Dynamic Greeks and Model Risk

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Risk and Stochastics Day, 19.3.07

Outline

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2. Option pricing in Lévy process driven models
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    - Pricing with integro-differential equations
3. Dynamic sensitivity analysis in Lévy process driven models
    - European vanilla option in a jump diffusion model
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5. Existence of derivatives
6. Numerical issues

1. Option pricing and sensitivity in the Black-Scholes model

Black-Scholes framework under EMM:

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= rS_t dt + \sigma S_t dW_t. \end{aligned}$$

Price function of European call at time t:

$$c(t,s) = e^{-r(T-t)}\mathbb{E}[\max(s-K,0)|S_t = s].$$

Use Itô:

$$c_t(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s) + rsc_s(t,s) - rc(t,s) = 0.$$

Terminal condition $c(T,s) = \max(s-K,0)$.

Standard Greeks

Explicit formula for price:

$$c(t,s) = sN(d_1) - Ke^{-r(T-t)}N(d_2),$$

with

$$d_{1,2} = \frac{\ln\left(\frac{s}{K}\right) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Sensitivities:

$$\Delta = \frac{\partial c}{\partial s}, \qquad \Gamma = \frac{\partial^2 c}{\partial s^2},$$
$$\rho = \frac{\partial c}{\partial r}, \qquad \mathcal{V} = \frac{\partial c}{\partial \sigma},$$
$$\theta = \frac{\partial c}{\partial t}.$$

Dynamic greeks

PDE for the price and vega:

$$c_t(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s) + rsc_s(t,s) - rc(t,s) = 0,$$
$$\mathcal{V}_t(t,s) + \frac{1}{2}\sigma^2 s^2 \mathcal{V}_{ss}(t,s) + rs\mathcal{V}_s(t,s) - r\mathcal{V}(t,s) + \sigma s^2 c_{ss}(t,s) = 0.$$

Terminal conditions:

$$c(T,s) = \max(s-K,0),$$

and

$$\mathcal{V}(T,s) = 0.$$

2. Option pricing in Lévy process driven models

The exponential Lévy model

$$B_t = e^{rt},$$
$$S_t = e^{rt+X_t}.$$

Driving process:

$$X_t = \gamma t + \sigma W_t + \int_{|x|\geq 1} x\mu(dt,dx) + \int_{|x|<1} x(\mu(dt,dx) - \nu(dx)dt).$$

Stock price dynamics:

$$dS_t = rS_{t-}dt + \sigma S_{t-}dW_t + \int_{-\infty}^{\infty} (e^x - 1)S_{t-}\,(\mu(dt,dx) - \nu(dx)dt).$$

Option Prices

European vanilla option price at time t under EMM:

$$c(t, S_t) = e^{-r\,(T-t)}\mathbb{E}[h(S_T)|S_t],$$

where

$$S_t = e^{rt+X_t}.$$

How can options in such a model be priced?

- Pricing via simulation
- Pricing with Fourier transform methods
- Pricing with numerical integration
- Pricing with integro-differential equations

Integro-differential equation approach

The discounted option price $\hat{c}(t, S_t) = e^{-rt} c(t, S_t)$ is martingale.
Use Itô:

$$\begin{aligned} d\hat{c}(t, S_t) =& e^{-rt} \Big[- rc(t, S_{t-})dt + c_t(t, S_{t-})dt + c_s(t, S_{t-})dS_t \\ & + \frac{1}{2} c_{ss}(t, S_{t-}) d[S, S]^c_t \\ & + \int_{-\infty}^{\infty} \big(c(t, S_{t-}e^x) - c(t, S_{t-}) - (e^x - 1) S_{t-} c_s(t, S_{t-}) \big) \mu(dt, dx) \Big]. \end{aligned}$$

PIDE:

$$\begin{aligned} & c_t(t, s) + rsc_s(t, s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t, s) - rc(t, s) \\ & \quad + \int_{-\infty}^{\infty} \nu(dx) \Big[c(t, se^x) - c(t, s) - (e^x - 1) s c_s(t, s) \Big] = 0. \end{aligned}$$

Terminal condition

$$c(T, s) = h(s), \quad s > 0.$$

Changes of variables

$$\begin{aligned} \tau &= T - t, \\ x &= \ln\left(\frac{s}{K}\right) + r\tau, \\ c(t,s) &= Ke^{-r\tau}u(\tau,x). \end{aligned}$$

New PIDE

$$\begin{aligned} u_\tau(\tau,x) = &\left(r - \frac{\sigma^2}{2}\right) u_x(\tau,x) + \frac{\sigma^2}{2} u_{xx}(\tau,x) \\ &+ \int \nu(dy)[u(\tau,x+y) - u(\tau,x) - (e^y - 1)u_x(\tau,x)]. \end{aligned}$$

New side condition

$$u(0,x) = \max(0, e^x - 1).$$

3. Dynamic sensitivity in Lévy models

Greeks of a vanilla option in the Merton Model

The Merton jump-diffusion model:

$$\begin{aligned} X_t &= \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \\ S_t &= S_0 e^{rt+X_t}, \\ Y_i &\sim N(\mu, \delta^2). \end{aligned}$$

Its Lévy measure:

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\delta^2}} dx.$$

To obtain **vega**, differentiate with respect to σ.

$$\mathcal{V} = \frac{\partial c}{\partial \sigma}, \qquad v = \frac{\partial u}{\partial \sigma}, \qquad \mathcal{V} = Ke^{-r\tau} v.$$

PIDE for u:

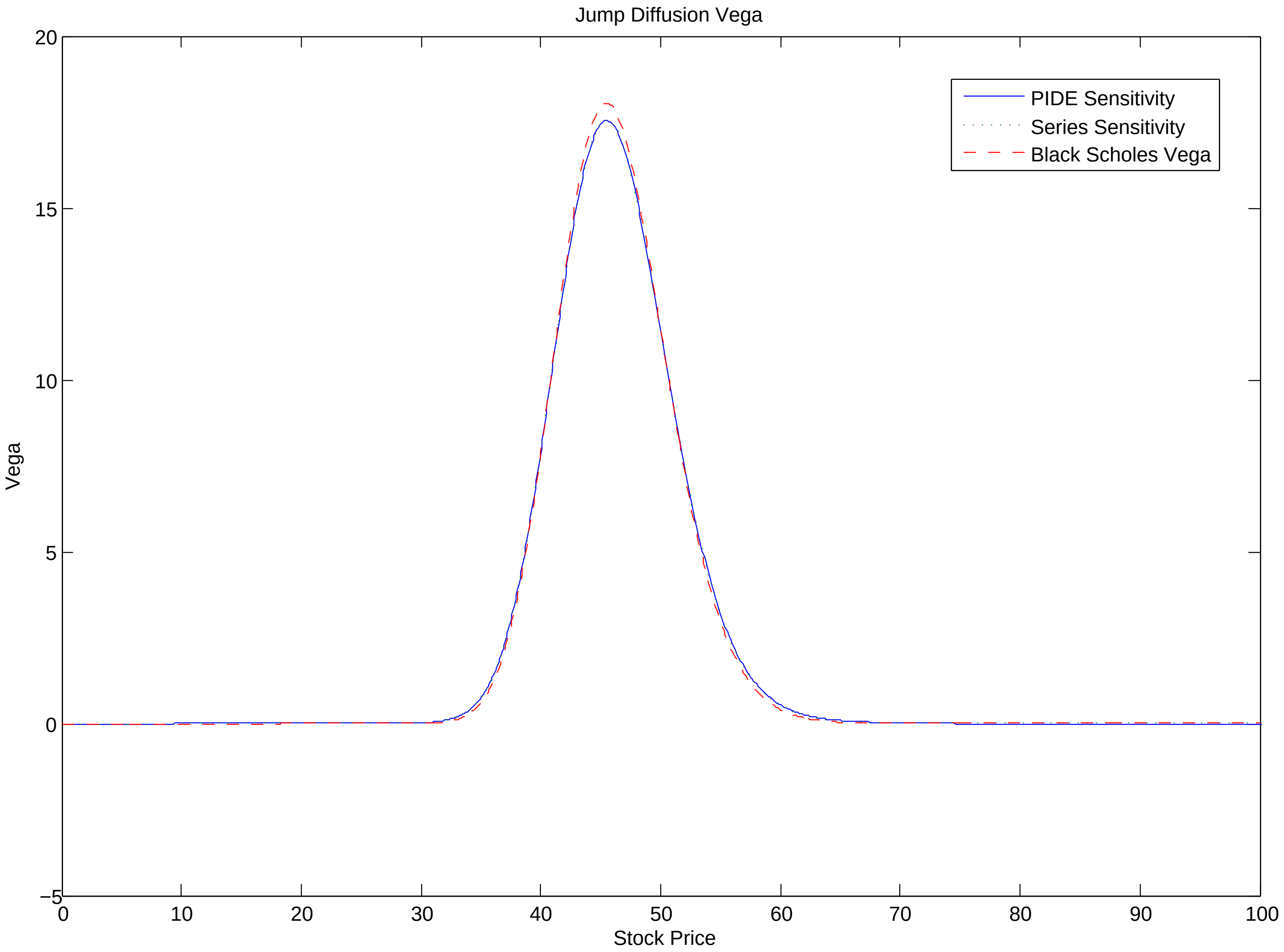
$$\begin{aligned} u_\tau(\tau, x) &= \left(r - \frac{\sigma^2}{2}\right) u_x(\tau, x) + \frac{\sigma^2}{2} u_{xx}(\tau, x) \\ &\quad + \int \nu(dy)[u(\tau, x+y) - u(\tau, x) - (e^y - 1) u_x(\tau, x)]. \end{aligned}$$

PIDE for v:

$$\begin{aligned} v_\tau(\tau, x) &= \left(r - \frac{\sigma^2}{2}\right) v_x(\tau, x) + \frac{\sigma^2}{2} v_{xx}(\tau, x) + \sigma(u_{xx}(\tau, x) - u_x(\tau, x)) \\ &\quad + \int \nu(dy)[v(\tau, x+y) - v(\tau, x) - (e^y - 1) v_x(\tau, x)]. \end{aligned}$$

Side conditions:

$$u(0, x) = \max(0, e^x - 1), \qquad v(0, x) = 0.$$


Jump Diffusion Vega
20
15
10
5
0
−5
Vega
0
10
20
30
40
50
60
70
80
90
100
Stock Price
PIDE Sensitivity
Series Sensitivity
Black Scholes Vega

Greeks of a vanilla option in the variance gamma Model

Variance gamma process Y_t:

$$Y_t = \theta Z_t + \sigma W_{Z_t}, \qquad Z_t \sim \Gamma\left(\frac{t}{\kappa}, \frac{1}{\kappa}\right).$$

The stock price:

$$S_t = S_0 e^{rt + Y_t + \gamma t}.$$

The Lévy measure:

$$\nu(dx) = \frac{1}{\kappa|x|} e^{Ax - B|x|} dx,$$

with

$$A = \frac{\theta}{\sigma^2}, \qquad B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}.$$

Calculate vega:

$$\mathcal{V} = \frac{\partial c}{\partial \sigma},$$
$$v = \frac{\partial u}{\partial \sigma},$$
$$\mathcal{V} = Ke^{-rT} v.$$

PIDE for the option price u:

$$u_\tau(\tau, x) = r u_x(\tau, x) + \int \nu(dy)[u(\tau, x+y) - u(\tau, x) - (e^y - 1)u_x(\tau, x)].$$

PIDE for the sensitivity v:

$$\begin{aligned} v_\tau(\tau,x) = {} & r v_x(\tau,x) \\ & + \nu(dy)[v(\tau,x+y) - v(\tau,x) - (e^y-1)v_x(\tau,x)] \\ & + \tilde{\nu}(dy)[u(\tau,x+y) - u(\tau,x) - (e^y-1)u_x(\tau,x)], \end{aligned}$$

where

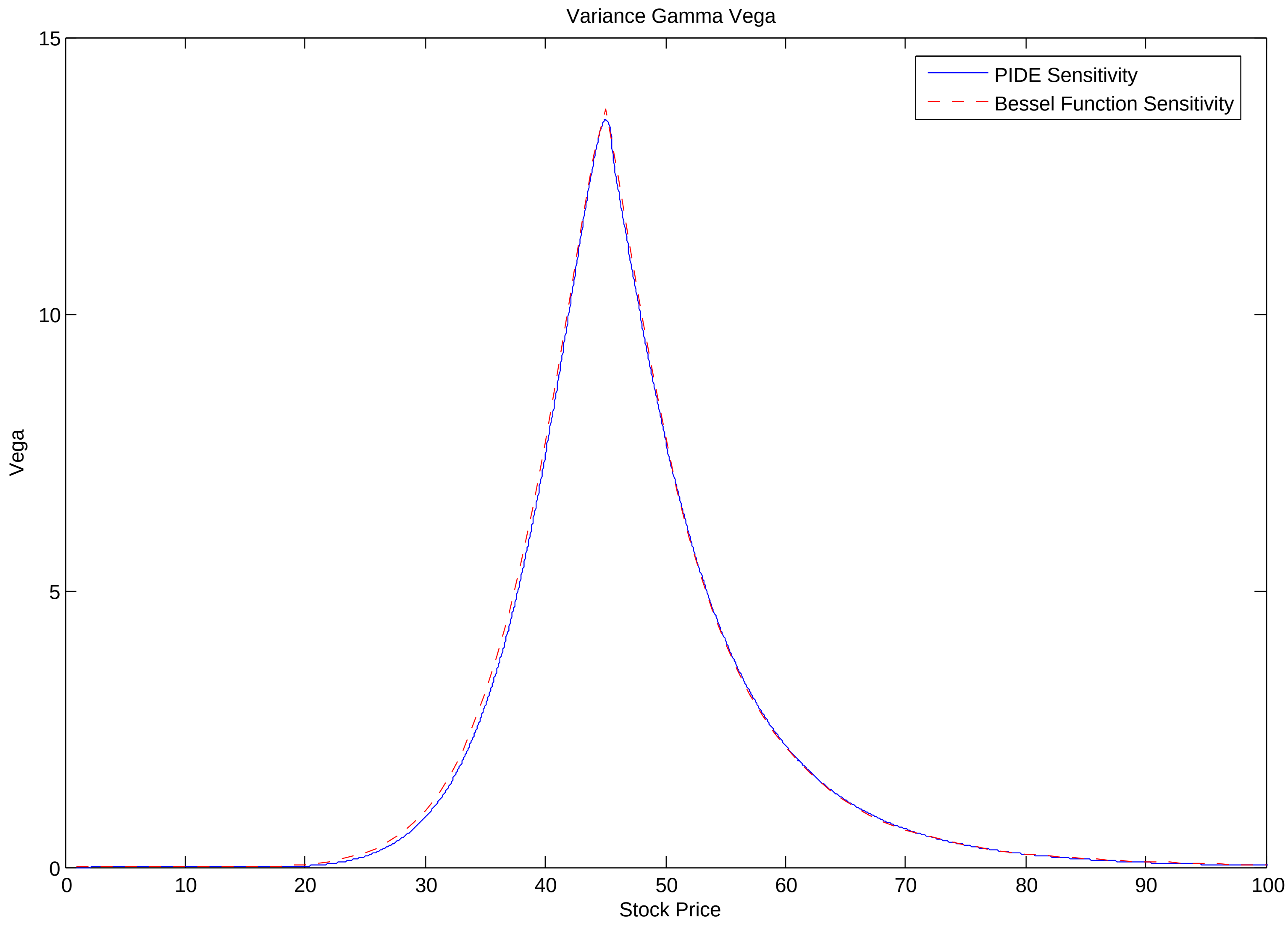
$$\tilde{\nu}(dy) = \left(-\frac{2\theta y}{\sigma^2} - \frac{2|y|}{\sqrt{\theta^2 + 2\sigma^2 \sigma\kappa}} + \frac{2\sqrt{\theta^2 + 2\sigma^2/\kappa}|y|}{\sigma^3} \right) \nu(dy).$$

Side conditions

$$u(0,x) = \max(0, e^x - 1), \qquad v(0,x) = 0.$$

Interesting new greeks are

- The sensitivity with respect to the volatility σ,
- The sensitivity with respect to the drift θ,
- The sensitivity with respect to the variance κ of the subordinating gamma process.


Variance Gamma Vega
PIDE Sensitivity
Bessel Function Sensitivity
Vega
Stock Price
0
5
10
15
0
10
20
30
40
50
60
70
80
90
100

Lookback option

Price function of a floating strike lookback put:

$$c(t,s,m) = e^{-r(T-t)}\mathbb{E}[M_T - S_T | S_t = s, M_t = m].$$

Discounted option price is a martingale

$$\hat{c}(t,S_t,M_t) = e^{-rt}c(t,S_t,M_t).$$

Apply Itô:

$$\begin{aligned} d\hat{c}(t,S_t,M_t) =& e^{-rt}\Bigg[-rc(t,S_{t-},M_{t-})dt + c_t(t,S_{t-},M_{t-})dt + c_s(t,S_{t-},M_{t-})dS_{t-} \\ &+ \frac{1}{2}c_{ss}(t,S_{t-},M_{t-})d[S,S]_t^c \\ &+ \int_{-\infty}^{\infty}\Big(c(t,S_{t-}e^x,\max(M_{t-},e^xS_{t-})) - c(t,S_{t-},M_{t-}) \\ &- (e^x-1)S_{t-}c_s(t,S_{t-},M_{t-})\Big)\mu(dt,dx)\Bigg]. \end{aligned}$$

PIDE:

$$c_t(t,s,m) + rsc_s(t,s,m) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s,m) - rc(t,s,m)$$
$$+ \int_{-\infty}^{\infty} \nu(dx)\Big[c(t, se^x, \max(m, se^x)) - c(t,s,m) - (e^x - 1)sc_s(t,s,m)\Big] = 0.$$

Terminal condition:

$$c(T,s,m) = m - s.$$

Auxiliary side condition:

$$c_m(t,s,m) = 0 \qquad \text{at} \quad s = m,$$
$$c(t,0,m) = me^{-r(T-t)}.$$

Option price function:

$$c(t, s, m) = mw(t, z),$$

where $Z_t = S_t/M_t$.
The derivatives in the new variables w and z are

$$c_s = w_z, \qquad c_{ss} = \frac{1}{m} w_{zz}, \qquad c_m = m - zw_z,$$

$$c(t, se^x, \max(se^x, m)) = m \max(ze^x, 1) w(t, \min(ze^x, 1)).$$

PIDE with two state variables:

$$w_t(t, z) - rzw_z(t, z) + \frac{1}{2}\sigma^2 z^2 w_{zz}(t, z) - rw(t, z)$$
$$+ \int_{-\infty}^{\infty} \nu(dx)[\max(ze^x, 1)w(t, \min(ze^x, 1)) - w(t, z) - (e^x - 1)(w_z(t, z))] = 0.$$

Terminal condition:

$$w(T, z) = 1 - z.$$

Auxiliary side condition:

$$w(t, z) = w_z(t, z) \qquad \text{at} \quad z = 1,$$
$$w(t, 0) = e^{-r(T-t)}.$$

Second transform of variables

$$\begin{aligned} \tau &= T - t, \\ x &= \ln z, \\ w(t,z) &= e^{-r\tau} u(\tau, x), \end{aligned}$$

New PIDE

$$\begin{aligned} u_\tau = & \left(r - \frac{\sigma^2}{2}\right) u_x(\tau, x) + \frac{1}{2}\sigma^2 u_{xx}(\tau, x) \\ & + \int_{-\infty}^{\infty} \nu(dy)[\max(e^{x+y}, 1) u(\tau, \min(x+y, 0)) - u(\tau, x) - (e^y - 1) u_x(\tau, x)]. \end{aligned}$$

Terminal condition

$$u(\tau, x) = 1 - e^x \qquad \text{at} \qquad t = T.$$

Auxiliary side conditions

$$\begin{aligned} u(\tau, x) &= 1, \qquad \text{for} \qquad x \to -\infty, \\ u_x(\tau, x) &= u(\tau, x) \qquad \text{at} \qquad x = 0. \end{aligned}$$

Use

$$\mathcal{V} = \frac{\partial c}{\partial \sigma}, \qquad v = \frac{\partial u}{\partial \sigma}, \qquad \mathcal{V} = me^{-r\tau} v$$

PIDE for vega:

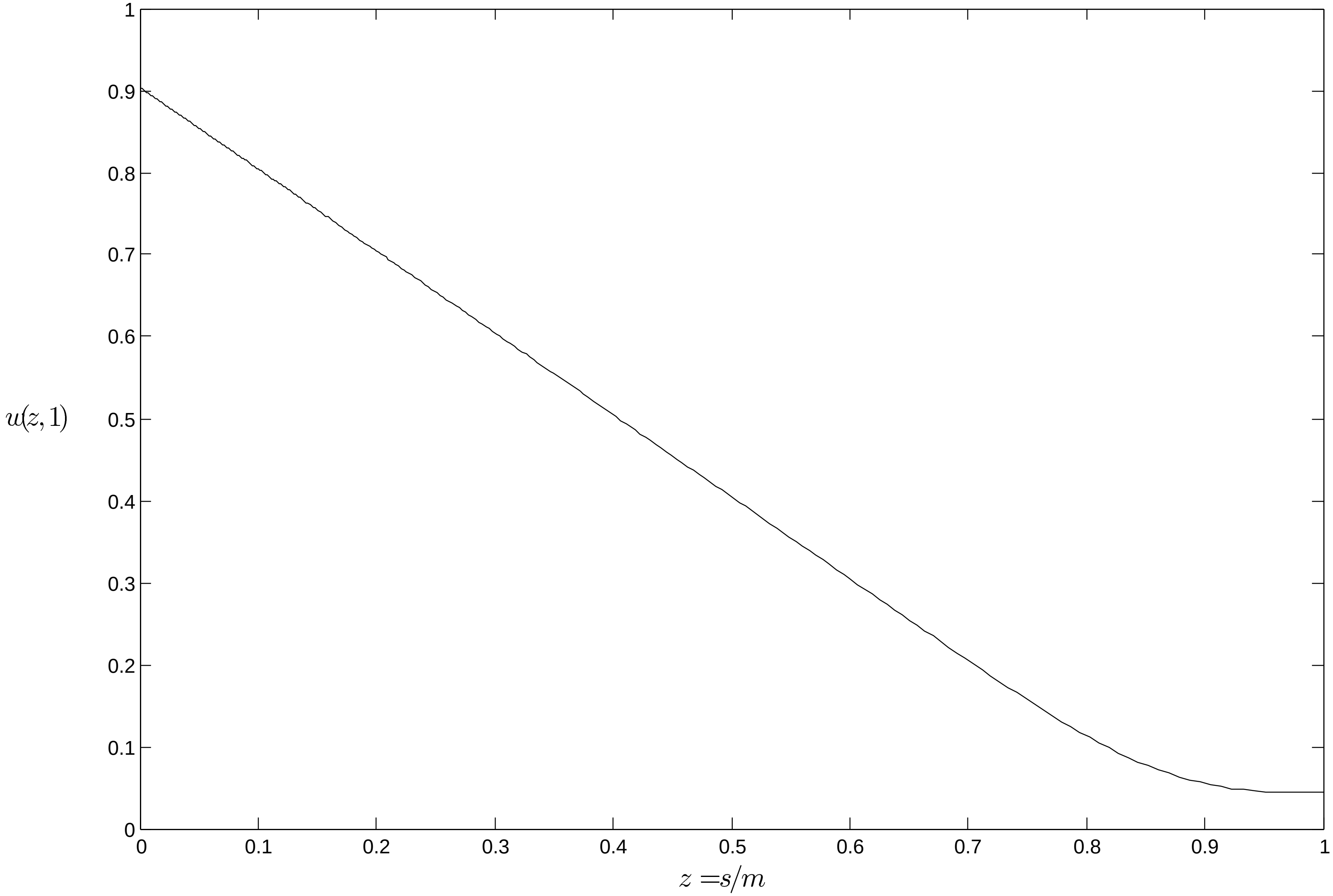
$$\mathcal{V}_\tau(\tau, x) = \left(r - \frac{\sigma^2}{2}\right) \mathcal{V}_x(\tau, x) + \frac{1}{2}\sigma^2 \mathcal{V}_{xx}(\tau, x) + \sigma(u_{xx}(\tau, x) - u_x(\tau, x))$$
$$+ \int_{-\infty}^{\infty} \nu(dy)[\max(e^{x+y}, 1)\mathcal{V}(\tau, \min(x+y, 0)) - \mathcal{V}(\tau, x) - (e^y - 1)\mathcal{V}_x(\tau, x)],$$

subject to the terminal condition

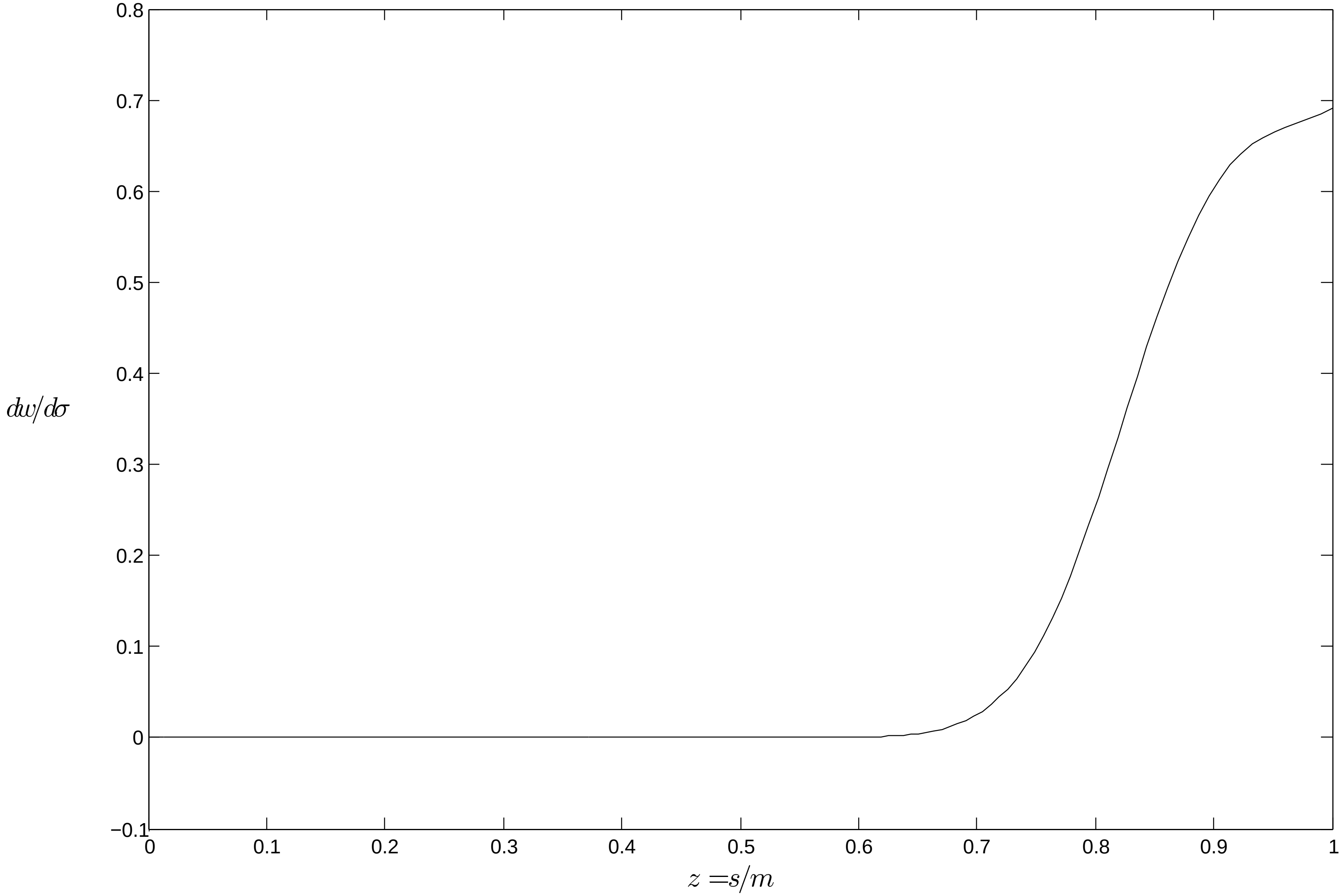
$$\mathcal{V}(\tau, x) = 0 \qquad \text{at} \qquad t = T.$$

Auxiliary side condition:

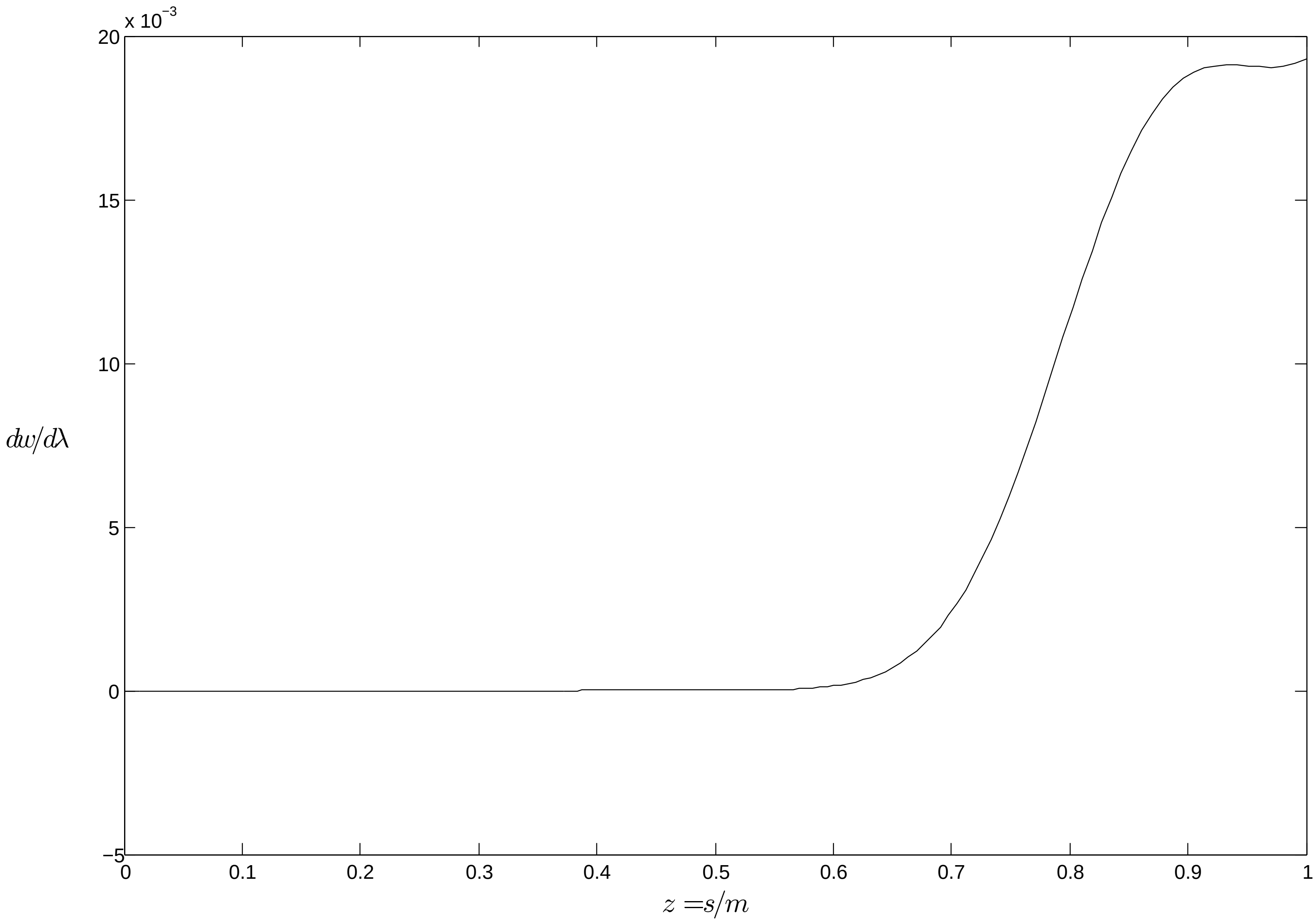
$$\mathcal{V}(\tau, x) = 0, \qquad \text{for} \qquad x \to -\infty,$$
$$\mathcal{V}_x(\tau, x) = \mathcal{V}(\tau, x) \qquad \text{at} \qquad x = 0.$$



Lookback option in the jump diffusion model $w(t,z) = \frac{1}{m}\, c(t,s,m)$



The sensitivity with respect to changes in σ of a lookback option in the jump diffusion model



The sensitivity with respect to changes in λ of a lookback option in the jump diffusion model

Other exotic options and models

The dynamic method works well for options where one can reduce the number of state variables to two.
For example:

- European vanilla options
- Many lookback options
- Asian options
- Some exchange options

It works for all types of Lévy processes:

- Jump-diffusion processes
- Infinite activity but finite variation processes
- Infinite variation processes

4. Two dimensional models and model risk

The basket option

Option on two Lévy processes driven price processes S_t and $\tilde{S}_t$

$$c(t,s,\tilde{s}) = e^{-r(T-t)}\mathbb{E}[(\alpha S_T + (1-\alpha)\tilde{S}_T - K)^+ | S_t = s, \tilde{S}_t = \tilde{s}],$$

PIDE for price

$$\begin{aligned} & c_t(t,s,\tilde{s}) - rc(t,s,\tilde{s}) + rsc_s(t,s,\tilde{s}) + r\tilde{s}c_{\tilde{s}}(t,s,\tilde{s}) \\ & + \frac{1}{2}\sigma_1^2 s^2 c_{ss}(t,s,\tilde{s}) + \frac{1}{2}\sigma_2^2 \tilde{s}^2 c_{\tilde{s}\tilde{s}}(t,s,\tilde{s}) + \rho\sigma_1\sigma_2 s\tilde{s}c_{s\tilde{s}}(t,s,\tilde{s}) \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(c(t,se^x,\tilde{s}e^y) - c(t,s,\tilde{s}) - s(e^x-1)c_s(t,s,\tilde{s}) \\ & - \tilde{s}(e^y-1)c_{\tilde{s}}(t,s,\tilde{s})\big)\nu(dx,dy) = 0, \end{aligned}$$

We are interested

$$\frac{\partial c}{\partial \alpha}.$$

Exponential Mixing

Stock price:

$$S_t = S_0 e^{rt+\alpha X_t+(1-\alpha)\tilde{X}_t}.$$

Option price function:

$$c(t,s) = e^{-r(T-t)}\mathbb{E}[(S_T-K)^+|S_t=s],$$

PIDE for price function

$$\begin{aligned}
c_t(t,s) - rc(t,s) + rsc_s(t,s) &+ \big(\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\rho\alpha(1-\alpha)\sigma_1\sigma_2\big)s^2 c_{ss}(t,s) \\
&+ \int_{-\infty}^{\infty} \big(c(t,s(\alpha e^x + (1-\alpha))) - c(t,s) - s(\alpha e^x - (1-\alpha) - 1)\big)\nu(dx) \\
&+ \int_{-\infty}^{\infty} \big(c(t,s(\alpha + (1-\alpha)e^x)) - c(t,s) - s(\alpha - (1-\alpha)e^x - 1)\big)\tilde{\nu}(dx) \\
&= 0.
\end{aligned}$$

Differentiate PIDE for price w.r.t α
Write $\frac{\partial c}{\partial \alpha} = \phi$.
PIDE for the sensitivity:

$$\begin{aligned}
\phi_t(t,s) - r\phi(t,s) + rs\phi_s(t,s) + (\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\rho\alpha(1-\alpha))s^2\phi_{ss}(t,s)& \\
+ (2\alpha\sigma_1^2 + 2(\alpha-1)\sigma_2^2 + 2\rho(1-2\alpha))s^2 c_{ss}(t,s)& \\
+ \int_{-\infty}^{\infty} \big(s(e^x-1)\phi(t, s(\alpha e^x + (1-\alpha))) - \phi(t,s)& \\
- s\alpha(e^x-1)\phi_s(t,s) - s(e^x-1)c_s(t,s)\big)\nu(dx)& \\
+ \int_{-\infty}^{\infty} \big(s(1-e^x)\phi(t, s(\alpha + (1-\alpha)e^x)) - \phi(t,s)& \\
- s(1-\alpha)(e^x-1)\phi_s(t,s) + s(e^x-1)c_s(t,s)\big)\tilde{\nu}(dx) = 0&,
\end{aligned}$$

subject to the side condition

$$\phi(T,s) = 0.$$

5. Existence of derivatives

First: The PIDE for the price function has to be well defined:

- The derivative(s) with respect to the stock price have to exist
- The time derivative has to exist
- The integral term has to converge

Second: Proof the existence of the derivatives with respect to parameters i.e. that

$$\partial_\alpha \int h(x,\alpha) F(dx,\alpha)$$

is well defined.

Consider

$$z(k) = e^{-rT}\mathbb{E}[\max(e^{rT+X_T} - e^k, 0)] - \max(1 - e^{k-rT}, 0).$$

Fourier transform:

$$\begin{aligned}\xi(v) &= \int_{-\infty}^{\infty} e^{ivk} z(k) dk \\ &= e^{ivrT}\frac{\phi_T(v-i) - 1}{iv(1-i+v)},\end{aligned}$$

ϕ_T is the characteristic function of X_T.
Option price function:

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\xi(v)dv + \max(1 - e^{k-rT}, 0).$$

Sensitivity:

$$c_\alpha = \lim_{h\to 0}\frac{c(\alpha+h) - c(\alpha)}{h}.$$

6. Numerical computations

Numerical solution of PIDE:

- Localise the PIDE to a bounded domain
- Truncate the integral
- Possibly approximate the small jumps

The PIDE then becomes

$$\begin{aligned} u_\tau(\tau,x) =& (r - \frac{\sigma^2}{2}) u_x(\tau,x) + \frac{\sigma^2}{2} u_{xx}(\tau,x) \\ &+ \int_{B_l}^{B_u} \nu(dy)[u(\tau,x+y) - u(\tau,x) - (e^y - 1)u_x(\tau,x)], \end{aligned}$$

where $x \in (-A, A)$.

Approximate the PIDE by a finite difference solution

$$u_i^n = u(n \cdot \Delta\tau, -A + i \cdot \Delta x)$$

and replace the derivatives by finite differences

$$\begin{aligned} \frac{\partial u(\tau,x)}{\partial\tau} &\to \frac{u_i^{n+1} - u_i^n}{\Delta\tau}, \\ \frac{\partial u(\tau,x)}{\partial x} &\to \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}, \\ \frac{\partial^2 u(\tau,x)}{\partial x^2} &\to \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}. \end{aligned}$$

Approximate integral term:

$$\nu_j = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu(dy),$$

$$\int_{B_l}^{B_u} \nu(dy) u(\tau, x+y) \to \sum_{j=K_l}^{K_u} \nu_j u_{i+j}.$$

Solve the PIDE on grid:
Use explicit-implicit solver:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = Du_i^{n+1} + Ju_i^n.$$

The diffusion part is

$$Du_i^n = \left(r - \frac{\sigma^2}{2}\right)\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \frac{\sigma^2}{2}\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}.$$

and the jump part is

$$Ju_i^n = \sum_{-K_l}^{K_r} \nu_j u_{i+j}^n.$$

The finite difference approximations of the Lévy measures are:
For the Merton model

$$\nu_j = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu(dy) = \frac{\lambda}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}} \Delta x.$$

For the variance gamma model:

$$\nu_j = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu(dy) = \frac{1}{\kappa |x_j|} e^{A x_j - B|x_j|} \Delta x,$$

where

$$x_j = -A + j \cdot \Delta x.$$

Conclusion

- The dynamic partial integro-differential approach for calculating prices and sensitivities works for a wide class of options in Lévy process driven models.
- For the options considered it is computationally faster than simulation methods and more widely applicable that Fourier transform methods.
- It produces option prices and sensitivities over a range of strikes or maturities.
- The approach can be applied to a wide spectrum of options, e.g. lookback options, Asian options and exchange options.

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