The Shapley index for games with r alternatives with a priori unions

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1 Introduction

Simple games serve as models in which a group of voters is to choose one of two candidates or alternatives. Voters can vote for or against. Interesting aspect of voting games is analysis of voter's power and one of the methods of its measuring is the Shapley index (Shapley & Shubik, 1954).

The classical game theory assumes that the probability of forming each coalition is equal. But in real life it is common that some players prefer to form coalition with particular players. This problem is described by the games with a priori unions introduced by Owen (Owen, 1977). Owen defined the Shapley index for such games and pointed out the construction that can be used to extend any index defined for simple games to a class of games with a priori unions.

The obvious restriction of simple game is that players can vote only for or against. To describe other voting - for example such one in which players can also avoid voting - we can use games with many alternatives introduced by Bolger (Bolger, 1993). Bolger showed some methods of measuring voter's power in such games. Particulary he defined the Shapley index for games with many alternatives.

This paper is about games with many alternatives extended with structure of a priori unions.

The paper presents four methods of extending Shapley index to class of games with *r* alternatives with a priori unions. Two of the methods are based on classical Owen construction applied to some cooperative game derived from game with *r* alternatives. The second pair consists in applying Owen construction directly to game with *r* alternatives. After analysis two methods were abandoned because they leaded to results contrary to a common sense. We show that two other methods give the same index.

2 Basic definitions

2.1 The cooperative game, the Shapley value

There is a set of players (voters) $N = \{1, 2, ..., n\}$. Each subset of that set is called a coalition. Set N is called the full coalition. The cooperative game in characteristic function form is a pair (N, w) such that $w : 2^N \longrightarrow \mathbb{R}$ fulfills

- $w(\emptyset) = 0$,

- $w(N) = 1$,
- $w(S \cup T) \geq w(S) + w(T), \text{ if } S \cap T = \emptyset$.

The value of cooperative game is informally the way the total prize (that is $v(N)$) is justly divided between players. Formally the value is a map q from the set of all cooperative games to the set $\bigcup_{k=1}^{\infty} \mathbb{R}^k$, such that for any n-person cooperative game from domain of q we have $q(v) = (q_1(v), q_2(v), ..., q_n(v)) \in \mathbb{R}^n$.

One of the most known and useful values of games is Shapley value(Shapley, 1953). The Shapley value φ for n-person cooperative game w and player i is given by formula

$$\varphi_i(w) = \sum_{C \subseteq N,\ i \in C} \frac{(|C|-1)!\,(n-|C|)!}{n!}[w(C) - w(C \backslash \{i\})],$$

when $|C|$ denotes the number of elements of C.

2.2 Simple game, the Shapley index

The simple game, also called the voting game, is a cooperative game taking values from $\{0, 1\}$.

Coalition C is called the winning coalition in simple game w, if $w(C) = 1$ and the losing coalition if $w(C) = 0$.

One of the methods of measuring the relative power in voting game is the Shapley index, denoted by S. The Shapley index is just the Shapley value for simple game:

$$S_i(w) = \sum_{C \subseteq N,\ i \in C} \frac{(|C|-1)!\,(n-|C|)!}{n!}[w(C) - w(C \backslash \{i\})].$$

In the rest of the paper, the index defined in this way we will call "the simple Shapley index ".

3 Simple game with a priori unions

In many situation some players are more likely to act together than others - they want to collaborate. Simple games with a priori unions model such situation in which, even before voting, some players decide to collaborate and preserve unanimity in voting.

The simple game with a priori unions is a pair (w, T) such that

- w is simple game with player set N,
- $T = \{T_1, T_2, ..., T_m\}$ is a priori union structure, i.e. a partition of N. Each $T_j \in T$ is an a priori union (precoalition), i.e. a set of players who have agreed to collaborate. Sets T_j are pairwise disjoint, nonempty and theirs sum is set N. The set of the a priori unions we denote by $M = \{1, 2, ..., m\}$.

Shapley index for the game with a priori unions was defined by Owen. For game (v, T) and player i it is given by formula

$$S_i(v,T) = \sum_{\substack{H \subseteq M \\ j \notin H}} \sum_{\substack{C \subseteq T_j \\ i \notin C}} \Big\{ \frac{|H|!(|M|-|H|-1|)!|C|!(|T_j|-|C|-1)!}{|M|!|T_j|!} \cdot \\ \cdot [v(Q \cup C \cup \{i\}) - v(Q \cup C)] \Big\},$$

when $Q = \bigcup\limits_{k \in H} T_k$.

To get the Shapley index for games with a priori unions we can also use the Owen construction. This construction for game (v, T) is composed of four following steps.

- Step 1. We form the quotient game $u = v/T$ with the player set $M = \{1, 2, ..., m\}$:

$$u(C) = v\Big(\bigcup_{j \in C} T_j\Big), \text{ where } C \subseteq M.$$

- Step 2. We construct the family of altered games $u_{T_j,K}$. For every $j \in M$ and every $K \subseteq T_j$ we define

$$u_{T_j,K}(C) = \begin{cases} u(C) & \text{if } j \notin C \\ v\Big(K \cup \bigcup\limits_{l \in C \setminus \{j\}} T_l\Big) & \text{if } j \in C. \end{cases}$$

- Step 3. For every $j \in M$ and every $K \subseteq T_j$ we define reduced game w_j on T_j by

$$w_j(K) = S_j(u_{T_j,K}).$$

- Step 4. The Shapley index S for player i of game with a priori unions (v, T) is given by

  $$S_i(v,T) = \varphi_i(w_j), \quad i \in T_j,\ i = 1,2,...n,\ j = 1,2,...,m,$$

  where φ denotes the Shapley value for cooperative game.

We can apply this construction to extend any index defined for simple games and also to extend any value defined for cooperative game.

The Owen construction applied to the Shapley value gets Shapley value with a priori unions.

4 Games with many alternatives

The simple games serve as model in which voter chooses one of two alternatives - he can vote for or against (the bill, the candidate etc.). There are two possible outcomes of the voting - the bill or candidate can pass or defeat. We generalize this model to situation in which voters choose on of r alternatives and the game can have r outcomes. The players are divided into sets voting for particular alternative and whether the coalition wins depends not only on its members (as in simple games) but also on what are other coalitions and what alternatives their vote for. Such a game we will call game with *r* alternatives.

4.1 Definition of game with r alternatives

There are set of voters $N = \{1, 2, ..., n\}$ and r alternatives. Each voter must choose one of the r alternatives. Let Γ_j be the set of voters who choose alternative j. The r-tuple $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_r)$ is called an arrangement of the n voters among the r alternatives. Sets Γ_j are pairwise disjoint, theirs sum is whole set N and some of them can be empty.

We call the pair (Γ_j, Γ) an embedded coalition.

We assume there is a decision rule which tell us which, if any, of the alternatives is chosen for the arrangement Γ. If for the arrangement $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_r)$ the j^{th} alternative is chosen, we say Γ_j wins with respect to the arrangement Γ and write $v(\Gamma_j,\Gamma)=1$; otherwise we set $v(\Gamma_j,\Gamma)=0$ and say Γ_j loses with respect to the arrangement Γ. The function v will be called a voting game with r alternatives if $v(T, \Gamma) = 0$ for $T = \emptyset$.

Example 1. We consider 4-person game v with 3 alternatives. There are voters 1, 2, 3, 4 with weights $w_1 = 10$, $w_2 = 20$, $w_3 = 30$, $w_4 = 40$, and alternatives "for"(Y), "avoid "(A) and "against "(No). Let $\Gamma_Y, \Gamma_A, \Gamma_{No}$ denote set of players voting for, avoiding from voting and voting against. Game v is defined as follows. In voting the alternative Y wins if the sum of weights of voters choosing Y is greater than 51. Alternatives No wins if Y doesn't and at the same time at least one voter chooses No. Alternative A never wins:

- $v(\Gamma_Y, (\Gamma_Y, \Gamma_A, \Gamma_{No})) = 1 \iff \sum_{j \in \Gamma_Y} w_j > 51,$
- $v(\Gamma_{No}, (\Gamma_Y, \Gamma_A, \Gamma_{No})) = 1 \iff v(\Gamma_Y, (\Gamma_Y, \Gamma_A, \Gamma_{No})) = 0 \wedge \Gamma_{No} \neq \emptyset,$
- $v(\Gamma_A, (\Gamma_Y, \Gamma_A, \Gamma_{No})) = 0$ for every $(\Gamma_Y, \Gamma_A, \Gamma_{No})$,
- for the others embedded coalitions v takes value 0.

4.2 The Shapley index for the game with r alternatives

E. Bolger (Bolger, 1993) presented some extension of the Shapley index to class of games with r alternatives and characterized it by some set of axioms. The voter's power is counted for every alternative separately. Therefore the index S for game with r alternatives is an r-tuple $S = (S^1, ..., S^r)$ and a S^j for $j = 1, 2, ..., r$ is the index relative to alternative j.

The Shapley index for voter i related to alternative j for n-person game with r alternatives v introduced by Bolger is given by

$$S^j{}_i(v) = \sum_{\Gamma:\, i \in \Gamma_j} \sum_{k \neq j} \frac{(|\Gamma_j| - 1)!\,(n - |\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}} [v(\Gamma_j, \Gamma) - v(\Gamma_j \backslash \{i\}, \alpha_{i\Gamma_k}\Gamma)].$$

This index is the extension of ordinary indices in the following sense: if we treat simple game as game with two alternatives: "for" and "against", then index with respect to alternative "for" equals the ordinary index for this game.

We could obtain the Shapley index, defined by Bolger, in other way - as a Shapley value φ for average game, that is for some cooperative game, which is connected with game v.

The average game relative to alterative j is a game with player set N defined as follow

$$v^j(S) = \frac{\sum\limits_{\Gamma\,:\,S=\Gamma_j} v(S,\Gamma)}{|\{\ \Gamma : S = \Gamma_j\}|} = \frac{\sum\limits_{\Gamma\,:\,S=\Gamma_j} v(S,\Gamma)}{(r-1)^{n-|S|}}, \quad \text{where} \quad S \subseteq N.$$

The value $v^j(S)$ is simple the average of values $v(S,\Gamma)$, where Γ is r-arrangement with $\Gamma_j = S$.

Theorem 1. The Shapley index defined by Bolger is the Shapley value for v^j, i.e. average game relative to alternative j connected with v:

$$S_i^j(v) = \varphi_i(v^j).$$

Proof.

$$\varphi_i(v^j) =$$

$$\sum_{S\subseteq N,\ i\in S} \frac{(|S|-1)!\,(n-|S|)!}{n!}[v^j(S) - v^j(S\backslash\{i\})] =$$

$$\sum_{S\subseteq N,\ i\in S} \frac{(|S|-1)!\,(n-|S|)!}{n!}\Big[\frac{\sum\limits_{\Gamma\,:\,S=\Gamma_j} v(S,\Gamma)}{(r-1)^{n-|S|}} - \frac{\sum\limits_{\Gamma\,:\,S\backslash\{i\}=\Gamma_j} v(S\backslash\{i\},\Gamma)}{(r-1)^{n-|S\backslash\{i\}|}}\Big] =$$

$$\sum_{S\subseteq N,\ i\in S} \frac{(|S|-1)!\,(n-|S|)!}{n!}\Big[\frac{(r-1)\sum\limits_{\Gamma\,:\,S=\Gamma_j} v(S,\Gamma)}{(r-1)^{n-|S|+1}} - \frac{\sum\limits_{\Gamma\,:\,S\backslash\{i\}=\Gamma_j} v(S\backslash\{i\},\Gamma)}{(r-1)^{n-|S|+1}}\Big] =$$

$$\sum_{S\subseteq N,\ i\in S} \frac{(|S|-1)!\,(n-|S|)!}{n!(r-1)^{n-|S|+1}}\Big[(r-1)\sum_{\Gamma\,:\,S=\Gamma_j} v(S,\Gamma) - \sum_{\Gamma\,:\,S\backslash\{i\}=\Gamma_j} v(S\backslash\{i\},\Gamma)\Big] =$$

$$\sum_{S\subseteq N,\ i\in S} \frac{(|S|-1)!\,(n-|S|)!}{n!(r-1)^{n-|S|+1}}\Big[\sum_{\Gamma\,:\,S=\Gamma_j}\sum_{k\neq j} v(S,\Gamma) - \sum_{\Gamma\,:\,S=\Gamma_j}\sum_{k\neq j} v(S\backslash\{i\},\Gamma_{i\,\Gamma_k})\Big] =$$

$$\sum_{S\subseteq N,\ i\in S}\sum_{\Gamma\,:\,S=\Gamma_j}\sum_{k\neq j} \frac{(|S|-1)!\,(n-|S|)!}{n!(r-1)^{n-|S|+1}}\Big[v(S,\Gamma) - v(S\backslash\{i\},\Gamma_{i\,\Gamma_k})\Big] =$$

$$S_j^j(v).$$

$\square$

Remark 1. In case of cooperative game with r alternatives (i.e. game defined for r-arrangements, which values are non-negative numbers not necessarily equal 0 or 1) we can consider Shapley value φ^j relative to alternative j. The formula for that value is the same as for the Shapley index relative to alterative j. The Shapley value relative to alternative j has also the property from the theorem 1, i.e. $\varphi_i^j(u) = \varphi_i(u^j)$.

5 Definition of game with r alternatives with a priori unions

The simple game with a priori unions is defined as the pair (w, N) composed of simple game and a priori union structure. Similarly we define the game with r alternatives with a priori unions.

Let's v denotes game with r alternatives with player set N and $T = \{T_1, T_2, ..., T_n\}$ is a priori union structure, i.e. a partition of N. The pair (v, N) we call the game with r alternatives with a priori unions.

Example 2. We consider 4-person game v with 3 alternatives from the example 1 and a priori union structure $T = \{\{1, 4\}, \{2, 3\}\}$. Game (v, T) is a 4-person game with 3 alternatives with a priori unions T.

6 The Shapley index for game with many alternatives with a priori unions

In this chapter we show four methods of extending the Shapley index to a class of games with r alternatives with a priori unions. In following examples, which depict use of those methods, we omit the long and complicated calculations.

The Shapley index for game (v, T) is denoted as $S(v, T)$. The symbol $S_i^j(v, T)$ denotes the Shapley index relative to alterative j for game (v, T).

In method directly based on the average game (method I *a*) we use the definition of average game v^j and the theorem 1. The theorem 1 says that the Shapley index for game with r alternatives we can obtain as the Shapley value of some cooperative game associated with game v, i.e. the average game relative to alternative j. In method directly based on the average game (method I *a*) we act similarly. First we change game with r alternatives for some cooperative game and to this game we apply the Owen construction of the Shapley value with a priori unions. It appears that the Shapley index obtained in method I *a* has some bad properties. We can improve this solution with some modification of the average game - we obtain method based on the modified average game (method I *b*).

Two other methods (II *a* and II *b*) are based on the Owen construction adjusted to game with r alternatives. In step 2 of the construction players from outside of the selected coalition (denoted in those methods as K) and belonging to given a priori union T_l must chose alternative different from j. For simple game, where there are only two alternatives, they can do this in one way. In case of game with more than two alternatives they have more possibilities. We consider two cases. In case *a* (method based on the Owen construction without unions preservation) every voter from set $T_l \setminus K$ can chose any alternative $k \neq j$. In case *b* (method based on the Owen construction with unions preservation) all voters from z $T_l \setminus K$ must chose the same alternative $k \neq j$. The case a leads to situation in which value of some cooperative game for empty set is nonnegative. This is contradiction. In case *b* we don't have such problems. In the example we get the same results for methods Ia and IIa - the question is whether is this the general property or just coincident? We show that those two methods lead to the same index.

6.1 Methods based on the average game (methods I *a* and I *b*)

In this paragraph we present two methods of computing the Shapley index S relative to alternative j for game with r alternatives with a priori unions based on the average game.

6.1.1 The method directly based on the average game (method I *a*)

The method I *a* has three steps.

- Step 1. We construct cooperative game v^j, i.e. the average game relative to alternative j connected with game v:

$$v^j(S) = \frac{\sum_{\Gamma\,:\,S=\Gamma_j} v(S,\Gamma)}{|\{\ \Gamma : S = \Gamma_j\}|} = \frac{\sum_{\Gamma\,:\,S=\Gamma_j} v(S,\Gamma)}{(r-1)^{n-|S|}},$$

  for every $S \subseteq N$.

- Step 2. We compute the Shapley value with a priori unions for game (v^j, T).

In this step we use the Owen construction discussed in the chapter 3.

- Step 3. We put $S_i^j(v,T) = \varphi_i(v^j,T)$.

Example 3. We consider 4-person game v with 3 alternatives from the example 2. Values of the Shapley index relative to alternative No computed using the method I *a* are presented below:

$$S_1 v^{No}(v,T) = \tfrac{11}{96}, \quad S_2 v^{No}(v,T) = \tfrac{19}{96}, \quad S_3 v^{No}(v,T) = \tfrac{30}{96}, \quad S_4 v^{No}(v,T) = \tfrac{36}{96}.$$

Results we get are contrary to the intuition. A priori unions $\{1,4\},\{2,3\}$ are symmetrical in game (v,T) because the weight of each of them equals 50, and players $1,4$ and $2,3$ are symmetrical in theirs a priori unions - they play equivalent roles. In such a situation "good " Shapley index should take value $\dfrac{1}{4}$ for each of them.

We can modify the method based directly on the average game, more precisely modify the definition of the average game, to improve the index.

6.1.2 The method based on the modified average game (method I *b*)

In the modified average game relative to alternative j, denoted by $\tilde{v}^j$, we add a priori union structure. In computing $\tilde{v}^j(S)$ we consider smaller set of r-arrangements and we take into account only this ones in which

- set S chooses alternative j,
- each a priori union T_l which is disjoint with S chooses the same arrangement $k \neq j$,
- each a priori union T_l such that $S \cap T_l \neq \emptyset$ and $T_l \setminus S \neq \emptyset$, chooses two arrangements: j and $k \neq j$. Moreover voters from $T_l \cap S$ choose alternative j and voters from $T_l \setminus S$ alternative k.

The set of r-arrangements fulfil conditions above we denote by U.

The modified average game $\tilde{v}^j$ relative to alternative j for game (v,T) is defined as follows. For $S \subseteq N$

$$\tilde{v}^j(S) = \frac{\sum\limits_{\Gamma \in U} v(S,\Gamma)}{|U|},$$

where

$$U = \{\Gamma : S = \Gamma_j \ \wedge \ \forall_{l \in M}\, \exists_{k \neq j}\, T_l \setminus S \subseteq \Gamma_k\}.$$

The cardinality of set U is

$$|U| = (r-1)^{|p \in M\colon\, T_p \not\subseteq S|},$$

so

$$\tilde{v}^j(S) = \frac{\sum\limits_{\Gamma \in U} v(S,\Gamma)}{(r-1)^{|p \in M\colon\, T_p \not\subseteq S|}}.$$

The method based on the modified average game (method I *b*) has three steps.

- Step 1. We construct cooperative game $\tilde{v}^j$, i.e. the modified average game relative to alternative j for game v.
- Step 2. We compute the Shapley value with a priori unions for game $(\tilde{v}^j, T)$.

  In this step we use the Owen construction discussed in the paragraph 3.
- Step 3. We put $S_i^j(v,T) = \varphi_i(\tilde{v}^j, T)$.

Example 4. We consider 4-person game v with 3 alternatives from the example 2. Values of the Shapley index relative to alternative No computed according to the method I *b* are equal

$$S_1^{No}(v,T) = \tfrac{1}{4}, \quad S_2^{No}(v,T) = \tfrac{1}{4}, \quad S_3^{No}(v,T) = \tfrac{1}{4}, \quad S_4^{No}(v,T) = \tfrac{1}{4}.$$

The results we get now are consistent with intuition.

6.2 Direct computation methods (methods II *a* and II *b*)

In this paragraph we present two methods of computing the Shapley index for games with r alternatives with a priori unions which based on generalization of the Owen construction. This construction is discussed in the paragraph 3.

In the step 2 of the Owen construction we construct altered games by replacing a priori union T_j with its subset denoted by K. In the case of simple game we have straight situation - voters from $T_l \cap K$ vote for the bill and voters from $T_l \setminus K$ vote against it. In the case of games with more than two alternatives voters from $T_l \cap K$ chose alternative j but voters from $T_l \setminus K$ have more than one option. We consider two cases

- method II a (the method based on the Owen construction without unions preservation): each voter from set $T_l \setminus K$ can chose any alternative $k \neq j$,
- method II b (the method based on the Owen construction with unions preservation): all voters from z $T_l \setminus K$ must chose the same alternative $k \neq j$.

Now we discuss those two methods.

6.2.1 The method based on the Owen construction without unions preservation (met. II *a*)

There is n-person game (v, T) with r alternatives with a priori unions T. The method of computing the Shapley index based on the Owen construction without unions preservation has the following steps.

• Step 1. We form the quotient game $u = v/T$ with the player set $M = \{1, 2, ..., m\}$. Let's $\Upsilon = (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)$ is any r-arrangement of set M. We define game u as follows. For $j = 1, 2, ...m$

$$u(\Upsilon_j, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = v(T_{\Upsilon_j}, (T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_r})),$$

where $T_X = \bigcup_{k \in X} T_k$, for any $X \subseteq M$.

• Step 2. We construct the family of altered games $u_{T_l,K}$ for r-arrangements Υ of set M. For every $l \in M$ and every $K \subseteq T_l$ we define value $u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r))$, where $C = \Upsilon_j$, as follows.

- If $l \notin C$, then
$$u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = u(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)),$$
- if $l \in C$ and $K = T_l$, then
$$u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = u(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)),$$
- if $l \in C$ and $K \neq T_l$, then in r-arrangement $(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_j}, ..., T_{\Upsilon_r})$ = $(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_C, ..., T_{\Upsilon_r})$ of set N we replace the set $T_l \subseteq T_C$ with set K and voters from $T_l \setminus K$ we add to some sets T_{Υ_k} where $k \neq j$. Next we compute the average of values of game v taken for matching embedded coalitions:
$$u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = \frac{\sum_{\Gamma \in X} v(T_{C \setminus \{l\}} \cup K, \Gamma)}{|X|} = \frac{\sum_{\Gamma \in X} v(T_{C \setminus \{l\}} \cup K, \Gamma)}{(r-1)^{|T_l \setminus K|}},$$

where

$$X = \{\Gamma : \ (\Gamma_j = T_{C\setminus\{l\}} \cup K), \ (\forall_{k\neq j} T_{\Upsilon_k} \subseteq \Gamma_k), (\forall_{i\in T_l\setminus K} \exists_{h\neq j} \ i \in \Gamma_h)\}.$$

• Step 3. For every alternative $j \in \{1, 2, ..., r\}$, every player $l \in M$ and every set $K \subseteq T_l$ we define reduced game w_l^j on T_l. The value $w_l^j(K)$ is equal the Shapley value of player l in game $u_{T_l,K}$ relative to alternative j:

$$w_l^j(K) = \varphi_l^j(u_{T_l,K}).$$

• Step 4. The Shapley index S for player i relative to alternative j of game with r alternatives with a priori unions (v, T) is given by

$$S_i^j(v, T) = \varphi_i(w_l^j),$$

for

$$i \in T_l, \ i = 1, 2, ...n, \ l = 1, 2, ..., m, \ j = 1, 2, ..., r.$$

Example 5. We consider 4-person game from the example 1 but now with a priori unions $T = \{\{1, 2\}, \{3\}, \{4\}\}$. During calculating of the Shapley index relative to alternative No as one of the intermediate results in step 3 we get $w_1^{No}(\emptyset) = \varphi_1(u_{T_1,\emptyset}^{No}) = \frac{1}{48} \neq 0$. There is a contradiction, because each cooperative game must take value zero for empty set. Thus we have to reject the method II *a*.

6.2.2 The method based on the Owen construction with unions preservation (met. II *b*)

There is n-person game (v, T) with r alternatives with a priori unions T. The method of computing the Shapley index based on the Owen construction without unions preservation has the following steps.

• Step 1. We form the quotient game $u = v/T$ with the player set $M = \{1, 2, ..., m\}$. Let's $\Upsilon = (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)$ is any r-arrangement of set M. We define game u as follows. For $j = 1, 2, ...m$

$$u(\Upsilon_j, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = v(T_{\Upsilon_j}, (T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_r})).$$

• Step 2. We construct the family of altered games $u_{T_l,K}$ for r-arrangements Υ of set M. For every $l \in M$ and every $K \subseteq T_l$ we define value $u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r))$, where $C = \Upsilon_j$, as follows.

- If $l \notin C$, then

$$u_{T_l,K}(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r)) = u(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r)),$$

- if $l \in C$ and $K = T_l$, then

$$u_{T_l,K}(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r)) = u(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r)),$$

- if $l \in C$ and $K \neq T_l$, then in r-arrangement $(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_j}, ..., T_{\Upsilon_r})$ = $(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_C, ..., T_{\Upsilon_r})$ of set N we replace the set $T_l \subseteq T_C$ with set K and the whole set $T_l \setminus K$ we add to one of sets T_{Υ_k} for $k \neq j$. Next we compute the average of values of game v taken for matching embedded coalitions:

$$u_{T_l,K}(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r)) = \tfrac{1}{r-1}\cdot$$

$$\cdot\Big[v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1} \cup (T_l \setminus K), T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r}\Big)\Big)+$$

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2} \cup (T_l \setminus K), ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r}\Big)\Big)+$$

...

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}} \cup (T_l \setminus K), T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r}\Big)\Big)+$$

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}} \cup (T_l \setminus K), ..., T_{\Upsilon_r}\Big)\Big)+$$

...

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r} \cup (T_l \setminus K)\Big)\Big)\Big].$$

• Step 3. For every alternative $j \in \{1, 2, ..., r\}$, every player $l \in M$ and every set $K \subseteq T_l$ we define reduced game w_l^j on T_l. The value $w_l^j(K)$ is equal the Shapley value of player l in game $u_{T_l,K}$ relative to alternative j:

$$w_l^j(K) = \varphi_l^j(u_{T_l,K}).$$

• Step 4. The Shapley index S for player i relative to alternative j of game with r alternatives with a priori unions (v, T) is given by

$$S_i^j(v, T) = \varphi_i(w_l^j),$$

for

$$i \in T_l,\ i = 1, 2, ...n,\ l = 1, 2, ..., m,\ j = 1, 2, ..., r.$$

Example 6. We consider 4-person game with r alternatives with a priori unions from the example 2. Values of the Shapley index relative to alternative No computed using method II *b* are equal

$$S_1^{No}(v, T) = \tfrac{1}{4}, \quad S_2^{No}(v, T) = \tfrac{1}{4}, \quad S_3^{No}(v, T) = \tfrac{1}{4}, \quad S_4^{No}(v, T) = \tfrac{1}{4}.$$

We can notice that results for game from the example 2 obtained with method I *b* (example 4) and method II *b* (example 6) are equal. It is not accidental situation - the following theorem is satisfied.

Theorem 2. The method based on the modified average game (method I *b*) and the method based on the Owen construction with unions preservation (method II *b*) lead to the same index.

Proof. We show in detail steps of two methods and we prove that two functions - $\tilde{u}_{T_l,K}^j$ from method I *b* and $u_{T_l,K}^j$ from method II *b* are identical. We show that this fact implies identity of the Shapley index obtained with those two methods.

There is game (v, T) with r alternatives with a priori unions.

The method I *b* based on the modified average game has the following steps.

- Step 1. We construct the modified average game cooperative game $\tilde{v}^j$ relative to alternative j for game (v, T). For coalition $S \subseteq N$ the game $\tilde{v}^j$ takes value

$$\tilde{v}^j(S) = \frac{\sum\limits_{\Gamma \in U} v(S, \Gamma)}{|U|} = \frac{\sum\limits_{\Gamma \in U} v(S, \Gamma)}{(r-1)^{|p \in M:\ T_p \not\subseteq S|}},$$

where

$$U = \{\Gamma : S = \Gamma_j \ \wedge \ \forall_{l \in M}\ \exists_{k \neq j}\ T_l \setminus S \subseteq \Gamma_k\}.$$

- Step 2. We compute the Shapley value with a priori unions for game $(\tilde{v}^j, T)$.

  In this step we use the Owen construction discussed in the paragraph 3. Steps of this construction we denote by numbers 2.1-2.4.

  - Step 2.1. We form the quotient game $\tilde{u}^j$ defined for all subsets $C \subseteq M$

$$\tilde{u}^j(C) = \tilde{v}^j(T_C) = \frac{\sum\limits_{\Upsilon:\ C=\Upsilon_j} v(T_C, (T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_r}))}{(r-1)^{|M \setminus C|}}.$$

  - Step 2.2. We construct the family of altered games $\tilde{u}^j_{T_l,K}$. For every $l \in M$, every $K \subseteq T_l$ and every $C \subseteq M$ we define $\tilde{u}^j_{T_l,K}(C)$:
    * if $l \notin C$, then

$$\tilde{u}^j_{T_l,K}(C) = \tilde{u}^j(C) = \frac{\sum\limits_{\Upsilon:\ C=\Upsilon_j} v(T_C, (T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_r}))}{(r-1)^{|M \setminus C|}},$$

    * if $l \in C$ and $K = T_l$, then

$$\tilde{u}^j_{T_l,K}(C) = \tilde{u}^j(C) = \frac{\sum\limits_{\Upsilon:\ C=\Upsilon_j} v(T_C, (T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_r}))}{(r-1)^{|M \setminus C|}},$$

    * if $l \in C$ and $K \neq T_l$, then

$$\tilde{u}^j_{T_l,K}(C) = \tilde{v}^j(T_{C \setminus \{l\}} \cup K).$$

      Taking into account the definition of the average game relative to alternative j for game $\tilde{v}$ we get

$$\tilde{v}^j(T_{C \setminus \{l\}} \cup K) =$$

$$= \frac{\sum\limits_{\Gamma \in Y} v(T_{C \setminus \{l\}} \cup K, \Gamma)}{|Y|} = \frac{\sum\limits_{\Gamma \in Y} v(T_{C \setminus \{l\}} \cup K, \Gamma)}{(r-1)^{|M \setminus C|+1}},$$

where

$$Y = \{\Gamma : \Gamma_j = T_{C\setminus\{l\}} \cup K,\ (\forall_{p\neq l} \exists_{k\neq j}\, T_p\ \subseteq \Gamma_k), \exists_{h\neq j} T_l \setminus K \subseteq \Gamma_h\},$$

thus

$$\tilde{u}^j_{T_l,K}(C) = \frac{\sum_{\Gamma\in Y} v(T_{C\setminus\{l\}} \cup K, \Gamma)}{(r-1)^{|M\setminus C|+1}}.$$

- Step 2.3. We define reduced games $\tilde{w}^j_l$. For every $l \in M$ and every $K \subseteq T_l$
$$\tilde{w}^j_l(K) = \varphi_l(\tilde{u}^j_{T_l,K}).$$
- Step 2.4. The Shapley index S for player i of game with a priori unions $(\tilde{v}^j, T)$ is given by
$$\varphi_i(\tilde{v}^j, T) = \varphi_i(\tilde{w}^j_l),$$
where $i \in T_l$.

- Step 3. We put $S^j_i(v, T) = \varphi_i(\tilde{v}^j, T)$.

The method II *b* based on the Owen construction with unions preservation has the following steps.

- Step 1. We form the quotient game $u = v/T$ for r-arrangements Υ of set M:

$$u(\Upsilon_j, \Upsilon) = u(\Upsilon_j, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = v(T_{\Upsilon_j}, (T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_r})).$$

- Step 2. We construct the family of altered games $u_{T_l,K}$ for r-arrangements Υ of set M. For every $l \in M$ and every $K \subseteq T_l$ we define value $u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r))$, where $C = \Upsilon_j$, as follows.

  - If $l \notin C$, then
  $$u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = u(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)),$$
  - if $l \in C$ and $K = T_l$, then
  $$u_{T_l,K}(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)) = u(C, (\Upsilon_1, \Upsilon_2, ..., \Upsilon_r)),$$
  - if $l \in C$ and $K \neq T_l$, then

$u_{T_l,K}(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r)) = \frac{1}{r-1}\cdot$

$$\cdot\Big[v\Big(T_{C\setminus\{l\}}\cup K,\Big(T_{\Upsilon_1}\cup(T_l\setminus K),T_{\Upsilon_2},...,T_{\Upsilon_{j-1}},T_{C\setminus\{l\}}\cup K,T_{\Upsilon_{j+1}},...,T_{\Upsilon_r}\Big)\Big)+$$

$$+v\Big(T_{C\setminus\{l\}}\cup K,\Big(T_{\Upsilon_1},T_{\Upsilon_2}\cup(T_l\setminus K),...,T_{\Upsilon_{j-1}},T_{C\setminus\{l\}}\cup K,T_{\Upsilon_{j+1}},...,T_{\Upsilon_r}\Big)\Big)+$$

...

$$+v\Big(T_{C\setminus\{l\}}\cup K,\Big(T_{\Upsilon_1},T_{\Upsilon_2},...,T_{\Upsilon_{j-1}}\cup(T_l\setminus K),T_{C\setminus\{l\}}\cup K,T_{\Upsilon_{j+1}},...,T_{\Upsilon_r}\Big)\Big)+$$

$$+v\Big(T_{C\setminus\{l\}}\cup K,\Big(T_{\Upsilon_1},T_{\Upsilon_2},...,T_{\Upsilon_{j-1}},T_{C\setminus\{l\}}\cup K,T_{\Upsilon_{j+1}}\cup(T_l\setminus K),...,T_{\Upsilon_r}\Big)\Big)+$$

...

$$+v\Big(T_{C\setminus\{l\}}\cup K,\Big(T_{\Upsilon_1},T_{\Upsilon_2},...,T_{\Upsilon_{j-1}},T_{C\setminus\{l\}}\cup K,T_{\Upsilon_{j+1}},...,T_{\Upsilon_r}\cup(T_l\setminus K)\Big)\Big)\Big].$$

- Step 3. For every alternative $j\in\{1,2,...,r\}$, every player $l\in M$ and every set $K\subseteq T_l$ we define reduced game w_l^j on T_l.

  $$w_l^j(K)=\varphi_l^j(u_{T_l,K}).$$

  By virtue of property of the Shapley value for games with many alternatives discussed in the paragraph 4, in the theorem 1 we have

  $$\varphi_l^j(u_{T_l,K})=\varphi_l(u_{T_l,K}^j),$$

  thus

  $$w_l^j(K)=\varphi_l(u_{T_l,K}^j).$$

- Step 4. The Shapley index S for player i relative to alternative j of game with r alternatives with a priori unions (v,T) is given by

  $$S_i^j(v,T)=\varphi_i(w_l^j),$$

  for

  $$i\in T_l,\ i=1,2,...n,\ l=1,2,...,m,\ j=1,2,...,r.$$

If we show that the altered game $\tilde{u}^j_{T_l,K}$ from the step 2 (2.2) of method based on the modified average game is equal the average game $u^j_{T_l,K}$ from the step 4 of the method based on the Owen construction with unions preservation, it will prove that reduced games $\tilde{w}^j_l$ from method I *b* and w^j_l from method II *b* are equal, thus values of the Shapley index $S^j_i(v,T)$ obtained with both methods are equal.

We compute value of the average game for $u_{T_l,K}$ and set $C \subseteq M$ relative to alternative j, i.e. $u^j_{T_l,K}(C)$ and check that it is equal to value $\tilde{u}^j_{T_l,K}(C)$.

We have

- if $l \notin C$
  or if $l \in C$ and $T_l = C$, then

$$u^j_{T_l,K}(C) =$$

$$= \frac{\sum\limits_{\Upsilon: C=\Upsilon_j} u_{T_l,K}(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r))}{|\Upsilon : C = \Upsilon_j|} =$$

$$= \frac{\sum\limits_{\Upsilon:\ C=\Upsilon_j} u(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r))}{|\Upsilon : C = \Upsilon_j|} =$$

$$= \frac{\sum\limits_{\Upsilon:\ C=\Upsilon_j} v(T_C,(T_{\Upsilon_1},T_{\Upsilon_2},...,T_{\Upsilon_r}))}{|\Upsilon : C = \Upsilon_j|} =$$

$$= \frac{\sum\limits_{\Upsilon:\ C=\Upsilon_j} v(T_C,(T_{\Upsilon_1},T_{\Upsilon_2},...,T_{\Upsilon_r}))}{(r-1)^{|M\setminus C|}}$$

$$= \tilde{u}^j_{T_l,K}(C)$$

  First equality comes from the definition of the average game relative to alternative j, second one from the definition of game $u_{T_l,K}$ and third from the definition of the quotient game u.

- if $l \in C$ and $T_l \neq C$, then

$$u^j_{T_l,K}(C) = \frac{\sum\limits_{\Upsilon:C=\Upsilon_j} u_{T_l,K}(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r))}{|\Upsilon : C = \Upsilon_j|} =$$

$$= \frac{1}{(r-1)^{|M\setminus C|}} \sum_{\Upsilon:C=\Upsilon_j} u_{T_l,K}(C,(\Upsilon_1,\Upsilon_2,...,\Upsilon_r)) =$$

$$= \frac{1}{(r-1)^{|M\setminus C|}} \cdot \frac{1}{(r-1)} \cdot \sum_{\Upsilon:C=\Upsilon_j}$$

$$\Big[v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1} \cup (T_l \setminus K), T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r}\Big)\Big) +$$

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2} \cup (T_l \setminus K), ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r}\Big)\Big) +$$

...

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}} \cup (T_l \setminus K), T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r}\Big)\Big) +$$

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}} \cup (T_l \setminus K), ..., T_{\Upsilon_r}\Big)\Big) +$$

...

$$+v\Big(T_{C\setminus\{l\}} \cup K, \Big(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., T_{\Upsilon_{j-1}}, T_{C\setminus\{l\}} \cup K, T_{\Upsilon_{j+1}}, ..., T_{\Upsilon_r} \cup (T_l \setminus K)\Big)\Big)\Big].$$

In this formula we sum values of function v for embedded coalitions of N as $(T_{C\setminus\{l\}} \cup K, \Gamma)$. Every occurring r-arrangement arises from r-arrangement $(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., \underbrace{T_C}_{j}, ..., T_{\Upsilon_r})$ by replacing set $T_l \subseteq T_C$ with set K and adding set $T_l \setminus K$ to some T_{Υ_h}, where $h \neq j$. All r-arrangements $(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., \underbrace{T_C}_{j}, ..., T_{\Upsilon_r})$ are used.

In formula for the value $\tilde{u}^j_{T_l,K}(C)$ appeared in method II *b*, i.e.

$$\tilde{u}^j_{T_l,K}(C) = \frac{\sum_{\Gamma \in Y} v(T_{C \setminus \{l\}} \cup K, \Gamma)}{(r-1)^{|M \setminus C|+1}},$$

under the sign of the sum also occur values of v for embedded coalitions as $(T_{C\setminus\{l\}} \cup K, \Gamma)$ for some Γ. In each of r-arrangements set $T_{C\setminus\{l\}} \cup K$ chooses alternative j, each a priori union T_p, where $p \neq l$, as a whole chooses one alternatives $k \neq j$ and whole set $T_l \setminus K$ chooses one alternative $h \neq j$.

We can see, that every such r-arrangement is built from some r-arrangement $(T_{\Upsilon_1}, T_{\Upsilon_2}, ..., \underbrace{T_C}_{j}, ..., T_{\Upsilon_r})$, where $l \in C$, by substitution of set $T_l \subseteq T_C$ with set K and adding set $T_l \setminus K$ to some set T_{Υ_h}, where $h \neq j$.

Thus, every r-arrangement that appears in the formula for $\tilde{u}^j_{T_l,K}(C)$ appears also in the formula for $u^j_{T_l,K}(C)$. The number of r-arrangement in both formulas is the same and equals $(r-1)^{|M\setminus C|+1}$ - what means that the same r-arrangements Γ appear in all the sums, so the same values of function v on embedded coalitions $(T_{C\setminus\{l\}} \cup K, \Gamma)$. In both formulas we have the same factor $\frac{1}{(r-1)^{|M\setminus C|+1}}$. It follows that $u^j_{T_l,K}(C) = \tilde{u}^j_{T_l,K}(C)$.

We showed that for every $l \in M$ and every $K \subseteq T_l$ hold $u^j_{T_l,K} = \tilde{u}^j_{T_l,K}$. Thus for every game with many alternatives with a priori unions the Shapley value calculated with method based on modified average game is identical to the one calculated with method based on the Owen construction with unions preservation.

□

7 Summary

The paper presents two pairs of methods for calculation of the Shapley index for games with many alternatives with a priori unions. The methods of the first pair are based on the classical Owen construction with a priori unions on some cooperative game derived from game with many alternatives. Methods of the second pair use construction analogous to the Owen construction directly on game with many alternatives. Inside the pair the methods differ in that, that for one of them some restriction is imposed.

The restriction is that the voters from broken coalition have to vote for the same alternative. No boundary condition leads to the results contrary to common sense. The methods with restriction give better results and lead to the same index.

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