Strange Accumulators[a]

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In astronomy, we see many irregular structures manifested in the form of strong inhomogeneities of physical variables like density, temperature, or magnetic field. The appearance of strong concentrations of these variables, in the presence of what frequently must be highly turbulent conditions, poses an interesting question whose answer probably lies in the nature of turbulence itself. Indeed, G. I. Taylor, seventy years ago, wrote that turbulence is the strong concentration of vorticity. How does this work?

Fluid dynamicists often refer to a mechanism that they call Batchelor-Prandtl expulsion to explain inhomogeneities; the nature of this process has recently been clarified by Rhines and Young.[1] Solar physicists know of the process as it applies to the development of magnetic inhomogeneities on the solar surface.[2] However, no unanimity seems to exist among solar physicists on the explanation of the fine-scale structures on the sun.[3] There even seems to be disagreement about the proper description of the topology of field lines.

Dynamical systems theory shows us a simple way to look at field lines that may teach us about their structures. Though the approach does not explain how such fields arise in hydromagnetics (MHD), it may nevertheless be valuable in our thinking about the possibilities that confront us when we look at a complicated situation like that of the solar surface.

Consider a vector field $\boldsymbol{B}(x, t)$ whose structure we want to look at. This may be a magnetic or vorticity field whose field structure may be too complicated to pull out of numerical solutions of the MHD equations (for example). Thus, this is an example of how the theoretical developments of chaos theory may help in astrophysics by adding to the ways we have of thinking about problems, even when the data may not be good enough to confirm or deny the presence of chaos.

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Let X be a function of an independent variable s such that

$$\frac{dX}{ds} = \boldsymbol{B}(X(s), t) \tag{1}$$

for fixed t. If $\boldsymbol{B}$ should happen to be a velocity field, we would identify s and t and integrate equation 1 to get the particle paths (as has been done for fluid flows in two dimensions).[4,5] However, if $\boldsymbol{B}$ is some other field, we can look at a snapshot of it by regarding t as a parameter that we hold fixed during a run in which we study equation 1 as a dynamical system with s as the "time". The trajectory in X-space will give us a picture of the field. We can even go farther and cut the X-space with a plane and simply plot the points where the trajectory pierces it. By studying typical maps of the plane into itself, we can get an idea of the possibilities that lie in store with real fields, $\boldsymbol{B}$.

Perhaps the most typical, typical map is the standard map:[6]

$$x_{i+1} = x_i - k \sin 2\pi y_{i+1}, \quad y_{i+1} = x_i + y_i. \qquad \text{mod } 1 \tag{2}$$

This is an area-preserving map that seems a qualitatively appropriate choice when (the solenoidal) $\boldsymbol{B}$ does not vary much in the direction normal to the surface of section. For given k (not too large) and for suitable initial coordinates, this map gives rise to regular behavior in the form of periodic orbits lying on tori. The cross sections of these tori in the surface of section form island chains that are like the cat's-eyes of the fluid dynamicist. Between the islands are saddle points, in the neighborhood of which there are smaller islands, and so on.

A trajectory that gets in amongst the small islands will go for many twists before reemerging into the chaotic sea. This sojourn is responsible for long-range correlations of the successive points on the orbits of equation 2.[7,8] Meiss and Ott[9] have modeled the tendency of an orbit to linger in the reefs, while MacKay, Meiss, and Percival,[10] among others, have studied the mechanism of escape from them.

Our own naive point of view about the long times spent by particles in the reefs of the map is to liken these regions to the attractors of dissipative theory. One of the simplest examples of a dynamical system with an attractor is the Landau equation for $A(t)$:

$$\frac{dA}{dt} = A - A^3. \tag{3}$$

This system has attractors at $A = \pm 1$ and a repeller at $A = 0$. If we differentiate this system once and substitute, we get

$$\frac{d^2A}{dt^2} = -\frac{\partial V}{\partial A}, \tag{4}$$

where

$$V = -\tfrac{1}{2} A^2(A^2 - 1)^2. \tag{5}$$

The original system (equation 3) is contained in equations 4 and 5 if we impose suitable initial conditions.

Where the original Landau equation had either attractors or repellers, the new system (which is Hamiltonian) has saddle points. In the neighborhood of these points, the motion is very slow. We call sets of these points accumulators because a representative point moving in this system will tend to spend more time in the neighborhood of these points. Hence, the points in an orbit will tend to accumulate there. In systems with stochasticity and island chains, we can have quite a complex of such points; together, these make an accumulator with very fine texture. The presence of such complicated accumulators, we suggest, gives rise to strong inhomogeneities in field structures.

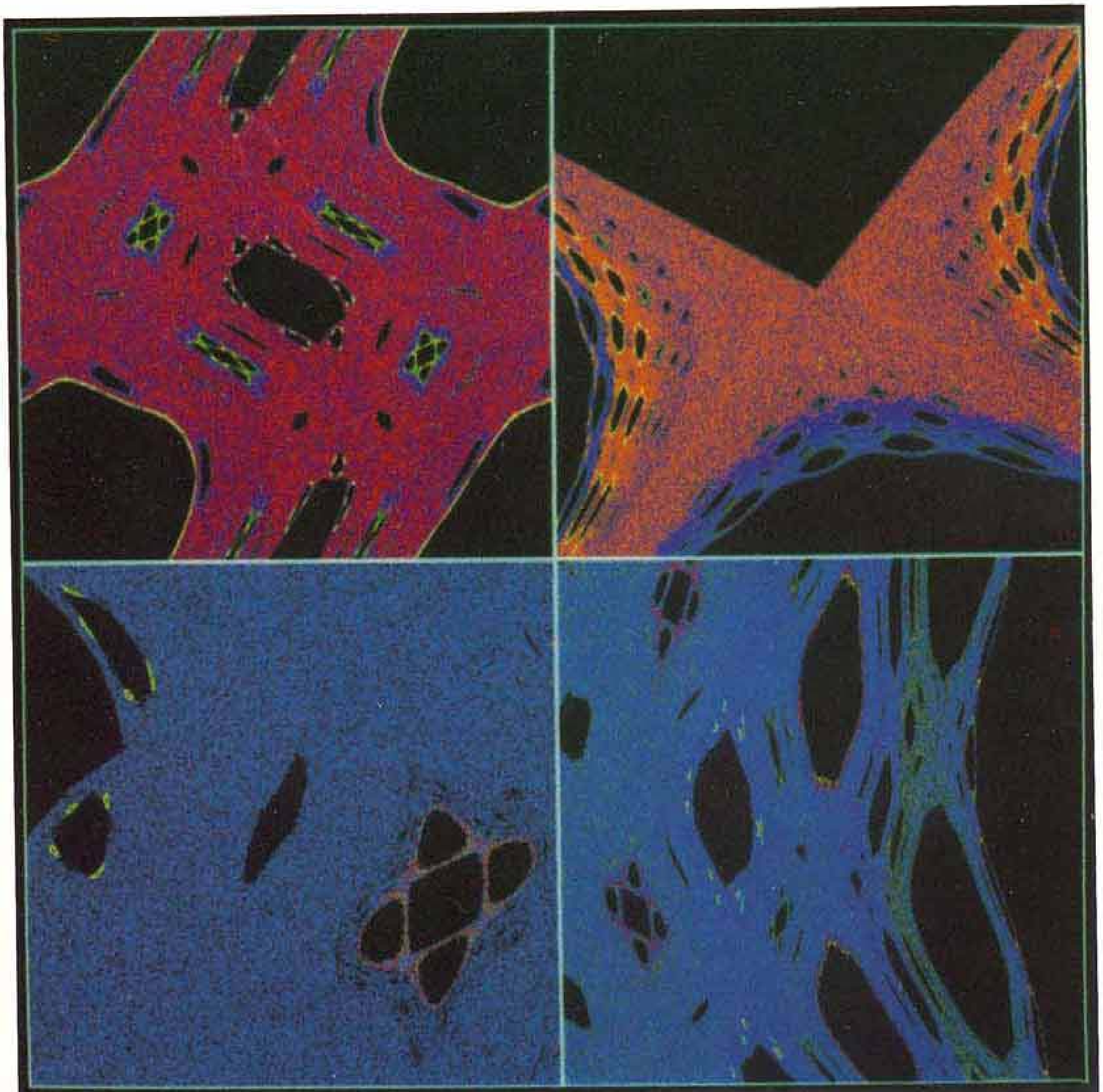

FIGURE 1. The panels on the left show the visitation histogram for the numerically one-to-one standard map ($k = 1.25$). The lower panel is a blowup. On the right, simulations with the Hénon[13] conservative map ($\cos \alpha = 0.240$) are shown. The number of visitations, as indicated by each color, varies slightly from panel to panel for photographic reasons. In the upper left panel, black means no visitation, brown is 1 to 4, blue is 5 to 9, red is 10 to 19, green is 20 to 29, and yellow is 30 to 50. White partitions were visited between 51 and 316 times. All histograms are calculated on a 2^9 by 2^9 grid.

In the left panels of the accompanying figure (FIGURE 1), we show the result of a long numerical integration based on the map of equation 2. The calculation for the standard map was done on a finite ($2^{20} \times 2^{20}$) mesh that preserves the 1–1 character of the map of equation 2, as in the work of Rannou[11] (see Miller and Prendergast[12] for the use of "noise-free" methods in dynamical astronomy). The result of this method (with a large, but finite number of grid points) is that every orbit calculated is periodic. Indeed, any deterministic simulation on a digital computer tends, in finite time, to a periodic orbit. If we are using this result to visualize field lines, this means that we are computing only closed field lines. In this case with an explicit grid, phase volumes are exactly conserved.

FIGURE 1 shows coarse-grained histograms of the number of visits to each grid square obtained in two different simulations. The panels on the left show histograms from a periodic 2,764,949-iteration orbit for the numerically one-to-one standard map with $k = 1.25$ (the lower panel being a blowup on a finer grid). The panels on the right similarly illustrate a double-precision calculation for the Hénon area-preserving, quadratic map (the field chosen is after figure 14 of Hénon[13]). In the Hénon case, we have followed the mapping for 2^{27} iterations. The calculations for the Hénon case, unlike those for the standard map, are not one-to-one numerically; hence, the structure shown in the right panels may be transient. The range in density shown is a factor of several hundred. Already, from these first results, we can see why the discussants at the solar physics meeting were in some doubt about what to call a flux tube in the solar surface. The notion of flux tube will often be useful only locally, but then the tube may splay out into the surrounding stochastic regions.

The deficiency in this type of study is that it is a little too generic. We want to be able to derive actual maps for real situations. This can be done when the vector field being studied is a real velocity field. Indeed, Chaikin *et al.*[5] have set up real flows corresponding closely to their maps and have found excellent agreement for the motion of advected particles. The flow in that case involved time-dependent open streamlines. We may also get open streamlines by studying the motion of particles drifting through a convecting fluid.[14,15] These studies suggest formation of strong inhomogeneities in the particle densities, but the results are, as yet, still limited.

In summation, we can say that this note is a conference publication *par excellence.* We are suggesting a way for modeling processes that seem too complex for detailed simulation without yet offering any quantitative predictions for astrophysics. Nevertheless, our qualitative results from the numerical experiments on long-period orbits seem to have surprised many of those who have seen them. Thus, it seems worth pointing out our conclusion that complicated accumulators are at the origin of strongly inhomogeneous concentrations of advected fields in flows. These fields need not be passive. Though the fields are dynamic and feed back on the flow, we may still expect the structure to be given by maps, even if we do not know them explicitly. This is ultimately a useful way to think about inhomogeneous fields.

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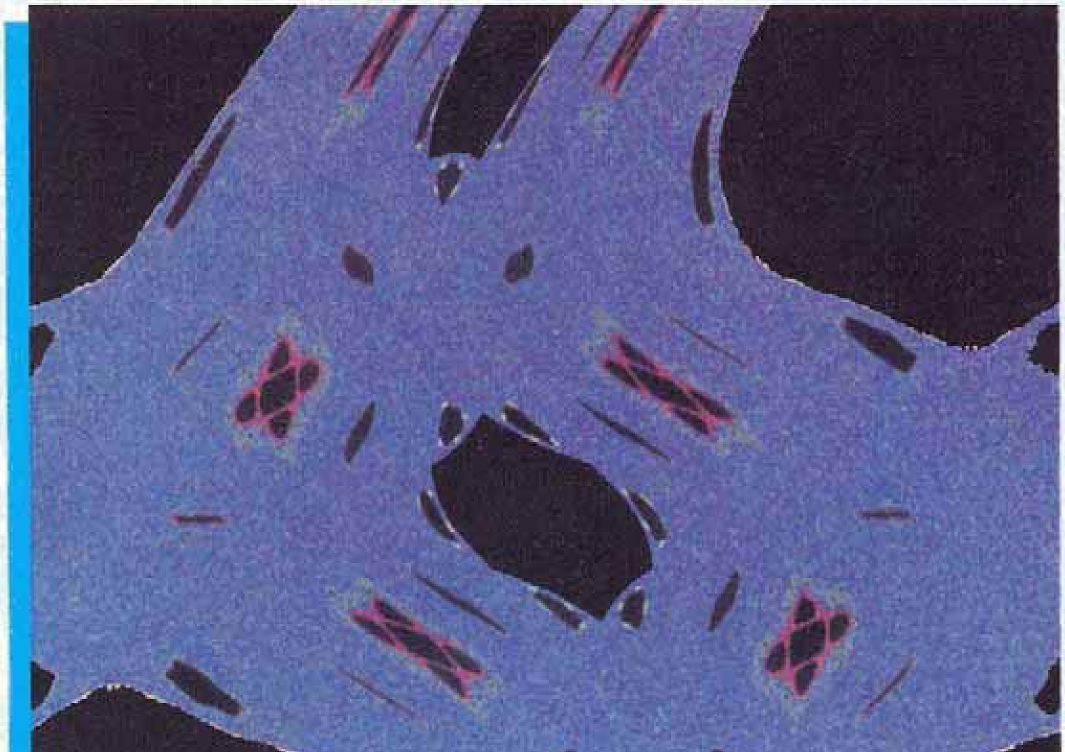

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