$ "weighs" a block, and $\displaystyle\int_{\Omega} \mathbb{P}(d\omega)$ is the process of "weighing" all the blocks.

What if we took a block, and broke off a small piece? The small fragment we will call i, and the rest of the block we will call $\omega^{N\setminus\{i\}}$. If you still wanted to weigh the block, you could weigh the i piece first, and then the $\omega^{N\setminus\{i\}}$ piece second. Likewise, if we broke a piece off of every block in Ω, we can still get the total weight of Ω by first weighing all the i pieces, and then all the $\omega^{N\setminus\{i\}}$ pieces. Expressing this idea using the integral notation gives,

$$\int_{\omega\in\Omega} \mathcal{W}(\omega')\ \mathbb{P}(d\omega) = \int_{\omega^{N\setminus\{i\}}} \int_{i} \mathcal{W}(\omega')\ \mu(di)\ \lambda(d\omega^{N\setminus\{i\}}).$$

In the above equation, we've replaced the $\mathbb{P}$ "weighing" machine with two new weighing machines; μ which specialises in weighing the small fragments, and λ which weighs the remainder of a block.[4] Therefore we can express $\Pr(DC_i^\delta)$ as,

$$\Pr(DC_i^\delta) = \Pr(\text{Winning}) - \int_{\omega^{N\setminus\{i\}}} \int_{i} \mathcal{W}(\omega')\ \mu(di)\ \lambda(d\omega^{N\setminus\{i\}}).$$

There's one more change to make, recall that ω' is explicitly constructed as ω with player i changing its vote to "no". If we take ω', and break it into two fragments, i and $\omega^{N\setminus\{i\}}$, we can rename the i part as i_{no} (to represent that i is no longer a variable, but now a constant "no"). Therefore, we have,

$$\Pr(DC_i^\delta) = \Pr(\text{Winning}) - \int_{\omega^{N\setminus\{i\}}} \int_{i} \mathcal{W}(\omega^{N\setminus\{i\}} \times i_{no})\ \mu(di)\ \lambda(d\omega^{N\setminus\{i\}}).$$

Examine the inner integral over the variable i. The term inside, $\mathcal{W}(\omega^{N\setminus\{i\}} \times i_{no})$, is constant with respect to i, so it can be brought outside of this integral to give,

$$\Pr(DC_i^\delta) = \Pr(\text{Winning}) - \int_{\omega^{N\setminus\{i\}}} \mathcal{W}(\omega^{N\setminus\{i\}} \times i_{no}) \int_{i} \mu(di)\ \lambda(d\omega^{N\setminus\{i\}}).$$

The specialised "weighing" machine μ is a sigma finite marginal measure constructed to ensure that $\int_i \mu(di) = 1$. Therefore,

$$\Pr(DC_i^\delta) = \Pr(\text{Winning}) - \int_{\omega^{N\setminus\{i\}}} \mathcal{W}(\omega^{N\setminus\{i\}} \times i_{no})\ \lambda(d\omega^{N\setminus\{i\}}).$$

[4]The new "weighing" machines have been simplified for the purposes of this paper, they are actually sigma finite marginal measures, and are more correctly given by $\mu_{d\omega^{N\setminus\{i\}}}(di)\ \lambda(d\omega^{N\setminus\{i\}})$.

As we are concerned with a-priori voting power, it is sensible to assume that the voters decide how they will vote independently.[5] This means that a group of voters will not change their minds if they find out that player i has changed its mind. As a consequence we can state that $\lambda(d\omega^{N\setminus\{i\}}) = \lambda_{i_{no}}(d\omega^{N\setminus\{i\}})$.[6] Thus,

$$\Pr(DC_i^\delta) = \Pr(\text{Winning}) - \int_{\omega^{N\setminus\{i\}}} \mathcal{W}(\omega^{N\setminus\{i\}} \times i_{no})\ \lambda_{i_{no}}(d\omega^{N\setminus\{i\}}).$$

By the measure theoretic interpretation of probability we know that $\int_{\omega^{N\setminus\{i\}}} \mathcal{W}(\omega^{N\setminus\{i\}} \times i_{no})\ \lambda_{i_{no}}(d\omega^{N\setminus\{i\}})$ is the probability $\Pr(\text{Winning} \mid i_{no})$. Hence,

$$\Pr(DC_i^\delta) = \Pr(\text{Winning}) - \Pr(\text{Winning} \mid i_{no}).$$

9.1.2. *Discussion.* This means that any voting power technique that calculates Decreasing Criticality δ is simply calculating the unconditional probability of the game being winning, less the conditional probability of it being winning, given that player i has voted "no".

Crucially, this is true irrespective of the underlying probability model assumed by the technique. Just like counting blocks, it doesn't matter if the voting scenarios are distributed uniformly, or in funny sized clumps. A Decreasing Criticality δ based technique will still be calculating $\Pr(\text{Winning}) - \Pr(\text{Winning} \mid i_{no})$.

9.2. IC_i^δ**.** In a similar fashion we can create a super-measurer for the Increasing Criticality δ voting power measure. If we should happen to do so we would arrive at the following result.

$$\Pr(IC_i^\delta) = \Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning}).$$

So any voting power technique based upon Increasing Criticality δ, irrespective of underlying probability model, is simply calculating the conditional probability of a game being winning, given that player i has voted "yes", less the unconditional probability of the game being winning.

9.3. TC_i^δ**.** There is no need to create a new super-measurer for Total Criticality δ, because the expression for Total Criticality is simply the sum of both Increasing and Decreasing Criticality.

$$\Pr(TC_i^\delta) = \Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no}).$$

[5] This is not entirely necessary, but the mathematics required to relax this condition is far beyond the scope of this mathematically-light paper.

[6] This notation may seem strange if you haven't come across it before. In essence $\lambda(d\omega^{N\setminus\{i\}}) = \lambda_{i_{no}}(d\omega^{N\setminus\{i\}})$ is a mathematical way of saying that the other players behave the same way, irrespective of player i.

Therefore, any voting power technique that calculates Total Criticality δ, such as the Banzhaf measure, or the Straffin index, is simply calculating the conditional probability of the game being winning, given that player i has voted "yes", less the conditional probability of the game being winning, given that player i has voted "no".

9.4. DC_i^0**.** Creating a super-measurer for Decreasing Criticality 0 would give the following result.

$$\Pr(DC_i^0) = \Pr(\text{Winning}) - \Pr(\text{Winning} \mid i_{no}).$$

Therefore, any voting power technique that calculates Decreasing Criticality 0, like Deegan-Packel, Holler-Packel, and Johnston, are simply calculating the unconditional probability of the game being winning, less the conditional probability of it being winning, given that player i has voted "no".

9.5. IC_i^0**.** Creating a super-measurer for Increasing Criticality 0 is a little more involved, so let's look at how this would be done.

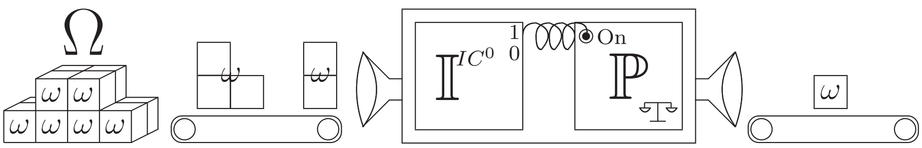

FIGURE 5. Increasing Criticality 0

We already know how the $\mathbb{P}$ machine works, so let's focus upon the $\mathbb{I}^{IC_i^0}$ machine. By definition, for a little ω to be Increasing Criticality 0 we require three conditions to hold true.

Condition 1 - As this is a Criticality 0 measure, the ω being measured must have player i already expressing zero support (i.e. voting "no").

Condition 2 - The ω being measured must be losing.

Condition 3 - A modified version of ω must be winning. The modified ω is the same as the original ω, except that player i has changed its vote, and is now voting "yes". We'll call this new voting scenario ω'.

If we let the indicator function $\mathbb{I}^{i_{no}}$ be the indicator of player i voting "no", then the indicator function for Increasing Criticality 0 is given by,

$$\mathbb{I}^{IC_i^0}(\omega) = \mathbb{I}^{i_{no}}(\omega)\left(\mathcal{W}(\omega') - \mathcal{W}(\omega)\right).$$

The proof of this is easily given by the following truth table.

$\mathbb{I}^{i_{no}}(\omega)$	$\mathcal{W}(\omega')$	$\mathcal{W}(\omega)$	$\mathbb{I}^{i_{no}}(\omega)\left(\mathcal{W}(\omega') - \mathcal{W}(\omega)\right)$
0	0	0	0
0	1	1	0
0	1	0	0
1	0	0	0
1	1	1	0
1	1	0	1

(The construction of ω' ensures that it is not possible for a ω' to be losing while ω is winning).

Integrating $\mathbb{I}^{IC_i^0}(\omega)$ over the set Ω gives the following result,

$$\Pr(IC_i^0) = \Pr(i_{no}) \times \left(\Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no})\right).$$

Therefore, any voting power technique that calculates Increasing Criticality 0, such as the Shapley-Shubik index, is simply calculating the conditional probability of the game being winning, given that player i has voted "yes", less the conditional probability of the game being winning, given that player i has voted "no", multiplied by the probability of player i voting "no".

9.5.1. *Discussion.* Before moving on, let's examine the $\Pr(IC_i^0)$ result in greater detail. I suggest that, of all the possible voting power measures, this is the least useful, and arguably, most flawed. It's all to do with the fact that $\Pr(IC_i^0)$ is directly proportional to $\Pr(i_{no})$.

To understand why this is such a problem, think about a game where instead of voting "yes" or "no", the player can abstain (i.e. any real life game). Arguably, if a player does not have any inherent bias, their probability of voting "no" should now tend towards $\frac{1}{3}$. Now, let's imagine a game where the player can abstain, or vote "yes", "no", and "maybe".[7] In this scenario the probability of voting "no" should tend towards $\frac{1}{4}$.

In the most general case, where a player expresses their vote by selecting from a continuous range of options (for example, if they had to rate their approval for a motion with a percentage), then $\Pr(i_{no}) \to 0$, and accordingly send $\Pr(IC_i^0) \to 0$. In other words, any voting power technique based upon $\Pr(IC_i^0)$ will report that all players, even dictators, have zero voting power.

And herein lies the problem for the Shapley-Shubik index. While most detractors of this technique will question the validity of the probability model, it turns out that, irrespective of probability model, the underlying measure itself is flawed.

[7]Abstention, is not the same as "maybe", see Das (2008) for details.

10. Summary

This paper argued that there should be a clear distinction made between a voting power measure, and the techniques used to calculate them. The current status quo in voting power research is to convolute these two different ideas, to the detriment of the subject.

Using a simple block counting example, this paper showed how it is possible to construct a simple measuring machine to calculate any statistic. The key idea behind this example was that the measuring machine still calculated the same statistic even when the distribution of blocks changed. Adapting the measuring machine to calculate voting power created a measuring machine capable of calculating any voting power measure, irrespective of underlying probability model.

Using this machine, it became a simple matter to show that the Banzhaf measure is calculating,

$$\Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no}).$$

And that the Shapley-Shubik index is calculating,

$$\Pr(i_{no}) \times \left(\Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no})\right).$$

Which brings us to the main premise of this paper, and the fundamental flaw inherent in the Shapley-Shubik index. The Shapley-Shubik index is directly proportional to the probability of the player voting "no". While this may not seem like a flaw, it is important to understand why it is.

You see, in the most general types of voting games, the probability of voting "no" will tend towards zero, which in turn will send the Shapley-Shubik index towards zero. This gradually erodes the information content of this statistic; making it eventually report that both null players and dictators have the same minimal voting power. An undesirable property for any reasonable voting power statistic.

Appendix A. Questions

This appendix anticipates some potential questions that may have arisen for the reader, during the course of this paper.

A.1. **Can't this work be summed up as nothing more than moving the probability model from the statistic to the game?** Correct! That is a brilliant observation. But it is important to understand its relevance. In almost every other field of science the probability model is a property of the system being analysed, and not of the statistic. Returning to our block counting example, you would have thought it very strange, if, instead of calculating the probability of a blue block, I claimed to be calculating a Bernard measure when the blocks were uniformly distributed. Or that I was calculating a Steven-Seagel index when the blocks were distributed in clumps.

A.2. **You imply that the Banzhaf measure is fundamentally different to the Shapley-Shubik index, how can this be?** Quite simple, if we examine these techniques independent of their probability model, we can see quite clearly what they are calculating.

The Banzhaf measure is calculating,

$$\Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no}).$$

And the Shapley-Shubik index is calculating,

$$\Pr(i_{no}) \times \left(\Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no})\right).$$

A.3. **You suggest that the Shapley-Shubik index is different from the Straffin (Homogeneity) measure, how is that possible?** Good question. The Straffin index, both Homogeneity and Independence assumption, is calculating,

$$\Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no}).$$

Whereas, the Shapley-Shubik index is calculating

$$\Pr(i_{no}) \times \left(\Pr(\text{Winning} \mid i_{yes}) - \Pr(\text{Winning} \mid i_{no})\right).$$

But I guess your real question is why did Straffin claim they were the same? Straffin proposed a Total Criticality measure, and then showed how it could be calculated for two different underlying probability models. The first was essentially a uniform distribution, and the second distribution he called homogeneous. He was able to show that his homogeneous distribution gave an answer numerically equal to a Shapley-Shubik index.

The key flaw in his argument was that he only examined games in which every player must vote "yes" or "no". If he examined more realistic games, in which players could abstain, then this numerical identity would disappear, and it would be quite obvious that the Straffin Homogeneity index and the Shapley-Shubik index are inequivalent.

A.4. **How is it possible that the Shapley-Shubik index can be flawed, but the Straffin Homogeneity measure is perfectly okay?** An even better question! In many ways the answer to this is similar to the previous answer. These techniques are only numerically equivalent within voting games in which every player must vote "yes" or "no". In this narrow scenario, the flawed Shapley-Shubik index, happens to give a result numerically identical to the Straffin Homogeneity index. However, as the games become increasingly complex the Straffin index will continue to give sensible answers, but the Shapley-Shubik index will start to produce answers that tend towards zero.

A.5. **Given that Shapley-Shubik and Straffin agree in simple "yes/no" games, the Shapley-Shubik index must have some intrinsic merit, right?** Not really. It's a bit like having two clocks, one working and one broken. Once every 12 hours the broken clock and the working clock will numerically agree. But this agreement doesn't mean that the broken clock isn't flawed, it just means that it occasionally happens to show the right time.

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