Testing Proposer and Voter Rationality*

Tasos Kalandrakis†

November 12, 2010

[PRELIMINARY & INCOMPLETE]

Abstract

Consider data that for each of a finite number of periods record a status quo policy, a proposal offered by some known committee member (possibly different across periods), and the votes for or against the proposal by all committee members. Can we refute the hypothesis that the proposer offers the best possible proposal, that the voters vote optimally according to an unobserved utility function, and that each utility function is strictly concave? We derive necessary and sufficient conditions that take the form of a system of equalities and inequalities that are polynomial in the unknowns. We apply quantifier elimination to derive equivalent conditions that involve only the known location of the voting alternatives in the one-dimensional case. These conditions impose testable restrictions on the data that are significantly more potent than the restrictions derived by voter rationality and concavity alone (as studied in Kalandrakis (2010)). In fact, we show that in one policy dimension these conditions imply testable restrictions on the data even if the proposal and status quo are unobserved.

1 Introduction

Many collective choice procedures share as common feature the following sequential interaction between a proposer and the members of a collective body (a committee, legislature, etc.): the proposer first puts forth a policy to replace the status quo (or chooses to maintain the status quo policy) and then the members of the committee vote in favor or against that proposal. Government formation in multiparty parliamentary systems is a prominent example of actual collective

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*Thanks to audiences at the 2010 APSA and the Wallis working paper series for helpful comments. I am solely responsible for all errors.

†Department of Political Science, University of Rochester. E-mail: kalandrakis@rochester.edu

choice processes for which the above procedure constitutes a reasonable (if not stylized) description, with the formateur in the role of the proposer and the political parties in the role of the voters. Modulo complications in the crafting of the agenda, legislative deliberation in most sophisticated legislatures must involve the careful design of new proposals prior to these proposals coming to a vote. It is no accident then that the interplay between proposers and voters is the focus of many theoretical models of collective choice. For example, this interaction exactly describes the object of study in the influential *agenda-setting* model of Romer and Rosenthal (1978) and constitutes a key building block in most non-cooperative models of collective decision-making (e.g., Baron and Ferejohn (1989)) since.

In this paper we assume that we have data generated over time from the above procedure. Specifically, in each of a finite number of periods we observe a status quo policy, a proposer, a proposal, the set of individuals voting in favor of the proposal, and the voting rule that was in effect in that period. Both the identity of the proposer and the voting rule may change over time. We assume that all proposers and voters have policy preferences represented by a utility function. We ask under what conditions are these data consistent with the hypothesis that all proposer and all voting decisions are optimal, that is, (i) the proposer optimizes (i.e., chooses the best possible proposal given what is likely to be approved by the voters) and (ii) each vote is cast in favor of the alternative that the individual voter (weakly) prefers?

It is not surprising that without restrictions on individual preferences these optimality conditions on the part of the proposer and voters imply no testable restrictions on the data. Thus, we add structure to the problem by assuming that individual preferences are represented by strictly concave utility functions.[1] We derive necessary and sufficient conditions in order for any data to be consistent with the hypotheses that proposers and voters optimize and have preferences represented by strictly concave utility functions. These conditions take the form of a system of polynomial equalities and inequalities that are quadratic in a set of unknown quantities that amount to utility levels and supergradients of the individuals' utility functions, and Lagrange multipliers as well as putative locations of alternative candidate proposals.

Verification of these conditions on a computer yields at the same time an instance of the utility functions of the individuals. These conditions can serve as direct tests of the stipulated

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[1] All results hold if we instead assume merely concave utility functions coupled with a limited shared weak preference condition.

hypotheses but can also provide a basis for the non-parametric estimation of individual preferences and prediction of future choices. This is a standard component of the research agenda of the revealed preference theory of the consumer from finite data sets developed in economics with the pioneering work of Afriat (1967), and subsequently Varian (1982) and others. The work of Brown and Matzkin (1996) extends this approach to explore the testable implications of competitive economic equilibria. A recent literature explores testable implications of Nash equilibrium and its variants. The present contribution lies primarily in the game-theoretic, rather than the decision theoretic side, of this literature. Indeed, our necessary and sufficient conditions can be thought of as characterizing data generated from equilibria of two different non-cooperative game forms: agenda-setting games as in Romer and Rosenthal (1978), or (under additional assumptions) dynamic games with endogenous status quo as in Baron (1996), Kalandrakis (2004), etc.

Kalandrakis (2010) derives necessary and sufficient conditions in order for voting data to be consistent with the hypothesis that individual voters maximize a (quasi-)concave utility function. These conditions impose testable restrictions on voting data when the voting alternatives are observed. When the voting alternatives are not observed, then Kalandrakis shows that the hypotheses of vote optimality and convexity of preferences are not testable with any finite data even in one dimension. In fact, such data impose no testable restrictions on the joint location of voters' ideal points. These results foreclose the possibility for the non-parametric estimation of voter preferences in the absence of knowledge on the location of the voting alternatives. Kalandrakis shows that this is the case in two or more dimensions even if it is known that certain alternatives are recurring in the voting agenda (though their location is still unknown).

In the present study, we strengthen the assumptions on the data generating process by assuming that one of the two alternatives in the voting agenda arises as a result of optimization on the part of a proposer. As we have already pointed out, many datasets incorporate this extra information in a natural way. The optimization calculus of the proposer significantly constrains the likely location of the proposals and provides indirect information about the preferences of other committee members by ruling out alternative proposals as acceptable by a winning coalition of the committee. As a result, the necessary and sufficient conditions we derive in this study are significantly stronger than those derived in Kalandrakis (2010). In fact, we show that in the one-dimensional case these conditions impose testable restrictions on the data even if the location of the voting alternatives is unknown.

In what follows, we first set up preliminaries and notation. We derive tests of proposer/voter rationality in the ensuing section. We focus on the special environment of one dimensional politics in section 4. In section 5 we consider the same questions assuming the policies are unobserved.

2 Data and Axioms

2.1 Data

Assume the existence of data generated by a set $N = \{1, \ldots, n\}$ of $n \geq 2$ individuals. Each period $t = 1, \ldots, T$ a subset of these individuals convene in order to decide on a policy drawn from d-dimensional Euclidean space $\mathbb{R}^d$, with $d \geq 1$.[2] As analysts we observe a sextet:

$$(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t),$$

for each period t. Specifically,

- $N_t \subseteq N$ is the set of individuals that can cast a vote in period t,
- $\mathcal{D}_t \subset 2^N_t$ is a non-empty collection of decisive coalitions, i.e., subsets of the set N_t whose approval is sufficient to change the status quo policy,
- $q_t \in \mathbb{R}^d$ is the status quo policy,
- $i_t \in N_t$ is the proposer,
- $z_t \in \mathbb{R}^d \cup \{z_0\}$ is the proposal offered by i_t, and
- $C_t \subseteq N_t$ is the set of individuals that vote in favor of the proposal.

Note that we allow the proposer to offer a null proposal $z_t = z_0 \notin \mathbb{R}^d$ which amounts to the proposer passing on the opportunity to offer a proposal. In these cases we follow the convention that $C_t = \emptyset$. We also require that $z_t \neq q_t$ for all t such that $z_t \in \mathbb{R}^d$, that is, the either the proposal is different than the status quo or the proposer passes. This may reflect a restriction on the data

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[2] It is straightforward to extend the analysis to the case policies must lie in a convex subset of $\mathbb{R}^d$ cut out by concave inequality constraints.

generating process or simply constitute a convention in the way we record the data, that is, either the proposer cannot offer $z_t = q_t$ or we record $z_t = z_0$ and $C_t = \emptyset$ whenever $z_t = q_t$. Lastly, we require a standard monotonicity assumption on the set of decisive coalitions so that for each t, if $C \in \mathcal{D}_t$ and $C \subseteq C' \subset N_t$ then $C' \in \mathcal{D}_t$.

We partition the data into three subsets according to whether the proposer's proposal is accepted, rejected, or the proposer passed in that period:

$$\mathcal{T}^a = \{t | C_t \in \mathcal{D}_t \text{ and } z_t \in \mathbb{R}^d\},$$

$$\mathcal{T}^r = \{t | C_t \notin \mathcal{D}_t \text{ and } z_t \in \mathbb{R}^d\},$$

$$\mathcal{T}^p = \{t | z_t = z_0\}.$$

We denote the set of policies that appear in the data either as status quo or as a proposal by

$$X = \bigcup_{t \in \mathcal{T}^a \cup \mathcal{T}^r} \{z_t, q_t\} \bigcup_{t \in \mathcal{T}^p} \{q_t\}.$$

The voting decisions of individual i in data $(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)_{t=1}^T$ define an irreflexive relation $V_i \subset X \times X$

$$V_i = \{(z_t, q_t) \mid i \in C_t, t \in \mathcal{T}^a \cup \mathcal{T}^r\} \cup \{(q_t, z_t) \mid i \in N_t \setminus C_t, t \in \mathcal{T}^a \cup \mathcal{T}^r\}.$$

We denote the collective decision reached in period t by x_t where

$$x_t = \begin{cases} z_t & \text{if} \quad t \in \mathcal{T}^a \\ q_t & \text{if} \quad t \in \mathcal{T}^r \cup \mathcal{T}^p. \end{cases}$$

Accordingly, the collective decision in period t is the proposal if a proposal is offered and receives the approval of a decisive coalition in that period, and the status quo if either the proposal did not garner the approval of a decisive coalition or if the proposer passed.

We assume that each individual $i \in N$ has preferences over X that are represented by a real-valued (but *unobserved*) utility function $u_i : \mathbb{R}^d \to \mathbb{R}$.[3] We denote a vector of such utility

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[3] Thus, in the analysis in the following section we eschew the question of when the data are consistent with the existence of such a representation.

functions by $u = (u_1, \ldots, u_n)$.

2.2 A Data Generating Process and a Test

One immediate interpretation of this setup is that the data are generated by equilibria of a sequence of agenda setting (Romer and Rosenthal (1978)) games $\Gamma(N_t, u; i_t, q_t, \mathcal{D}_t)$. In game $\Gamma(N_t, u; i_t, q_t, \mathcal{D}_t)$ the proposer i_t moves first offering a proposal or passing, and if a proposal is offered then all players N_t vote between the proposal and the status quo q_t. Thus, a strategy for the proposer is given by $z \in (\mathbb{R}^d \setminus \{q_t\}) \cup \{z_0\}$[4] and a strategy for each voter is represented by a function $v_i : \mathbb{R}^d \to \{yes, no\}$. Given such voting strategies $v = (v_i)_{i \in N_t}$, the set of proposals that can be approved by a decisive coalition is given by

$$A(v) = \{y \mid \{i \mid v_i(y) = yes\} \in \mathcal{D}_t\}.$$

A standard equilibrium concept for this game is subgame perfect Nash with weakly dominated voting strategies eliminated:

Definition 1. *An equilibrium for game $\Gamma(N_t, u; i_t, q_t, \mathcal{D}_t)$ is a profile of strategies (z, v) such that*

$$\begin{aligned}
&(1) \qquad & v_i(x) &= \begin{cases} yes & \textit{if} \quad u_i(x) > u_i(q_t) \\ no & \textit{if} \quad u_i(x) < u_i(q_t) \end{cases} \quad i \in N_t, x \in \mathbb{R}^d \setminus \{q_t\}, \\
&(2) \qquad & u_{i_t}(z) &\geq \sup\{u_{i_t}(x) \mid x \in A(v) \cup \{q_t\}\} \ \textit{if}\ z \in A(v), \\
&(3) \qquad & u_{i_t}(q_t) &= \sup\{u_{i_t}(x) \mid x \in A(v) \cup \{q_t\}\} \ \textit{if}\ z \notin A(v)
\end{aligned}$$

In the above interpretation, the data are generated by a sequence of static agenda setting games. Under what conditions can we refute the hypothesis that the data are generated from equilibria of these games? The following rationalizability criterion provides a test based on three axioms:

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[4] As discussed earlier, we can assume that a proposal strategy takes the form $z \in \mathbb{R}^d$ and that the data record z_0 and $C_t = \emptyset$ whenever the proposer chooses $z = q_t$.

Definition 2. *The utility functions* $u_i : \mathbb{R}^d \to \mathbb{R}, i \in N$ *rationalize the data* $(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)_{t=1}^T$ *if*

(VR) $u_i(x) \geq u_i(y)$ *for all* $(x, y) \in V_i$ *and all* $i \in N$.

(PR) $u_{i_t}(x_t) = \sup\{u_{i_t}(x) \mid x \in \{y \mid \{i \mid u_i(y) > u_i(q_t)\} \in \mathcal{D}_t\} \cup \{q_t\}\}$ *for all* $t = 1, \ldots, T$,

(UC) $u_i : X \to \mathbb{R}$ *is strictly concave for all* $i \in N$.

The *voter rationality* (VR) axiom requires that those voting in favor of the proposal prefer it over the status quo and those voting against the proposal prefer the status quo instead. Note that we only impose restrictions on the preference between the proposal and the status quo from the voting choices of the individuals that are active in each period t. This allows us to accommodate data that feature legislators or committee members that either are legitimately absent from the proceedings, or stay out of office for one term, etc. Furthermore, axiom (VR) does not presume that voters resolve indifference in favor of the proposal or the status quo.

The second axiom (PR) is a *proposer rationality* axiom. Note the two-pronged reach of the axiom. First, the proposer must be achieving her highest possible utility level when putting forth a proposal that garners support from a decisive coalition. Second, when the proposer maintains the status quo (either by passing or by proposing an alternative that fails at the voting stage) there must not exist an alternative proposal that is strictly preferred by all members of a decisive coalition that yields a higher utility level for the proposer.

Exacting as they may appear, axioms (VR) and (PR) impose no testable restrictions on finite data without some additional structure on the shape of individual utility functions. Here we insist on a non-parametric approach and limit ourselves to strict concavity using axiom (UC). We do note that we can weaken axiom (UC) in various directions maintaining convexity of individual preferences.

Axioms (VR), (PR), and (UC) constitute a test of the hypothesis that the data are generated by equilibria of a sequence of setter models in the following sense:

Theorem 1. *(i) If data* $(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)_{t=1}^T$ *are rationalized by* u *then there exists an equilibrium of the game* $\Gamma(N_t, u; i_t, q_t, \mathcal{D}_t)$ *such that the equilibrium proposal is* z_t *and it is approved by* C_t *for all* t.

(ii) If z_t, C_t are possible equilibrium outcomes of the game $\Gamma(N_t, u; i_t, q_t, \mathcal{D}_t)$ for each t and u satisfy (UC), then u rationalize the data $(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)_{t=1}^{T}$.

There is an alternative interpretation of the data generating process that we do not pursue here, namely, u could be interpreted as (endogenous) dynamic utilities emerging from dynamic bargaining games with endogenous status quo (e.g., Baron (1996), Kalandrakis (2004), etc.). In that case the axioms test whether the data are generated from an equilibrium of such a game with strictly concave endogenous dynamic payoff functions.

Axioms (VR) and (UC) (along with many variations) have been exhaustively studied by Kalandrakis (2010). In the next section we study testable implications when we combine these axioms with proposer rationality (PR).

3 Testable Restrictions

To extract the testable implications of (VR), (PR), and (UC) we will focus on the optimization conditions for the proposer. Not surprisingly, conditions necessary and sufficient for the proposer to optimize will entail some consideration of the minimum decisive coalitions available to the proposer. Accordingly, we define the set of coalitions that exclude the proposer i_t, are decisive in period t with the inclusion of the proposer i_t, and cease to be decisive if any member other than i_t is removed:

$$\mathcal{M}_t = \{C \subseteq N \setminus \{i_t\} \mid \{i_t\} \cup C \in \mathcal{D}_t \text{ and } \{i_t\} \cup C \setminus \{j\} \notin \mathcal{D}_t \text{ for all } j \in C\}.$$

Let m_t denote the cardinality of $\mathcal{M}_t$. Of these minimum decisive coalitions we distinguish the subset (possibly empty) of coalitions that are contained in the coalition of individuals that approved the proposer's proposal:

$$\mathcal{M}_t^a = \{C \in \mathcal{M}_t \mid C \subseteq C_t\}.$$

We follow the convention of indexing coalitions $C_t^c \in \mathcal{M}_t$ by $c = 1, \ldots, m_t^a, m_t^a + 1, \ldots, m_t$, and let the first m_t^a index values identify coalitions in $\mathcal{M}_t^a$, i.e., $C_t^c \in \mathcal{M}_t^a$ if and only if $c = 1, \ldots, m_t^a$ (with m_t^a possibly zero when $\mathcal{M}_t^a$ is empty).

We first derive necessary conditions that must hold in cases when the proposer's proposal is successful. The gist of these conditions is that if there exist strictly concave utility functions that rationalize such data then the proposal must solve the optimization problem:

$$P(q_t, i_t, C_t) \qquad\qquad \max u_{i_t}(x) \quad s.t.$$

$$u_j(x) \geq u_j(q_t), j \in C_t.$$

The fact that z_t must be indeed the solution to $P(q_t, i_t, C_t)$ follows from axiom (PR) which implies

$$u_{i_t}(z_t) \geq u_{i_t}(x), \forall x \in \{y \mid u_j(y) > u_j(q_t), j \in C_t\}.$$

If some $y \neq z_t$ solves $P(q_t, i_t, C_t)$ then (PR) is violated by any proposal $z' = \lambda y + (1-\lambda)z_t, \lambda \in (0,1)$ which is strictly preferred over the status quo by all members of C_t due to strict concavity. Thus, z_t must solve $P(q_t, i_t, C_t)$. Note that due to the strict concavity of u_i $P(q_t, i_t, C_t)$ has a unique solution and, since in addition $z_t \neq q_t$, Slater's constraint qualification is satisfied so that the Kuhn-Tucker conditions are necessary and sufficient for the program $P(q_t, i_t, C_t)$. Standard minimum decisive coalition arguments then easily lead us to conclude that, in fact, the proposal z_t must not only solve $P(q_t, i_t, C_t)$ but it must also solve any smaller program $P(q_t, i_t, C_t^c), c = 1, \ldots, m_t^a$ for otherwise the proposer would have a better proposal by removing some redundant coalition partner from her coalition. Similarly, it must be the proposer has no incentive to form a coalition different than C_t and propose the solution to one of the programs $P(q_t, i_t, C_t^c), c = m_t^a + 1, \ldots, m_t$. All these necessary conditions are summarized in the next lemma:

Lemma 1. *If the utility functions $u_i : X \to \mathbb{R}, i \in N$ rationalize the data $\{(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)\}_{t \in \mathcal{T}^a}$ then there exist*

$$\begin{aligned} u_i^{q_t}, u_i^{z_t}, \lambda_t^{cj} \in \mathbb{R}, d_i^{q_t}, d_i^{z_t} \in \mathbb{R}^d \qquad & t \in \mathcal{T}^a; i = 1, \ldots, n; c = 1, \ldots, m_t; j \in C_t^c \\ \lambda_t^{ci_t} \in \mathbb{R}, x_t^c \in \mathbb{R}^d, u_i^{x_t^c} \in \mathbb{R}, d_i^{x_t^c} \in \mathbb{R}^d \qquad & t \in \mathcal{T}^a; c = m_t^a + 1, \ldots, m_t; i = 1, \ldots, n \end{aligned}$$

such that:

$$d_{i_t}^{z_t} + \sum_{j \in C_t^c} \lambda_t^{cj} d_j^{z_t} = 0 \qquad t \in \mathcal{T}^a; c = 1, \ldots, m_t^a \tag{A.1}$$

$$\lambda_t^{ci_t} d_{i_t}^{x_t^c} + \sum_{j \in C_t^c} \lambda_t^{cj} d_j^{x_t^c} = 0 \qquad t \in \mathcal{T}^a; c = m_t^a + 1, \ldots, m_t \tag{A.2}$$

$$\lambda_t^{cj}(u_j^{z_t} - u_j^{q_t}) = 0 \qquad t \in \mathcal{T}^a; c = 1, \ldots, m_t^a; j \in C_t^c \tag{A.3}$$

$$\lambda_t^{cj}(u_j^{x_t^c} - u_j^{q_t}) = 0 \qquad t \in \mathcal{T}^a; c = m_t^a + 1, \ldots, m_t; j \in C_t^c \tag{A.4}$$

$$u_j^{x_t^c} - u_j^{q_t} \geq 0 \qquad t \in \mathcal{T}^a; c = m_t^a + 1, \ldots, m_t; j \in C_t^c \tag{A.5}$$

$$\lambda_t^{cj} \geq 0 \qquad t \in \mathcal{T}^a; c = 1, \ldots, m_t; j \in C_t^c \cup \{i_t\} \tag{A.6}$$

$$\lambda_t^{ci_t} + \sum_{j \in C_t^c} \lambda_t^{cj} > 0 \qquad t \in \mathcal{T}^a; c = m_t^a + 1, \ldots, m_t; j \in C_t^c \tag{A.7}$$

$$u_{i_t}^{z_t} - u_{i_t}^{x_t^c} \geq 0 \qquad t \in \mathcal{T}^a; c = m_t^a + 1, \ldots, m_t \tag{A.8}$$

Proof. Let the policies x_t^c be the unique solutions to the programs $P(q_t, i_t, C_t^c), c = m_t^a + 1, \ldots, m_t$. Set $u_i^{z_t} = u_i(z_t)$, $u_i^{q_t} = u_i(q_t)$, $u_i^{x_t^c} = u_i(x_t^c)$ and let $d_i^{q_t}, d_i^{z_t}, d_i^{x_t^c}$ be derivatives or supergradients (see Rockafellar) of the u_i at these points. Then $(A.1)$-$(A.7)$ are (part of) the Kuhn-Tucker (or Fritz-John) conditions for the programs $P(q_t, i_t, C_t^c), c = 1, \ldots, m_t$ with associated Lagrange multipliers λ_t^{cj} and $(A.8)$ ensures that the proposer can not obtain higher utility by coalescing with one of the coalitions $C_t^c, c = m_t^a + 1, \ldots, m_t$. □

We next turn to the cases the proposer's action leads to the preservation of the status quo. In these cases it follows that the status quo is the unique feasible solution to all the programs $P(q_t, i_t, C_t^c), c = 1, \ldots, m_t$. But this is equivalent to the status quo being a Pareto point for all the minimum decisive coalitions $C_t^c \cup \{i_t\}, c = 1, \ldots, m_t$. This implies a set of analogous equalities and inequalities must hold, which we state in the next lemma:

Lemma 2. *If the utility functions $u_i : X \to \mathbb{R}, i \in N$ rationalize the data $\{(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)\}_{t \in \mathcal{T}^a \cup \mathcal{T}^p}$ then there exist*

$$\lambda_t^{ci_t}, \lambda_t^{cj}, u_i^{q_t}, \in \mathbb{R}, d_i^{q_t} \in \mathbb{R}^d \qquad t \in \mathcal{T}^r \cup \mathcal{T}^p; i = 1, \ldots, n; c = 1, \ldots, m_t; j \in C_t^c$$

$$u_i^{z_t} \in \mathbb{R}, d_i^{z_t} \in \mathbb{R}^d \qquad t \in \mathcal{T}^r; i = 1, \ldots, n$$

such that:

$$(R.1) \qquad \lambda_t^{ci_t} d_{i_t}^{z_t} + \sum_{j \in C_t^c} \lambda_t^{cj} d_j^{z_t} = 0 \qquad t \in \mathcal{T}^r \cup \mathcal{T}^p; c = 1, \ldots, m_t$$

$$(R.2) \qquad \lambda_t^{cj} \geq 0 \qquad t \in \mathcal{T}^r \cup \mathcal{T}^p; c = 1, \ldots, m_t; j \in C_t^c \cup \{i_t\}$$

$$(R.3) \qquad \lambda_t^{ci_t} + \sum_{j \in C_t^c} \lambda_t^{cj} > 0 \qquad t \in \mathcal{T}^r \cup \mathcal{T}^p; c = 1, \ldots, m_t$$

Proof. As in the previous lemma, set $u_i^{z_t} = u_i(z_t)$, $u_i^{q_t} = u_i(q_t)$ and let $d_i^{q_t}, d_i^{z_t}$ be derivatives or supergradients of the u_i at these points. Conditions $(R.1)$ to $(R.3)$ characterize the status quo as a Pareto point for all the minimum decisive coalitions. □

Lemmas 1 and 2 state necessary conditions in order for the observed data to be rationalized. In the next Theorem we show that these conditions combined with a system of inequalities that follows from strict concavity and voter rationality are also sufficient for the rationalization of the data. As in the pioneering approach of Afriat, the key to this result is that when all the conditions are met we have enough information to construct the desired (in our case strictly concave) utility functions with the property that they share the values and the supergradients we have identified in the lemmas. As a result, voter and proposer rationality follow, establishing the rationalizability of the data.

Theorem 2. *There exist $u_i : X \to \mathbb{R}, i \in N$ that rationalize the data $(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)_{t=1}^T$ if and only if there exist*

$$\begin{aligned} \lambda_t^{cj} &\in \mathbb{R} && t \in \mathcal{T}^a; i = 1, \ldots, n; c = 1, \ldots, m_t; j \in C_t^c \\ \lambda_t^{ci_t} \in \mathbb{R}, x_t^c \in \mathbb{R}^d, u_i^{x_t^c} &\in \mathbb{R}, d_i^{x_t^c} \in \mathbb{R}^d && t \in \mathcal{T}^a; c = m_t^a + 1, \ldots, m_t; i = 1, \ldots, n \\ \lambda_t^{ci_t}, \lambda_t^{cj} & && t \in \mathcal{T}^r \cup \mathcal{T}^p; i = 1, \ldots, n; c = 1, \ldots, m_t; j \in C_t^c \\ u_i^x \in \mathbb{R}, d_i^x &\in \mathbb{R}^d && x \in X; i = 1, \ldots, n \end{aligned}$$

that satisfy $(A.1)$-$(A.8)$, $(R.1)$-$(R.3)$, and:

$$(V) \qquad u_i^x - u_i^y \geq 0 \qquad i = 1, \ldots, n; (x, y) \in V_i$$

$$(U) \qquad u_i^x + d_i^x (y - x) - u_i^y > 0 \qquad i = 1, \ldots, n; x \in Y; y \in Y \setminus \{x\}$$

where $Y = X \cup \{x_t^{m_t^a+1}, \ldots, x_t^{m_t}\}_{t \in \mathcal{T}^a}$.

Proof. The necessity of (A.1)-(A.8), (R.1)-(R.3) follows from the lemmas, that of (V) from voter rationality, and the necessity of (U) follows from the properties of strictly concave functions and their supergradients. We thus need to show sufficiency. To that end, note that the strict inequalities (U) allow us to construct strictly concave utility function u_i for each i and for some small enough $\epsilon > 0$ as follows (see, e.g., Matzkin and Richter (1991)):

$$u_i(y) = \min_{x \in Y}\{u_i^x + d_i^x(y-x) - \epsilon(y-x)'(y-x)\}.$$

Note that by construction $u_i(y) = u_i^y$ and that d_i^y is the derivative of u_i at y for all $y \in Y$. Thus, these utility function satisfy (UC) and because conditions (A.1)-(A.8) and (R.1)-(R.3) and (V) also hold they also satisfy (VR) and (PR). □

We have thus derived necessary and sufficient conditions in order for data $(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)_{t=1}^T$ to be rationalized. These conditions take the form of a set of equalities and inequalities ((A.1)-(A.8), (R.1)-(R.3), (V), (U)) that are multi-linear quadratic polynomials in the unknowns (utility levels, supergradients, Lagrange multipliers, and putative alternative proposals). Solving this system of equations in a computer is not trivial, but a number of algorithms can be adopted to this task (e.g., the algorithms of Brown and Kannan (2006) for the corresponding conditions for the rationalization of competitive equilibrium outcomes).

4 One Policy Dimension & Quantifier elimination

The necessary and sufficient conditions in the previous section involve a system of inequalities that is polynomial in the unknown unobserved quantities. The Tarski-Seidenberg theorem guarantees the existence of a procedure that reduces this system (via a rigorous algorithm) into an equivalent set of of conditions that involve only known quantities (essentially, the coefficients of the original polynomial system). Brown and Matzkin (1996) offer a discussion of these ideas and their connection with known results in the revealed preference theory of the consumer, and exemplify this approach by applying quantifier elimination to the polynomial conditions that characterize certain competitive equilibria.

When it comes to voter rationality and concavity alone, the necessary and sufficient conditions in Kalandrakis (2010) essentially constitute an instance of this exercise as becomes apparent from the following Theorem concerning the one-dimensional policy space:

Theorem 3. *Given irreflexive $V \subset X \times X$, $X \subset \mathbb{R}$ and finite, the following conditions are equivalent:*

(UVR_1) *There exists strictly concave $u : \mathbb{R} \to \mathbb{R}$ such that $u(x) \geq u(y)$ for all $(x, y) \in V$.*

(UVR_1') *There exist $u^x, d^x \in \mathbb{R}, x \in X$, such that*

$u^x - u^y \geq 0, (x, y) \in V$, *and*

$u^x + d^x(y - x) - u^y > 0, x, y \in X; x \neq y$.

(W_1) *For every $\{(x_1, y_1), (x_2, y_2)\} \subseteq V$*

$y_j < \min_i\{x_i\}$ *for some $j = 1, 2$, or*

$y_j > \max_i\{x_i\}$ *for some $j = 1, 2$, or*

$y_j = x_i, i, j = 1, 2, i \neq j$.

Note how all the unobserved unknown quantities from (UVR_1') are eliminated in condition (W_1).

In this section we apply quantifier elimination to the conditions $(A.1)$-$(A.8)$, $(R.1)$-$(R.3)$, (V), and (U) for the case $d = 1$. It turns out that we can show that these inequalities are equivalent to a condition analogous to (W_1) that tests the voter rationality axiom for an augmented record of voting decisions by the individuals, i.e., by adding certain voting decisions to the records V_i and testing whether condition (W_1) holds for these augmented voting records. To state this result, we need to develop some notation.

First, assume without loss of generality that $t' > t \Rightarrow x_{t'} > x_t$ for all $t, t' = 1, \ldots, T$. We first construct non-finite interim voting records

$$\bar{V}_i = V_i \bigcup \left(\bigcup_{t:i=i_t} \{z_t\} \times [z_t, q_t] \right).$$

These essentially state that the proposers not only prefer the proposal over the status quo but also the proposal to anything between the proposal and the status quo which follows from the convexity of individual preferences and proposer rationality.

Let $b > |x|$ for all $x \in X$. We define two quantities

$$\underline{b}_t^i = \begin{cases} \min\{y \mid (x,y) \in \bar{V}_i, x \le x_t, x < y\} & \text{if} \quad \{(x,y) \in \bar{V}_i \mid x \le x_t, x < y\} \neq \emptyset \\ b & \text{otherwise.} \end{cases}$$

and

$$\overline{b}_t^i = \begin{cases} \max\{y \mid (x,y) \in \bar{V}_i, x \ge x_t, x > y\} & \text{if} \quad \{(x,y) \in \bar{V}_i \mid x \ge x_t, x > y\} \neq \emptyset \\ -b & \text{otherwise.} \end{cases}$$

The quantities $\underline{b}_t^i, \overline{b}_t^i$ essentially determine whether we can add certain voting decisions for i in t involving the collective decision x_t and adjacent alternatives, without violating (UC) and (VR). Accordingly, we define

$$G_t^- = \{i \in C_t \mid \overline{b}_t^i < q_t\}$$

and

$$G_t^+ = \{i \in C_t \mid \underline{b}_t^i > q_t\}$$

These sets determine whether an individual i can have a negative or positive gradient, respectively, at alternative x_t. We take collections of subsets of these sets that can block a decision in favor of the proposal (by switching their vote) or block an improvement over the status quo for the proposer when the status quo is the outcome:

$$\mathcal{G}_t^- = \{C \subset G_t^- \setminus \{i_t\} \mid C_t \setminus C \notin \mathcal{D}_t\}$$

and

$$\mathcal{G}_t^+ = \{C \subset G_t^+ \setminus \{i_t\} \mid C_t \setminus C \notin \mathcal{D}_t\}.$$

We now define the alternatives that may be compared with x_t when we add voting decisions to the directly revealed ones in V_i:

$$l_t = \max\{y \mid y \in X \cup \{-b\}, y < x_t\}$$

and

$$r_t = \min\{y \mid y \in X \cup \{b\}, y > x_t\}$$

Next, we take combinations of voting decisions to add for each t that, when combined with (V) and (U), will ensure that the proposer optimizes or that the status quo is a Pareto point for all decisive coalitions.

$$\mathcal{G}_t = \{(\{i_t\}, A) \mid i_t \in G_t^+, A \in \mathcal{G}_t^-\} \cup \{(A, \{i_t\}) \mid A \in \mathcal{G}_t^+, i_t \in G_t^-\} \cup \{(\{i_t\}, \{i_t\})\}.$$

Not all of the above combinations are consistent with axioms (VR) and (UC). Specifically, in one policy dimension we cannot have a succession of a negative gradient and a positive gradient for any player. Accordingly, the possible votes to be added to the voting records are given by

$$\mathcal{G} = \{((A_t^+, A_t^-))_{t=1}^T \in \times_{t=1}^T \mathcal{G}_t \mid i \in A_t^- \Rightarrow i \notin A_{t'}^+ \forall t' > t\}$$

We are now ready to state necessary and sufficient conditions that are equivalent to $(A.1)$-$(A.8)$, $(R.1)$-$(R.3)$, (V), and (U) for the case $d = 1$ and do not involve any of the unobserved unknowns (u^x, d^x, λ, x^c):

Theorem 4. *If $d = 1$ the data $(N_t, \mathcal{D}_t, q_t, i_t, z_t, C_t)_{t=1}^T$ are rationalizable if and only if there exists $((A_t^+, A_t^-))_{t=1}^T \in \mathcal{G}$ such that the augmented voting record*

$$\hat{V}_i = \bigcup_{t \mid i \in A_t^+} \{(x_t, l_t)\} \bigcup_{t \mid i \in A_t^-} \{(x_t, r_t)\} \bigcup_{t \in \mathcal{T}^a \mid i \in (A_t^- \cup A_t^+) \setminus \{i_t\}} \{(q_t, x_t)\} \bigcup V_i$$

satisfies (W_1) for all $i \in N$.

Proof. TO BE ADDED. □

5 Unobserved Policies

The conditions derived in Theorems 2 and 4 impose testable restrictions on the data that reach well beyond those implied by voter rationality and concavity ((VR) and (UC)) alone. Kalandrakis (2010) has shown that the necessary and sufficient conditions for the latter are vacuously met if the policy alternatives in the voting agenda are unobserved. This is true even in one-dimensional policy spaces and independent of any added hypotheses on the joint location of voters' ideal points. This is problematic because in most political contexts it is hard to obtain credible data on the location of the voting alternatives (bills, legislation, etc.) in Euclidean space. In this section we consider the case the status quo policy q_t and any proposals z_t are unobserved so that the data in period t take the form

$$(N_t, \mathcal{D}_t, i_t, C_t).$$

Note that we assume (as is logically possible from the way we encode the data) that the analyst can discern whether the proposer offered some $z_t \neq q_t$ that was approved (when $C_t \in \mathcal{D}_t$), or whether she offered a proposal that was rejected (when $C_t \neq \emptyset$ and $C_t \notin \mathcal{D}_t$), or whether the proposer passed (when $C_t = \emptyset$). Our main result is a simple but potent necessary condition:

Theorem 5. *Assume $d = 1$. If there exist $u_i : X \to \mathbb{R}, i \in N$ that rationalize the data $(N_t, \mathcal{D}_t, i_t, C_t)_{t=1}^T$ then there exists a linear order $\succ$ on N such that*

$$\{j \in N_t \mid i_t \succ j\} \subseteq C_t \text{ or } \{j \in N_t \mid j \succ i_t\} \subseteq C_t,$$

for all $t \in \mathcal{T}^a$.

Proof. Consider $t \in \mathcal{T}^a$. Let $\hat{x}_j$ be ideal point of j (possibly $\pm\infty$). Either $z_t \in (q_t, \hat{x}_{i_t}]$ whence $j \in C_t$ for all j with $\hat{x}_j \geq \hat{x}_{i_t}$, or $z_t \in [\hat{x}_{i_t}, q_t)$ hence $j \in C_t$ for all j with $\hat{x}_j \leq \hat{x}_{i_t}$. Take $\succ$ to be any ordering of ideal points. $\square$

The theorem states that in order for the data to be rationalizable, it suffices that we can rank the individuals (in fact this ranking needs to coincide with the ranking of the ideal points of the individuals) so that whenever a proposer offers a proposal either all individuals ranked higher or all individuals ranked lower (or both) than the proposal approve it.

It is easy to show on the basis of this Theorem that there exist data $(N_t, \mathcal{D}_t, i_t, C_t)_{t=1}^T$ that are not rationalizable. We do this by example. In particular, let $N = \{1, \ldots, 5\}$ and let $M_i = \{\{j, k\} \subset N \setminus \{i\} \mid j \neq k\}$ and $N_t = N$ and $\mathcal{D}_t = \{C \mid |C| \geq 3\}$ for all t. Suppose that each $i \in N$ serves as the proposer for $|M_i|$ periods with $C_t = \{i\} \cup C$ once for each $C \in M_i$. By the symmetry of the data any order on N must violate the condition of Theorem 6 as any person ordered third in that ranking must form a coalition that excludes some higher ordered and lower ordered individual. Note the contrast of this result with Theorem 4 of Kalandrakis (2010).

The following theorem states that data with unobserved policies impose minimal restrictions on the data in the absence of added assumptions when the we assume the policy space comprises more than one dimension.

Theorem 6. *Assume* $d = 2$ *and* $\mathcal{T}^r = \emptyset$*. Any data* $\{(N_t, \mathcal{D}_t, i_t, C_t)\}_{t=1}^{T}$ *can be rationalized by some* $u_i : X \to \mathbb{R}, i \in N$*.*

Proof. Set $z_t = \hat{x}_{i_t}$ for all $t \in \mathcal{T}^a$, $q_t = \hat{x}_{i_t}$ for all $t \in \mathcal{T}^p$. Place all z_t, q_t on a circle. Follows from Theorem 6 in Kalandrakis (2010). $\square$

6 Conclusions

TO BE ADDED

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