Good-Deal Bounds

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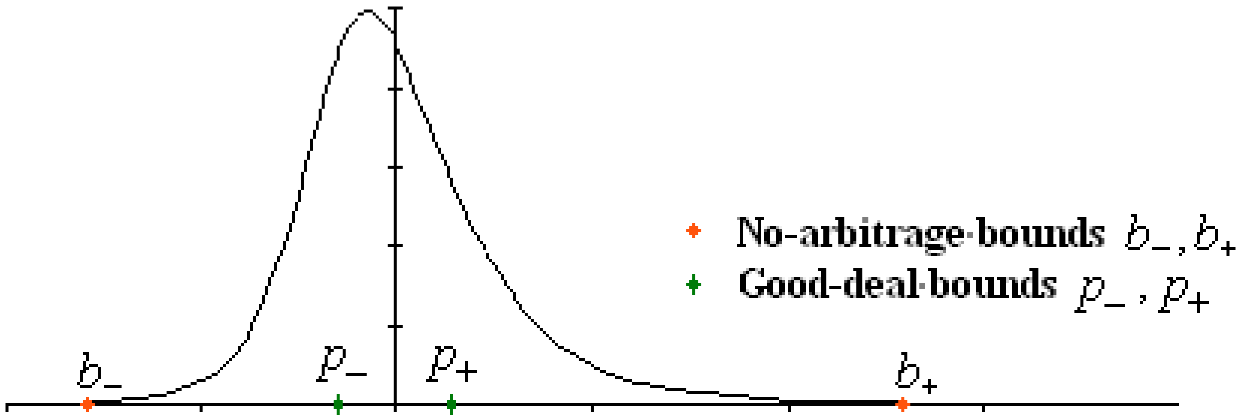
Motivation

"The difficulties of exactly replicating options positions suggest that we should put different bounds around the value of an instrument, depending on the market conditions, the assumptions and the securities that would be used to arbitrage any mispricing"

This is what good-deal bounds are intended to do:



Overview

We will look at:

- A brief history of good-deal bounds,
- Their areas of application,
- Alternative versions (including their pros and cons),
- Other issues

A Short History of Bounds

No-Arbitrage Bounds

Finding bounds on the values of derivatives is an old "art form":

- Merton (1973),
  - no arbitrage bounds,
- Perrakis and Ryan (1984), Levy (1985), Ritchken and Kuo (1989), Basso and Pianca (1994),
  - bounds based on stochastic dominance (or similar).
- Hobson (1998), Bounds on barriers using vanilla options.

Interest in this topic has intensified, with more interest in:

- Levy processes,
- exotic options,
- incomplete markets

"Good-Deal" Bounds

Early Work

"Good-Deal" bounds (or "No-Good-Deal" bounds?) were:

- introduced in 1996, by Cochrane and Saá-Requejo (Sharpe Ratio), and Bernardo, A and O Ledoit (E gain-loss ratio)
- modified by Hodges (Generalized Sharpe Ratio) in 1997,
- generalized to a more abstract setting
  by Cerny and Hodges, 1998 (Bachalier 2000 proceedings),
- related to Artzner *et al* "coherent risk measures"
  by Jaschke and Kuchler (2001)
  (also anticipated in Mejía-Pérez, 1998, and Hodges 1998).

More Recent Work on Good-Deal Bounds

General Theory

Cerny (2003), Staum, (2004), Bjork and I Slinko (2004)

Dynamic

Cont (2006), Frittelli and Scandolo (2006) etc.

Performance Measurement

Madan and McPhail (2000), Zakamouline (2007)

Mathematical Programming Formulations

Pınar (2006), Pınar and Salih (2006), etc.

General Theory of Good-Deal Pricing

The general framework of "no-good-deal" pricing places

- no-arbitrage, and
- representative agent equilibrium

at the two ends of a spectrum of possibilities.
[e.g. see Cerny and Hodges (2001)].

A **desirable claim** is one which provides a specific level of von Neumann-Morgenstern expected utility.
A **good deal** is a desirable claim with zero or negative price.

Within the analysis it is assumed that any quantity of the claim may be bought or sold.

Generalized Sharpe Ratio Equations

Claim utility: $U^*(p) = \sup_{\theta,h} E[U(W_0 + \theta(\tilde{X} - p) + \tilde{h}]$

where p is a prospective price for claim $\tilde{X}$, $\tilde{h}$ is a zero cost hedge, and W_0 is initial wealth.

The first order condition: $E[(\tilde{X} - p)U'] = 0$ establishes p as the price from the extremal measure induced by U'.

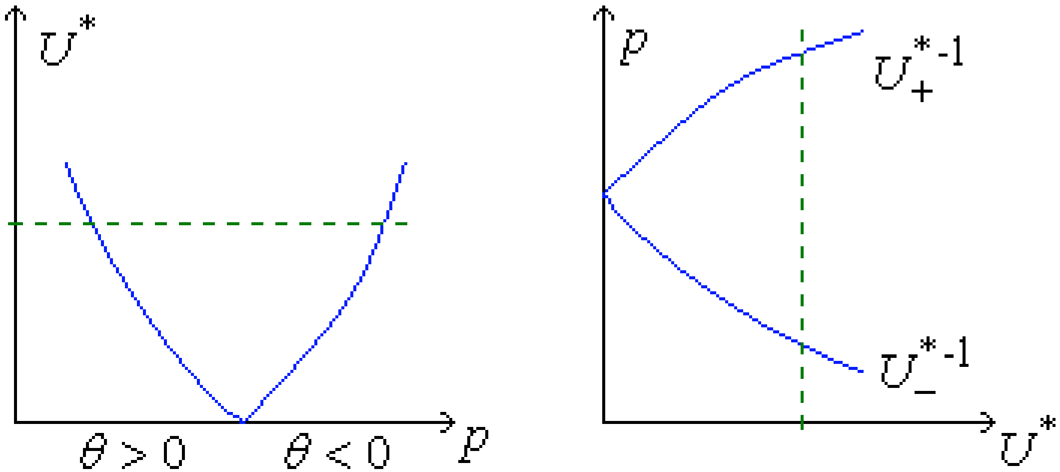
Benchmark utility: $U_B(SR) = \sup_{\theta} E[U(W_0 + \theta\tilde{\delta}_{SR})]$,

where $\tilde{\delta}_{SR}$ is a standardized opportunity[1] offering the stated Sharpe Ratio.

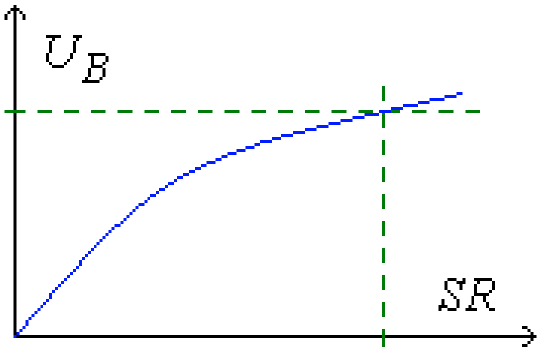
We construct the good-deal prices, p_+, p_-, to provide the same expected utility $U_B(SR)$.

The Construction

Claim utility:



Benchmark utility:



$$p_-(SR) = U_-^{*\,-1}(U_B(SR))$$

$$p_+(SR) = U_+^{*\,-1}(U_B(SR))$$

Applications

Broadly, there are applications to:

- Market equilibrium
- Statistical arbitrage/ Capital requirements, and
- Performance Measurement.

The settings vary:

- Static vs dynamic,
- Range of instruments available as hedges.

Incompleteness may be because of:

- Discrete time or Jumps,
- Costs, or
- Model Error.

Philosophy

The approach to calculate normative bounds for trading may be different from that to characterize equilibrium prices – even though the mathematics looks the same.

From a normative perspective, what matters is the expected utility of the hedged claim, and the reward for risk which this offers. This can even be applied to heuristic hedging methods – the better the hedge the tighter the bounds.

This is not quite the same as searching over a set of possible (assumed) market clearing pricing measures, for the extremal ones.

Alternative Utility Functions

Quadratic: Cochrane, 1996, including truncated at the bliss point.

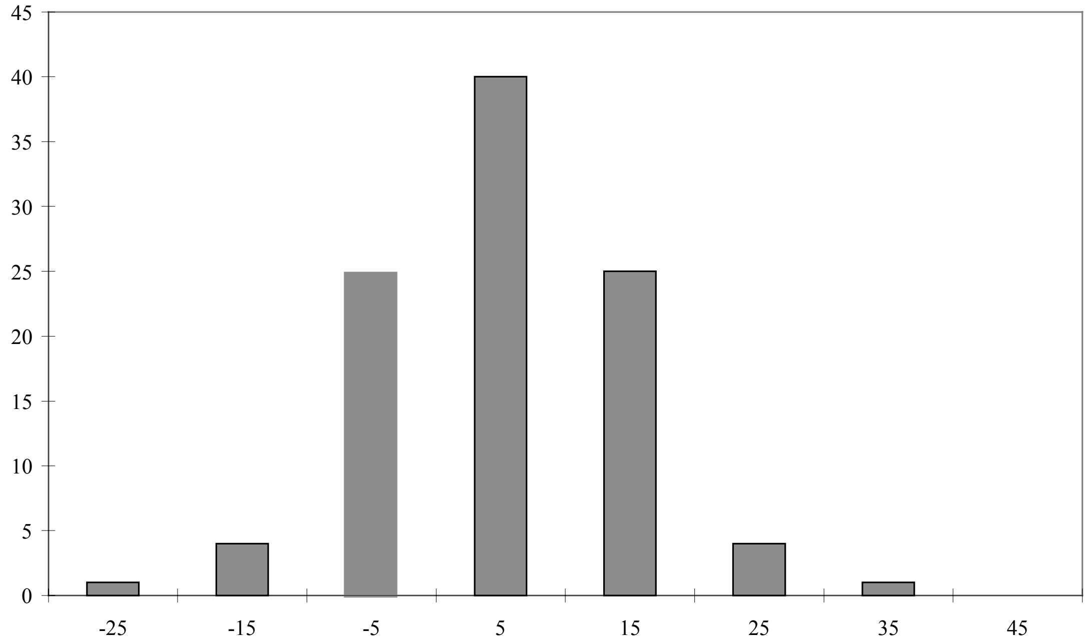
CARA: Hodges, Generalized Sharpe Ratio. 1998.

CRRA: Cerny, 2003.:

In the quadratic case, it is easy to see that Sharpe Ratios easily give rankings which contradict those from stochastic dominance:

In the following example, increasing the 35 payoff to 45 reduces the Sharpe Ratio from 0.500 to 0.493:

A Sharpe Ratio Paradox



μ = 5.00 → 5.10

σ = 10.00 → 10.34

S.R. = μ / σ = 0.50 → 0.493

Generalized Sharpe Ratio

Instead of quadratic utility, or as modified by Cochrane to truncate marginal utility at zero, Hodges (1998) suggested using CARA utility:

An investor with CARA utility can choose the quantity of the prospect to hold:

- it is defined to provide the usual Sharpe Ratio values for Normal distributions
- for non-Normal distributions, we provide a modification based on equating expected utility.

This measure restores the correct stochastic dominance ranking:

GSR = 0.498 $\rightarrow$ 0.500.

The Normal Case

A Normally distributed opportunity set provides forward outcomes distributed as $N(\mu T, \sigma^2 T)$.

To maximize $E[U(w)]$ with $U = -e^{-\lambda w}$ the choice problem is:

$$\underset{x}{Max} \; E[U] = -\exp\{-\lambda(\mu x T - ½\lambda\sigma^2 T x^2)\}$$

The first order condition is $\mu T - \lambda\sigma^2 T x = 0$, so $x = \dfrac{\mu}{\lambda\sigma^2}$.

Substituting into the expression for $E[U]$, we find

$$U^* = \underset{x}{Max} \, E[U] = -\exp\left\{-½\frac{\mu^2}{\sigma^2}T\right\}$$

which only depends on the Sharpe Ratio μ/σ and T (not on λ).

The General Case

In general, we optimize to find U^*, and then invert the previous formula:

$$U^* = \underset{x}{Max}\, E[U] = -\exp\left\{-\tfrac{1}{2}\frac{\mu^2}{\sigma^2}T\right\}$$

to find the Sharpe Ratio of the Normally distributed opportunity which would give the same level of expected utility, U^*.

This is given by:

$$GSR = \sqrt{\frac{-2}{T} ln(-U^*)}.$$

Computation

$$\underset{x}{Max}\ E[U]=\sum p_s \exp(-x r_s).$$

First order Condition :

$$\sum p_s r_s \exp(-x r_s) = 0 = f(x).$$

Solve using Newton - Raphson iteration for x with

$$f'(y) = -\sum p_s r_s^2 \exp(-x r_s).$$

We can do this easily on a spreadsheet.

Performance Measurement Revisited

Under a continuous diffusion process with a constant price of risk μ/σ , a CARA investor will have constant risk exposure.

The terminal distribution is Normal.

Hence, odd shaped distributions are **not** preferred.[2]

The Generalized Sharpe Ratio is robust in the sense that the maximum *ex ante* GSR **equals** the conventional Sharpe Ratio.

GSR "Good-Deal" Bounds

We solve the choice problem for an investor who maximizes

$E[U(w)]$ with $U = -e^{-\lambda w}$.

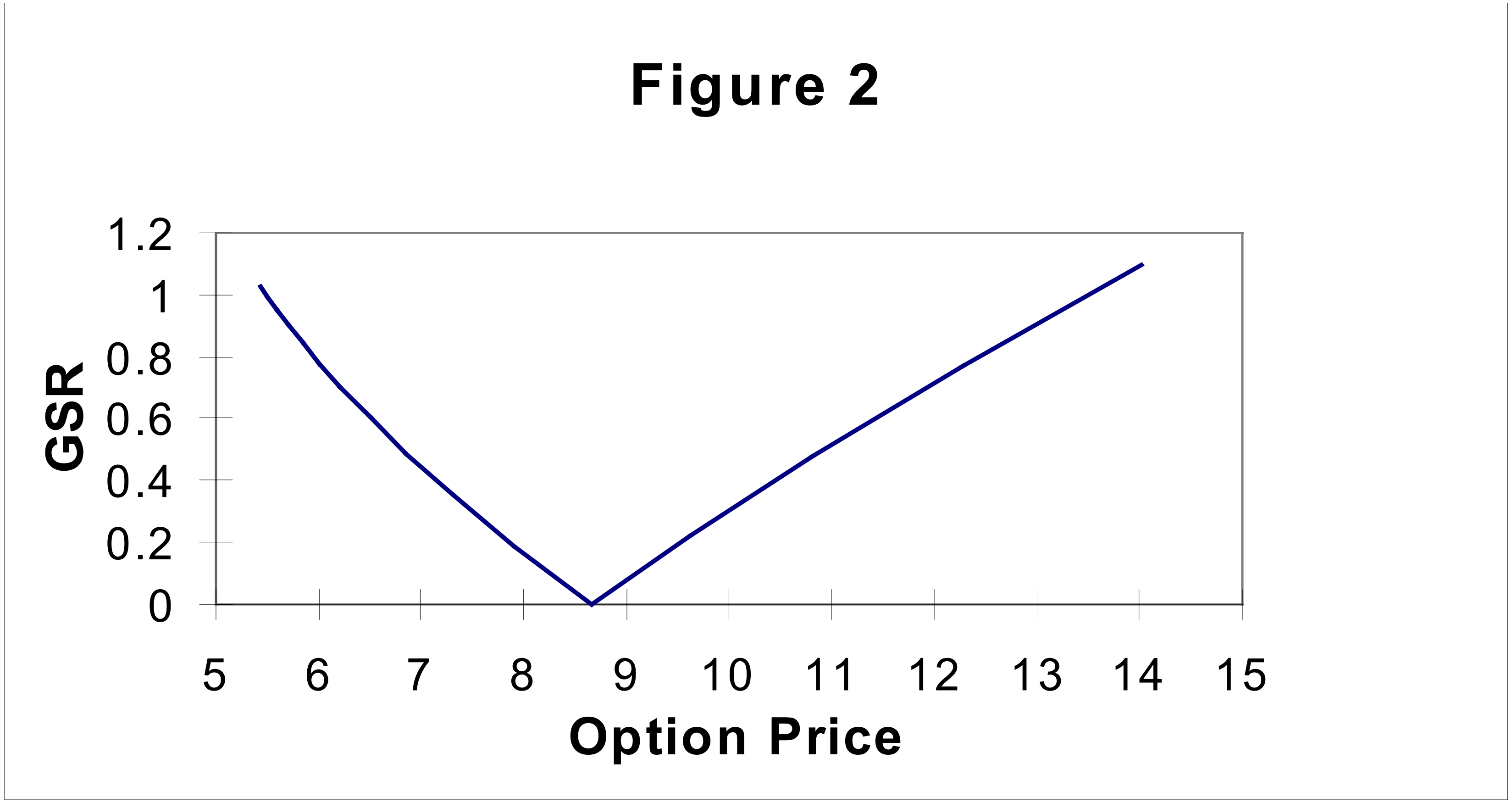
The investor buys x units of the contingent claim, and hedges with θ_t units of the underlying:

$$\underset{\theta_t, x}{\text{Maximise}} E[U] = -E\left[e^{-\lambda \left\{ \int_0^T \theta_t dS_t + x(C_T - C_0) \right\}} \right]$$

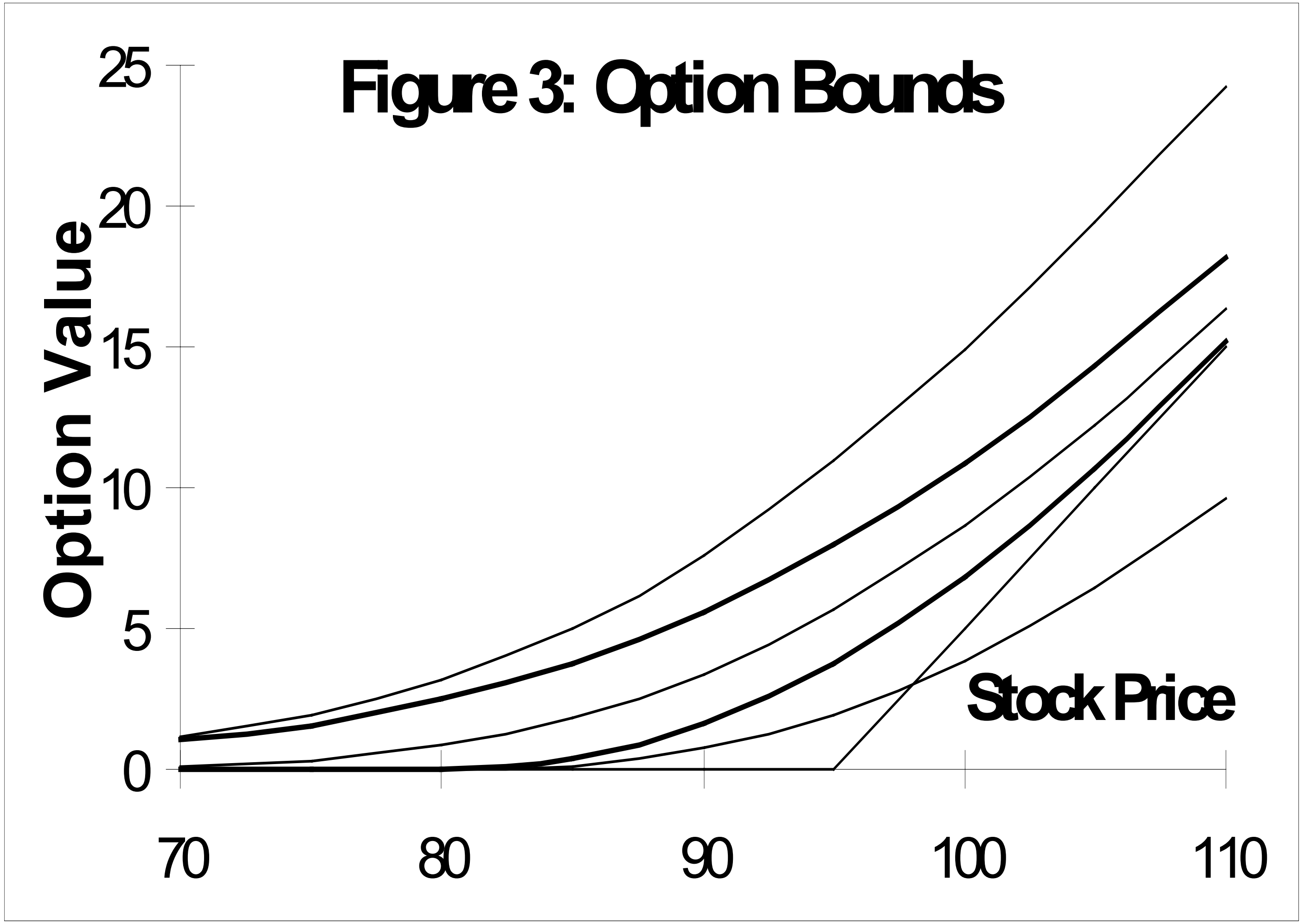
The value of the expected utility provides a GSR measure of the market opportunity provided by any particular C_0.

Conditional Bounds

We obtain valuation bounds which are much tighter than could be obtained by riskless arbitrage arguments.



Bounds at Different Asset Price Levels (GSR = ½)



See endnote [3]

Other Properties

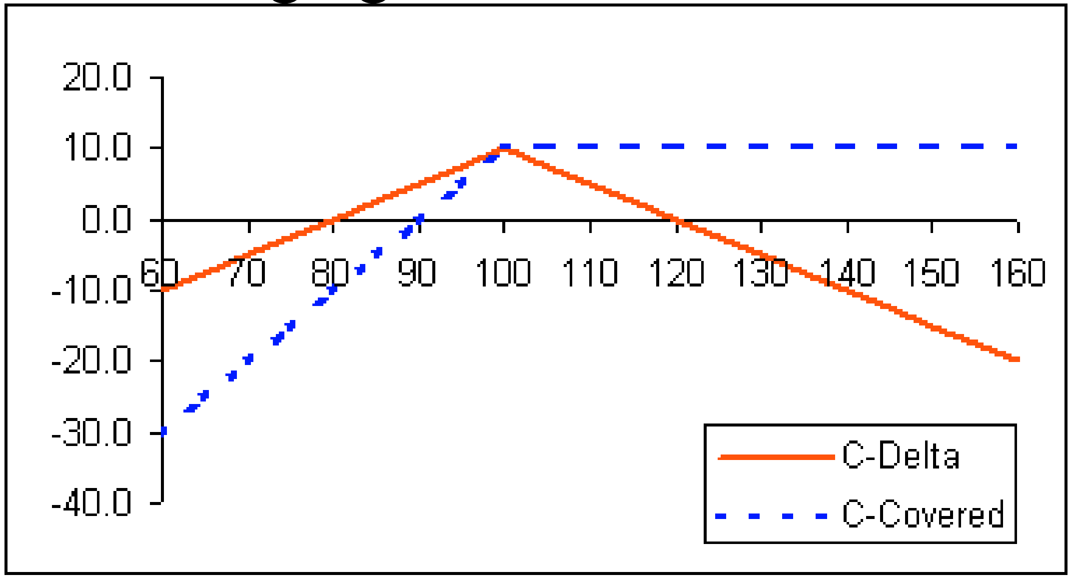
These GSR bounds defined by the class of negative exponential utility functions have a number of advantages and disadvantages:

- The bounds do not explicitly depend on risk aversion, or on wealth levels.
- Losses (negative wealth) are not ruled out
  - as they would be for power or log utility.
- Bounds respect the no-arbitrage limits.
- Some claims have very weak (and in some cases infinite) bounds.
  - in particular, any finite certain loss is preferred to a short position in a log-normal distribution.

Good-Deal Bounds and the Utility Function

The precise bounds obtained are sensitive to

- the choice of utility function, and
- on the kind of hedging which can be done.



Exponential utility precludes any delta hedge which gives a short lognormal position over finite time – though it will have a smaller standard deviation than the fully covered position.
CRRA utility precludes the possibility of negative future wealth.

CRRA: Changing x Affects W_0 not γ

With CARA utility, changing x simply changes the level of risk aversion.

With CRRA, it implicitly changes the initial wealth, W_0:

When $x \to \alpha x$,

$$\frac{(W_0 + x r_s)^{1-\gamma}}{1-\gamma} \to \frac{(W_0 + \alpha x r_s)^{1-\gamma}}{1-\gamma} = \frac{\alpha^{1-\gamma} (W_0 / \alpha + x r_s)^{1-\gamma}}{1-\gamma}.$$

The CRRA based good-deal bound searches across measures with the same exponent, but different wealth levels.

Summary: Characteristics of Utility Functions

Quadratic: does not satisfy stochastic dominance rankings (unless constrained): Sharpe Ratios must be modified.

CARA: scale and risk aversion are equivalent,
precludes uncovered short positions in log normal assets.
In transactions costs models, the tail of the distribution of costs incurred affects the shape of the control band.

CRRA: precludes losses (or at least negative wealth).

These properties suggest alternative functions may be advantageous.

An Alternative Utility Function

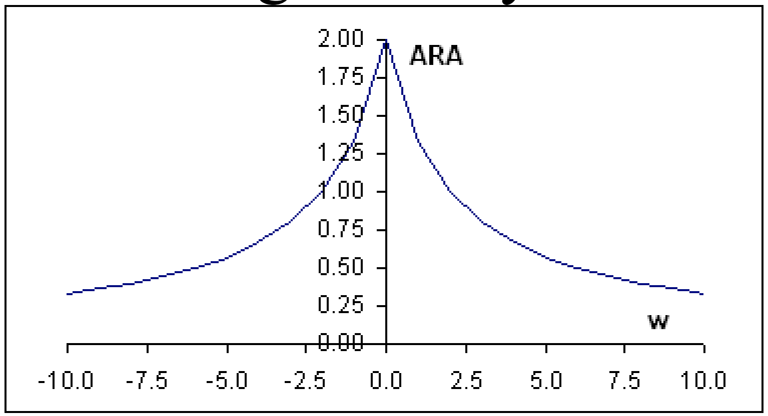
$$U(w) = \frac{1-(1-k_1 w)^{1+\gamma_1}}{k_1(1+\gamma_1)}, \qquad \text{for } w \le 0,$$

$$= \frac{(1+k_2 w)^{1-\gamma_2} - 1}{k_2(1+\gamma_2)}, \quad \left(\text{or } \frac{\ln(1+k_2 w)}{k_2} \text{ if } \gamma_2 = 1 \right) \text{ for } w \ge 0.$$

gives ARA asymptoptic to zero for $w \to \pm\infty$.

We need $k_1\gamma_1 = k_2\gamma_2$ to make ARA continuous at $w = 0$.

Without loss of generality we can choose $k_1 = 1$.



Here $k_1 = k_2 = ½; \quad \gamma_1 = \gamma_2 = 4.$

Conclusions

The no "good-deal" bound framework has been considerably extended from its original Sharpe Ratio definition.

It provides a powerful method for obtaining:

- Consistent measures of portfolio performance,
- Valuation bounds in incomplete markets
- Coherent risk measures for Value at Risk

It is computationally attractive.

Even in a single period setting the choice of utility function employed makes an important difference. This issues apply equally to bounds computed for capital adequacy purposes.

For normative work, one should also find a way of modelling the expected flow of future trading opportunities!!

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End Notes

[1] The benchmark may represent either a static distribution (e.g. from a buy and hold portfolio) or the outcome from an optimal continuous time policy. We will discuss this issue later.

[2] In this case the optimal continuous policy leads to a Normal distribution, i.e. the static and dynamic benchmarks are exactly equivalent. For other utility funcitons this is not the case – CRRA utilityonly leads to a 2 parameter lognormal distribution when there is no positive level of reserved wealth. However, the dynamic problem is easily solved using the Lagrange multiplier method to solve the stochastic control problem.

[3] The figure shows good-deal bounds on call options under negative exponential utility, if no intermediate trading can be undertaken before expiry. The loose bounds are for the case where even a static hedge cannot be placed, so the naked option payoff is experienced. The tighter bounds correspond to choosing an optimal static hedge consisting of the underlying and a bond. The upper bounds assume that a cap can be placed on the liability of shorting the option – without such a cap the unhedged case would be infinite, and the hedged one significantly weaker.