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Grantham Research Institute on Climate Change and the Environment

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# Laplace's Demon and Climate Change

Roman Frigg, Seamus Bradley, Hailiang Du and Leonard A. Smith[1]

## 1. Introduction: Laplace's Demon

Knowing what the future will bring is an age-old human desire. Yet it is a desire mortals find difficult to satisfy, and creatures endowed with appropriate powers tend to inhabit fictional landscapes. Among those creatures *Laplace's Demon* has gained notoriety. Laplace (1814, 4) invites us to consider a supreme intelligence who is able to identify all basic components of nature and the forces acting between them, and then observe these components' initial conditions. On the basis of this information the Demon knows the deterministic equations of motion of the world and uses his supreme computational power to solve them. The solutions of the equations of motion together with the initial conditions tell him everything he wants to know so that 'nothing would be uncertain and the future, as the past, would be present to [his] eyes' (*ibid*.). This operationally omniscient creature is now known as Laplace's Demon.

Let us give precise statement of the Demon's capabilities. In order to predict the future, the Demon possesses a mathematical model of the world. It is part of Laplace's original scenario that the model is a model of the *entire* world. However, nothing in what follows depends on the model being global in this sense, and so we consider a scenario in which the Demon's predicts the behaviour of a particular part or aspect of the world (which can but need not be the entire world). In line with much of the literature on modelling we refer to this part or aspect of the world as the *target system*. Mathematically modelling a target system amounts to introducing a *dynamical system* $(X, \phi_t, \mu)$, which represents that target system.[2] Unlike the target

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[2] Calling both the model and the target 'system' is unfortunate; we do so only in order to stick with conventionally used terminology. For a discussion of the anatomy of scientific modelling can be found in (Frigg 2010).

system, which is part of the material world, a dynamical system is a mathematical object. As indicated by the notation, a dynamical system consists of three elements. The first element, $X$, is the system's *state space*. When we take $(X, \phi_t, \mu)$ to represent a target system, the states in $X$ are taken to represent states of the target system. For instance, the state space of a particle moving along a line consists of all tuples $x = (q, p)$, where $q$ and $p$ are real numbers representing, respectively, the particle's position and momentum. The second element, $\phi_t$, is the *time evolution*: if the system is in state $x_0 \in X$ at time $t = 0$, then it is in $y = \phi_t(x_0)$ at some later time $t$; that is, $\phi_t$ tells us how the system's state changes in time. The state $x_0$ is called the system's *initial condition*. Often $\phi_t$ for a particular system is not formulated directly; instead we formulate the system's equation of motion and then $\phi_t$ is the solution of that equation. In the dynamical systems we are concerned with in this paper, the time evolution of a system is generated by the repeated application of a map $U$ at discrete time steps: $\phi_t = U^t$, for $t = 0,1,2,...$[3] The third element, $\mu$, is the system's Lebesgue measure: it allows us to say that parts of $X$ have certain size. In case $X$ is the real axis, $\mu$ is the length of an interval. The measure is used both to measure physical distance, and (as we will see) it plays a role in defining a probability density over $X$.

With this bit of formal apparatus in place, we can describe Laplace's Demon as a creature with the following capabilities:

(1) He has unlimited computational power: he is able to calculate instantaneously $y = \phi_t(x)$ for all $t$ and for any $x$.

(2) He has unlimited observational power: he is able to specify the *true* initial condition $x_0$.

(3) He has unlimited dynamical knowledge: he is able to formulate the *true* time evolution $\phi_t$.

[3] This is a common assumption. For an introduction to dynamical systems see (Arnold and Avez 1968).

If these conditions are met, it is indeed the case that 'nothing would be uncertain' to the Demon and 'the future, as the past, would be present to [his] eyes'.[4] In the modelling literature having the *true* $\phi_t$ is often referred to as the *Perfect Model Scenario*; so condition (3) says that the Demon has the perfect (or true) model.

Humans do not have any of the Demon's capabilities: most equations we can't solve, no measurement can ever reveal an exact initial condition, and for most systems idealisations are unavoidable when formulating equations. It is therefore no surprise that Laplace was quick to point out that the human mind 'will always remain infinitely removed' from the Demon's intelligence, of which it offers only a 'feeble idea' (*ibid*.). The interesting question in connection with Laplace's Demon is not whether we fail to perform at the Demon's level – of course we do. The interesting question is how exactly we fail and what the consequences of this failure are.

Laplace's own discussion emphasises the Demon's unlimited computational power and sees our main failure in being unable to do the calculations that the demon is able to do. This was indicative of the concerns of physicists and mathematicians at the time, when the focus was on developing techniques to solve differential equations. Interestingly, Laplace paid little attention to the other two conditions.[5] It is difficult to say in retrospect what exactly his and his contemporaries' attitude towards conditions (2) and (3) was, but given that only scant attention was paid to them, it cannot be far off the mark to assume that they were considered practical limitations of little theoretical interest.

The old view that the unavailability of exact initial conditions is no impediment to making successful predictions in a deterministic system was based on what is now known as the *strong principle of causality*. The principle says that if $y_0$, the initial condition used for calculations, is close enough to $x_0$, the true initial condition, then

[4] Laplace implies that the dynamics is invertible. We ignore cases in which this is not the case because we restrict attention on prediction.

[5] He briefly mentions that the Demon must 'comprehend all the forces by which nature is animated and respective situation of beings who compose it' (1814, 4). It is reasonable to assume that Laplace had the *true* forces (and hence the *true* equation of motion) and the *true* initial positions in mind, but he does not dwell on the point.

the trajectories originating in $x_0$ and $y_0$ stay together for all times – varying the initial conditions a little bit will not change the outcome very much. When confronted with the task of making a prediction, we have to assess what the required precision is and then make a sufficiently precise measurement of the initial condition. This may be challenging in practice, but in essence it is an engineering problem of no in-principle importance.

This way of thinking about initial conditions was debunked by Poincaré at the beginning of the last century. Poincaré's crucial insight was that if a system's dynamics is non-linear (and most systems are better modelled as non-linear), then even arbitrarily close initial conditions can diverge and end up taking very different paths – an effect now known as *sensitive dependence on initial conditions*.[6] So we cannot infer from the fact that initial conditions are similar that the later trajectories will be similar too. But if arbitrarily small variations in the initial condition can make a dramatic difference, then the strong principle of causality is wrong and we can no longer dismiss issues surrounding initial conditions a merely practical problem. In fact, the failure of the strong principle has wide-ranging consequences, and the study of these consequences is known as the *study of chaos*. So condition (2) is much less innocuous than it originally seemed to be.

Similar issues arise in connection with condition (3). Just as we cannot get our hands on the true initial condition, we cannot formulate the true equations of motion of a target system (if such equations exist at all). Idealisations, distortions, omissions and simplifications are inevitable. We model oblate spheroid planets as perfect spheres, sticky surfaces as frictionless, markets as having no transaction costs, and so on. This has of course long been recognised, but, as with condition (2), dismissed as a practical issue of no in-principle importance. However, this dismissal has rarely, if ever, been explicit. A tacit consensus has emerged that we are entitled to assume that if the equations of our model are close enough to the true equations, then the predictions the model makes are close enough. We call this assumption the *closeness-to-goodness*

[6] This is effect is often referred to as chaos, but in fact the relation between sensitive dependence and chaos is not straightforward; for a discussion of this point see (Smith 1998, Ch. 10).

*link*. This link is the 'model-level analogue' of the strong principle of causality – the reasoning behind both principles is that close enough is good enough.

Unlike the strong principle of causality, the closeness-to-goodness link has not yet attracted much attention. The central contention of this paper is that closeness-to-goodness link does not fare better then the strong principle of causality: even the *slightest* inaccuracy in the specification of the system's dynamics destroys the Demon's ability to predict the future. More specifically we argue that if a mathematical model is non-linear and if there is only a minuscule structural model imperfection, then treating model outputs as decision-relevant forecasts can be seriously misleading: the closeness-to-goodness link fails. And the worst is yet to come: this is the case even if we were to limit attention to making probabilistic forecasts. Unless the Demon has the *true* equations of motion, he cannot even make reliable probabilistic forecasts.

This is more than a pedantic addition to a contrived thought experiment. In fact, this addition has important implication for scientific practice because understanding the conditions that need to be met by the Demon to make reliable predictions teaches us important lessons about our own limitations. From planetary motion to nuclear fission, from inventory control to sea level rise, and from the growth of populations to the returns of an investment, and from short term weather forecasts to long term climate predictions, there is hardly a phenomenon that has not at one point or other been modelled mathematically, and the mathematical models are often used for the purpose of forecasting. Moreover, there is a general trend, aided by the availability of ever increasing computational power, of building ever larger and more complex mathematical models of an ever growing variety of systems. This trend is particularly prevalent in both climate science and in weather forecasting, where ever larger models are constructed and run with the aim of making specific predictions.

This raises the question of exactly what man-made models deliver: can they provide the results as advertised? This is where Laplace's Demon comes into play. The limits of the Demon are at once the limits of every mathematical modelling endeavour: the fact that the Demon loses his predictive powers if there is only a small inaccuracy in this specification of the system's dynamics has profound implications for what we can and cannot do with our mathematical models.

In Section 2 we introduce the Demon's freshman apprentice, who shares some but not all capabilities of the Demon. In Section 3 we let the apprentice offer bets in a concrete situation based on the logistic map, and document his failures. This shows in an exemplary manner what can go wrong if probabilities are used naively in an imperfect model scenario. While one counterexample is sufficient to refute a general view, it is important that the example used is not idiosyncratic and easily dismissed as being irrelevant to those cases we really care about. This question is addressed in Section 4, where we provide a general mathematical argument for the conclusion the problems described in Section 3 are generic and occur in a large class of systems. In Section 5 we discuss some concrete cases where the methodologies we criticise are used; most notably climate and weather models belong to this class. In Section 6 we discuss and dismiss a number of simple ways around the problem and in Section 7 we present our own tentative solution, which suggests abandoning probabilism and using non-probability odds instead. In Section 8 we draw some general conclusions.

## 2. Probabilistic Forecasting: The Demon's Apprentice

It is now time to meet the Demon's apprentices – the freshman apprentice and the advanced apprentice. Like the master, both apprentices can calculate $y = \phi_t(x)$ instantaneously for all $t$ and for any $x$. The senior apprentice also shares with the Demon the ability to know the *true* time evolution operator $\phi_t$, but has limited observational power and can specify the system's initial condition only with a certain margin of error (or, as physicists, would say: he only has noisy observations).[7] The freshman apprentice yet has to acquire the skills of the senior apprentice and can neither specify a precise initial condition nor know the true time evolution.

[7] This is the apprentice we have already encountered in (Smith 2007).

Both apprentices are aware of their limitations and come up with coping mechanisms. In order to overcome the limited knowledge about initial conditions, when making calculations, they both account for their uncertainty by considering a probability distribution $p_0(x)$ over relevant initial states, where the subscript indicates that the distribution describes their uncertainty about the initial condition at $t = 0$.[8] For the apprentices, therefore, it makes no sense to move a single precise initial condition forward in time; for them the relevant question is how initial probabilities change over the course of time. To answer this question they use $\phi_t$ to move $p_0(x)$ forward in time; that is, they calculate $p_t(x) := \phi_t[p_0(x)]$.[9]

This idea is simple and striking: if $p_0(x)$ provides them with the probability of finding the system's state at a particular place in $X$ at $t = 0$, then $p_t(x)$ is the probability of finding the system's state at a particular place at any later time $t$. We call the apprentices' view that decision-relevant probabilities for certain events to occur can be obtained by using $\phi_t$ to obtain forecast probabilities for events at later times the *default position*. The qualification 'decision-relevant' is crucial. The Default Position does not make the (trivial) statement that $p_t(x)$ is a probability distribution in a purely formal sense of being an object that satisfies the mathematical axioms of probability; the position is committed to the (non-trivial) claim that these probabilities are the true probabilities for outcomes in the world and that a rational decision maker should adjust his/her beliefs to these probabilities and act accordingly (assuming that there is no other pertinent evidence). In other words, the apprentices take $p_t(x)$ to

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[8] There is a question about what the true distribution is (Allen and Smith 1996); we set this issue aside and assume that in one way or another we can come by the true $p(x)$ (in the sense that it is a correct representation of our uncertainty). For a discussion of different kinds of uncertainty and their sources see (Bradley 2011), (Smith 2007), (Schiermeier 2010) and (Judd and Smith 2004)

[9] We use square brackets to indicate that $\phi_t[p_0(x)]$ is the propagating forward in time of the initial distribution $p_0(x)$. The time evolution of a distribution derives from the time evolution of a state as follows: $p_t(x) := \phi_t[p_0(x)] = \Sigma_i p_0(z_i)$, where the sum of $z_i$ reflects each of the states in $X$ which are mapped onto $x$ under $\phi_t$ (i.e. $\phi_t(z_i) = x$ for all $i$); if the time evolution is invertible this reduces to $p_t(x) = p_0(\phi_{-t}(x))$.

provide us with predictions about the future of sufficient quality that we ought to place bets, set insurance premiums, or make public policy decisions according to the it.

Both apprentices are content with that solution, but the freshman has a further obstacle to overcome: he is unaware of the true $\phi_t$. He has good sense for the target systems and is able to make idealisations and simplifications that are sound in the sense that they omit unnecessary detail while capturing the essence of the system, and he can write down an idealised time evolution without knowing the true $\phi_t$. The core of his response to his second limitation is captured in the slogan 'close enough is good enough': he limits himself to time evolutions that are generated by the iterative application of a map ($\phi_t = U^t$), and adopts the principle that if $U$ he comes up with is close enough to the target system's true $U$, then his model time evolution is not too different from the true time evolution and hence his $p_t(x)$ should not too different from the $p_t(x)$ one would get were one to use the true time evolution. Hence there is nothing wrong with making decisions using his $p_t(x)$. This is the main idea behind the *closeness-to-goodness link.*

A precise rendering of the closeness-to-goodness link goes as follows. Let $U_T$ be the Demon's map (where the subscript 'T' stands for 'true' and indicates that the Demon has the true model), and let $U_A$ be the Apprentice's approximate time evolution (where the subscript 'A' can stand either for 'Apprentice' or for 'approximate'). Then $\Delta_U := U_T - U_A$ is the difference between the two maps. Furthermore let $p_t^T(x) := \phi_t^T[p_0(x)]$ be probabilities obtained under the true time evolution (where $\phi_t^T = U_T^t$) and $p_t^A(x) := \phi_t^A[p_0(x)]$ the probabilities the result from the approximate time evolution (where $\phi_t^A = U_A^t$). Then $\Delta_p(x,t) := p_t^T(x) - p_t^A(x)$ is the difference between the two. The closeness-to-goodness link then says that if $\Delta_U$ is small, then

$\Delta_p(x,t)$ is small too for all time $t$, presupposing an appropriate notion of being small.[10]

The Demon and his apprentices now get into a discussion about the validity of their positions. The senior apprentice claims that while his inability to identify the true initial condition prevent him from making exact forecasts, his probability forecasts are good in the sense that if the initial probability distribution is decision relevant, then all future probability distributions are as well; that is, his $p_t(x)$ is decision relevant provided that $p_0(x)$ is. Since we have agreed above to set issues with determining $p_0(x)$ aside, the senior apprentice's position is correct. As long as we are content with a probability forecast, knowing the true time evolution and being able to make calculations as fast as we please, the resulting probability forecasts are decision relevant.

The freshman now claims that he can achieve the same even without knowing the true dynamics. He thinks that both the Demon and the senior apprentice use a sledgehammer to crack nuts and that one can achieve the same result with fewer resources because the closeness-to-goodness link makes knowledge of the true time evolution obsolete. The Demon disagrees. He distrusts the closeness-to-goodness link, which he regards as unfounded and potentially dangerous hand-waving. So he challenges the freshman apprentice to establish the utility of his methods.

In the next section we describe the challenge and report on the outcomes. Before doing so two points are worth emphasising. First, we will see that the Demon is right: the closeness-to-goodness link will break. This establishes our central conclusion: in order to foresee the future the Demon must know the true dynamics of the system; even the slightest inaccuracy in the specification of the system's dynamics limits his ability to predict the future. In Section 3 we establish this conclusion with a specific example. The use of an example makes the problem that arises when basing predictions on the closeness-to-goodness link more vivid than an abstract argument,

[10] The notion of $\Delta_U$ being small can be explained in different ways without altering the conclusion. Below we quantify $\Delta_U$ in terms of the maximal one-step error.

and, from a logical point of view, all we need to debunk a universal rule is a striking counterexample. In Section 4 we nevertheless provide a general argument based on a mathematical theorem in support of our conclusion. This is to put worries to rest that our conclusion is an artefact of the specific example which does not generalise to all models of interest.

Second, those with limited interest in fictional narratives should rest assured that our fiction is less distant from reality that we would like it to be. In fact, the freshman apprentice's methodology is modelled on real methodologies, and in Section 5 we mention examples showing that the Apprentice's methodology is the (tacit and unacknowledged) background methodology of many scientific endeavours, among them some approaches to predicting the local effects of climate change. So the problems we describe are problems not only for the demonic apprentices; they concern equally working scientists.

## 3. The Apprentice's Adventures

The Demon believes that the Default Position combined with closeness-to-goodness link causes havoc: $p_t^A(x)$ need not be the true probability distribution (or reflect that distribution in some sense), and taking $p_t^A(x)$ as a guide to actions can be ruinous. The Demon aims to highlight the problems with the method by presenting the freshman apprentice (from now simply the 'Apprentice') with a case where one can explicitly see that $p_t^A(x)$ need not be the true probability distribution. This is enough to refute the Apprentice's position, which has it that $p_t^A(x)$ *always* is the decision relevant probability distribution.

The Demon challenges the Apprentice to model a simple situation in ecology: the evolution over time of the size of a population of fish in a pond. To this end they introduce the population ratio density $\rho_t$: the number of fish per cubic meter at time $t$ divided by the maximum number of fish the pond could accommodate per cubic meter. Hence $\rho_t$ lies in the unit interval $[0,1]$. Then they both go away and study the situation.

After a while they reconvene and compare notes. The Apprentice's dynamics is given by the well-know *logistic map*,

$$\rho_{t+1} = 4\rho_t(1-\rho_t)\,, \tag{1}$$

where the difference between times $t$ and $t+1$ is a generation. For ease of presentation we assume that the fish reproduce weekly and hence $t$ is measured in units of weeks. In terms of dynamical systems, the Apprentice proposes as system with $X = [0,1]$ as the state space. The initial condition is the population density at time $t = 0$, $\rho_0$. The time evolution $\phi_t^A$ is generated by iteratively applying the generative map given by Equation (1); hence $4\rho_t(1-\rho_t)$ is $U_A$. The Lebesgue measure is the usual geometrical length.

The Demon has the true dynamical system based on the same state space and measure, but with a slightly different generative map:

$$\tilde{\rho}_{t+1} = 4\tilde{\rho}_t(1-\tilde{\rho}_t)\left[(1-\varepsilon)+\frac{4}{5}\varepsilon(\tilde{\rho}_t^{\,2}-\tilde{\rho}_t+1)\right], \tag{2}$$

where $\varepsilon$ is a parameter taken here to be 0.1. We call this the *quartic map*. The right hand side of Equation (2) is $U_T$ and hence iteratively applying his generative map (Equation 2) yields $\phi_t^T$.

It is immediately clear that the Apprentice's model is just missing a small perturbation; for $\varepsilon \to 0$ the Demon's map converges towards the Apprentice's. Figure 1 shows both $U_T$ and $U_A$ for $\varepsilon = 0.1$, making it obvious how small the difference between the two is.

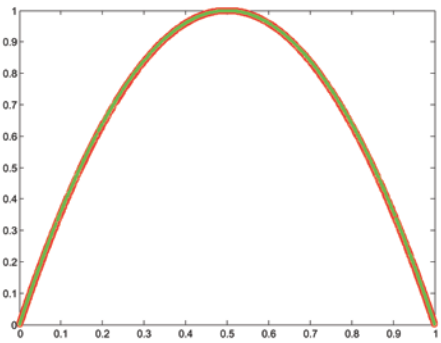


Figure 1: Equation 1 in green and Equation 2 in red with $\rho_t$ and $\tilde{\rho}_t$ on the *x*-axis and $\rho_{t+1}$ and $\tilde{\rho}_{t+1}$ on the *y*-axis.

The maximum one-step error of the model is $5\times10^{-3}$ at $x\cong 0.85344$, where $\rho_{t+1}\cong 0.50031$ and $\tilde{\rho}_{t+1}\cong 0.49531$. This is about 1/100 and hence it is reasonable to say that $\Delta_U$ is small. Applying the closeness-to-goodness link, the happy Apprentice now expects $\Delta_p$ to be small too. That is, starting with the same initial probability distribution $p_0(x)$, he would expect $p_t^T(x)$ and $p_t^A(x)$ to be least broadly similar. We will now see that the Apprentice is mistaken.[11]

Since it is impossible to calculate $p_t^T(x)$ and $p_t^A(x)$ with pencil and paper, we resort to computer simulation. To this end, we partition the system's state space – the unit interval $[0,1]$ – into 32 cells, which, in this context, are referred to as *bins*. These bins are now the atoms of our event space: in what follows we calculate the probabilities of the system's state $x$ being in a bin. This is of course not same as calculating a continuous probability distribution, but since nothing in what follows hangs on the difference between a continuous distribution and one over bins and for the sake of notational ease we refrain from introducing a new variable (and new probabilities) for the bins and from now on take '$p_t^T(x)$' and '$p_t^A(x)$' to refer to the probabilities of

[11] Our argument does not trade on worries about $p_0(x)$. We can assume that the initial distribution gives us the true probabilities and that setting one's degrees of belief in accordance with these probabilities would be rational. The core of our concern is what happens with these probabilities under the time evolution of the system.

bins. Similarly, a computer cannot handle analytical functions (or real numbers) and so we represent $p_0(x)$ by an ensemble of points. Specifically, we consider an ensemble of 1024 initial conditions. We first draw a random initial condition (according the invariant measure of the logistic map). By assumption this is the true initial condition of the system at $t=0$; it is indicated designated by the cross in Figure 2a. We then draw 1023 points randomly around the true initial condition according to a Gaussian distribution. These 1024 points form our distribution, which is shown in Figure 2a. Dividing the numbers on the *y*-axis by 1024 yields an estimate of the probability for the system's state to be in a particular bin.

We now evolve all these points forward both under the dynamics of the system and of the model. Figures 2b-2d show how many points there are in each bin after two, four and eight weeks respectively. Again, dividing these numbers by 1024 gives $p_t^T(x)$ and $p_t^A(x)$ at $t=2$, $t=4$ and $t=8$.

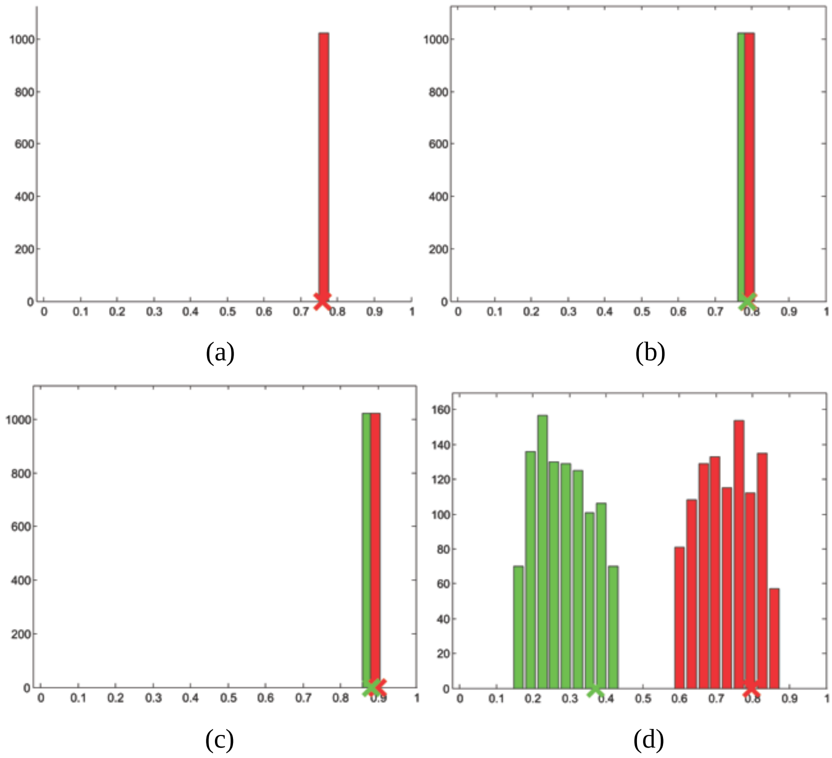

Figure 2: The evolution of the initial probability distribution under the Apprentice's approximate dynamics (green) and the Demon's true dynamics (red). The crosses mark the true initial condition and their time evolutions.

While the two distributions overlap relatively well after two and four weeks, they are almost completely disjoint after two months. Hence, these calculations show the failure of the closeness-to-goodness link because we have case where the smallness of $\Delta_U$ does not imply $\Delta_p$ is small too for all $t$; in fact for $t = 8$ $\Delta_p$ is as large as can be because there is no overlap at all between the two distributions.[12]

This shows that even if a non-linear model is extremely close to the true dynamics (remember that the maximal one step-error is $5\times10^{-3}$!), then predictions, probabilistic and deterministic alike, can break down. Hence, simply moving an initial distribution forward in time under the dynamics of a good model need not yield decision-relevant outcomes.

One could object that the presentation of our case is biased in various ways. The first alleged bias is the use of an eight week forecast: had we used two or four week forecasts instead, the apprentice's endeavours would have been successful. While this may be true in our specific example, in real modelling scenarios we cannot compare model outputs with the true occurrences and affirm that we are fine at $t = 4$. In fact, if we were able to calculate the evolution of the probabilities under the true dynamics we would not use a model in the first place! Outside the thought experiment, the only thing we have is the model, which we know to be imperfect in various ways. Our tail shows that model-probabilities and probabilities in the world *can* come unstuck dramatically, and as long as we have no means of telling *when* this happens, we'd better be on guard.

Another alleged bias is the choice of the particular initial distribution shown in Figure 2a. This distribution, so the argument goes, is cleverly chosen to drive our point home but most other distributions would not be misleading in such a way. Our results, so the argument continues, only shows that unexpected results can occur every now and then, but that is not enough for a wholesale rejection of the closeness-to-goodness link.

[12] We are operating with an intuitive notion of difference here; below we make this notion precise in terms of relative entropy.

There is of course no denying that the above calculations rely on a particular initial distribution, but that realisation does not rehabilitate the closeness-to-goodness link. We repeated the same calculations with 2048 different initial distributions (chosen randomly according to the invariant measure of the logistic map), and so we obtain 2048 pairs of $p_t^T(x)$ and $p_t^A(x)$ for $t=2$, $t=4$ and $t=8$. So far we operated with an intuitive notion of overlap of two distributions. But in order to analyse the 2048 pairs of distributions we need a formal measure of the overlap of two distributions. We choose the so-called relative entropy, which is defined as:

$$S(p_t^A \mid p_t^T) := \int_0^1 p_t^A \log\left(\frac{p_t^A}{p_t^T}\right) dx,$$

where ln is the natural logarithm (i.e. the logarithm *to the* base *e*).[13] The integral of course becomes a sum over the bins of the partition. The relative entropy provides a measure for the overlap of two distributions. If the distributions overlap perfectly, then $p_t^A = p_t^T$ and the entropy is zero; the more different the distributions, the higher the value of $S(p_t^A \mid p_t^T)$. Figure 3 shows a histogram of the relative entropy of our 2048 distributions.

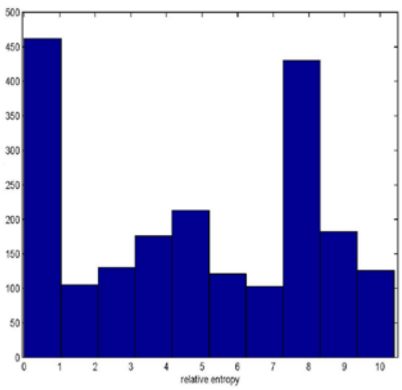


Figure 3 – Histogram of the relative entropy of 2048 pairs of distributions at $t=8$.

[13] For a discussion of relative entropy and information theory see (Curd and Thomas 1991).

The histogram shows that the Apprentice's probabilities are in line with the Demon's only in about a quarter of the cases. Almost half of the pairs of distributions have relative entropy 7 or more. The two distributions shown in Figure 2d have a relative entropy of 8.23.[14] So our histogram shows that at $t = 8$ almost half of all distribution pairs are as disconnected as the ones on Figure 2d, and hence are seriously misleading.

Observing how probabilities come unstuck and calling distributions 'off track' and 'seriously misleading' has intuitive force, but what exactly is the real damage? To answer this question we observe the Apprentice's next endeavour. Still not convinced by the Demon's arguments he opens the Pond Casino. The Pond Casino functions like a normal casino in that it offers bets at certain odds on certain events. Let $A$ be an event that can obtain in whatever game is played in the casino. The odds $o(A)$ the casino offers on $A$ is the ratio of payout to stake. If, for instance, the casino offers $o(A) = 2$ ('two for one'), this means that a punter who bets £1 on $A$ gets £2 back when $A$ obtains. Within the context of standard probability theory odds are usually taken to be the reciprocals of probabilities: $o(A) = 1/p(A)$. When flipping a coin, for instance, the probability for heads is 0.5, and if you bet £1 on heads and win, you get £2 back.[15] The Apprentice follows this convention when converting the probabilities of his model into odds. However, we emphasise that it is by no means necessary, or even advantageous, to construe the relation between probabilities and odds in this way and we will discuss alternatives in Section 7.

[14] In our calculations the lowest probability we assign to a bin is 1/(1024*32) when there is no ensemble member in the bin. If the true probability for that bin is 1, then the entropy would be 10.4. Hence 10.4 is the maximum value of the entropy.

[15] We use so-called *odds-for* throughout this paper. They give the ratio of total payout to stake. *Odds-to* give the ration of net gain to stake (net gain is the payout minus the stake paid for the bet). Odds-for and odds-to are interdefinable: if the odds-for for an event are $a/b$, then the odds-to are $(a-b)/b$. Since in this case odds-for are equal to $1/p(A)$, the odds-to are $1-p(A)/p(A)$ which is equal to $p(\neg A)/p(A)$, where $\neg A$ is 'not $A$'.

The Apprentice's casino is different from a normal casino in that the events on which punters can place bets are not outcomes of the spinning of a roulette wheel or any other traditional gambling device but future values of $\rho_t$. The Apprentice takes the above division of the unit interval into 32 bins and takes these as the basic events (corresponding to the slots on the roulette wheel in a 'normal' casino). He then offers odds on these events based on $p_t^A(x)$

More specifically, playing a 'round' in the Pond Casino at time $t$ amounts to placing a bet at $t$ on bin $B_i$, where the outcome is whether the system is in $B_i$ at $t+4$ (that is, a round is played with a four-step forecasts). So if you bet, say, on $B_{31}$ at $t=3$ the you win if $\rho_{t=7}$ is in $B_{31}$. The odds *offered by the casino* on this event are determined by a four-step forecast using $p_t^A(x)$. By contrast, the even that obtains at $t=7$ is determined according to the true distribution $p_t^T(x)$ because what happens in the pond is of course not influenced by the Apprentice's predictions.

Now a group of nine punters enters the casino. Each has £1000 and they adopt a simple strategy. In every round, the first punter bets 10% of his wealth of on events with probability in the interval $(0.5, 1]$. We call this strategy fractional betting (with $f = 1/10$) for the probability interval $(0.5, 1]$.[16] The second punter does the same with events with probability in $(0.25, 0.5]$, the third with event with probability in the interval $(0.125, 0.25]$, and so on with $(1/16, 0.125]$, $(1/32, 1/16]$, $(1/64, 1/32]$, $(1/128, 1/64]$, $(1/256, 1/128]$, $[0, 1/256]$. The minimum bet the casino accepts is £1; so if a punter's wealth falls below £1 he is effectively broke and can't play any more.

We now use the same initial distribution as above (shown in Figure 2). The Pond Casino offers odds on the events based on the Apprentice's calculations; i.e. based on $p_t^A(x)$. The outcomes of bets are of course determined by the true dynamics; i.e. $p_t^T(x)$. We now generate a string of outcomes based on $p_t^T(x)$ and trace the punters'

[16] The argument does not depend on fractional betting, which we chose for its simplicity. Our conclusions are robust in that they hold for other betting strategies (Smith et al. 2012).

wealth, which as a function of the number of rounds played. The result of this exercise is seen in Figure 4.

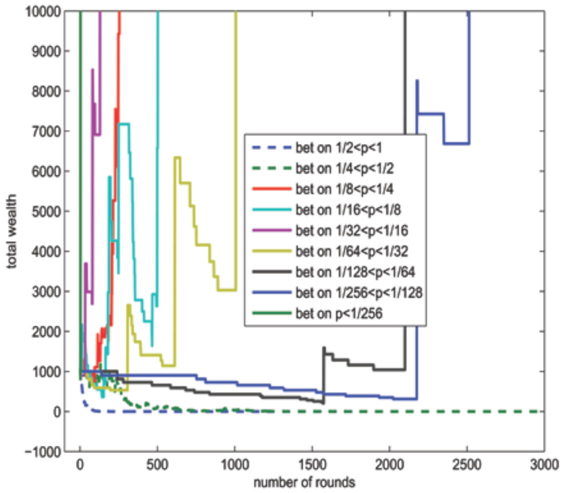


Figure 4 – Wealth of punters as a function of the number of rounds played.

We see that the punters have the time of their lives. Three of them make huge gains pretty soon, and further four follow suit a bit later. After 2500 rounds, seven out of nine punters have increased their wealth at least ten-fold, while only two of them have gone bust. So the punters take a huge amount of money off the casino!

One could now try to mitigate the suggestive force of this course of events by pointing out that it may well be an incident of bad luck for the casino: we generated a string of events by random draws according to $p_t^T(x)$, but this allows for the unlikely events to happen and so the casino lost all that money is in fact due to a low probability event happening.

To counter this objection we consider the same 2048 randomly chosen initial probability distributions as above. For each of these we let the game take place as before. If the above was a rare special event, then one would expect to see different results in the other 2047 runs. Since producing another 2048 plots like the one seen in Figure 4 would be rather cumbersome and since our focus is on the long term fate of the casino, we assume that casino starts with a capital of £1,000,000 and now calculate the time-to-bust. Figure 5 shows how the casinos are doing.

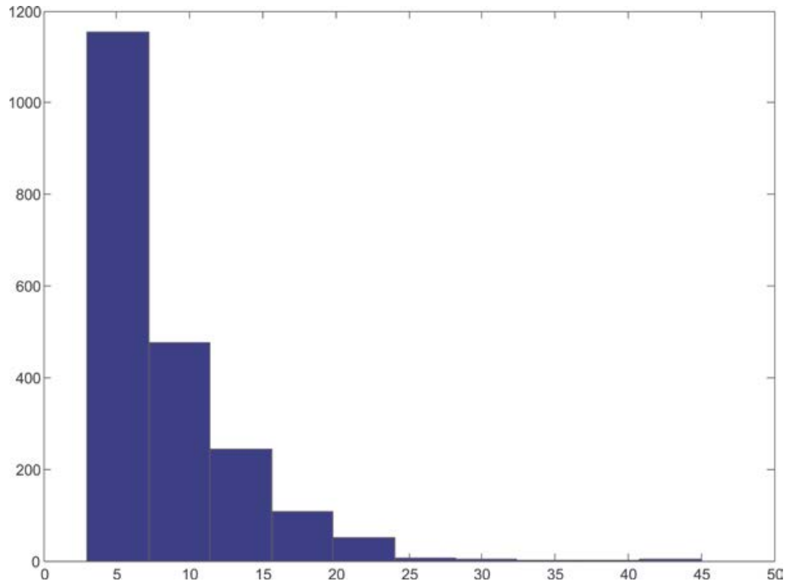


Figure 5 – Histogram of time-to-bust for 64 runs with outcomes determined according to $p_t^T(x)$.

Again the picture is a sobering one. Most casinos go bust after just a few rounds, and the last one is going out of business after 40 rounds. So we see that offering odds on $p_t^A(x)$ is disastrous!

The moral is that if you offer odds according to the probabilities of an imperfect non-linear model you are bound to go bust pretty quickly, and this even when playing against simple-minded punters.

Frustrated with his failures, he can't help and starts peeping over the demons shoulder and to get the exact initial condition. He convinces the Demon to repeat the entire casino adventure, but rather than moving probability distributions forward in time he now calculates the trajectory of the exact initial condition (which he gleans from the Demon). This, he thinks, will guarantee him a success. For want of space we do not follow his further adventures in detail, and in fact there is no need to. A look at Figure 2 suffices to realise that he has set himself up for yet another fiasco. The red cross in Figure 2a denotes the true initial condition (the so-called *verification*); the red crosses in Figures 2b-2c denote the time evolution of the true initial condition under the true dynamics while the green crosses denote the evolution of the true initial condition under the Apprentice's model. We see that the trajectories of the true initial condition under the two dynamical laws is vastly different, and any prediction generated with the model is, once again, seriously misleading.

So making forecasts with exact initial conditions rather than probability distributions does not convert failure into success. The lesson to be learned is that nothing short of the Demon's capabilities will deliver what we are after: reliable forecasts.

## 4. From Example to Proof

An obvious line of criticism would be to argue that the problems described in the last section are specific to the logistic map and do not generalise to other systems, and in particular don't generalise to interesting cases like climate models. We are, so the argument goes, guilty of overselling our case when making claims about all non-linear models on the basis of the most specific of cases, and a more tempered attitude to wards inductive generalisation would urge caution. Unfortunately the above problems cannot be dismissed so easily. In fact there is a general mathematical argument for the conclusion that the above phenomena occur in all structurally imperfect non-linear models.[17]

Let us return to the scenario of Section 2 and consider the true $\phi_t^T$ and the Apprentices idealised $\phi_t^A$, and, as before, we assume that $\phi_t^A$ is idealised and simplified and hence not identical with $\phi_t^T$. Also in keeping with the above set-up, let us assume that there is finite observational resolution: we cannot observe the precise initial condition but only assert that the system's initial condition is somewhere in an interval $I(x)$ around the condition $x$ which is the outcome of our measurement.[18] States in $I(x)$ cannot be distinguished by measurement in that that we cannot observe

[17] What follows is a simplified version of the argument in (Judd and Smith 2004). See also (Smith 2002). Furthermore, in what follows we only consider parts of the state space that the systems actually visit.

[18] This is the most simple type of observational noise, arguments below generalise to more complex forms of observational noise.

whether the system is in $y$ or $z$ for any $y, z \in I(x)$. We call such states non-distinct.[19]

Now consider $y \in I(x)$, $y \neq x$ at a given time, say $t_0$; that is, $x$ and $y$ are non-distinct at $t_0$. We can now ask the question whether $x$ and $y$ remain non-distinct in the future if $x$ evolves under the dynamics of the target system and $y$ under the dynamics of the model. Let us denote the set of states that are thus non-distinct by $I_\infty(x)$. More formally: $y \in I_\infty(x)$ iff $\phi_t^T(x)$ and $\phi_t^A(y)$ are non-distinct for all $t \geq 0$. Judd and Smith (2004) prove a theorem making a very general statement about pseudo-orbits of imperfect systems. The theorem itself need not occupy us here. What matters to us is that it implies what we call Proposition 1 (*ibid*., 231):

> Proposition 1: For a chaotic model with structural model error the following holds true: if there exists a non-empty $I_\infty(x)$, then $\phi_t^T$ and $\phi_t^A$ are identical.

This is logically equivalent to the statement that if $\phi_t^T$ and $\phi_t^A$ are not identical then there is no non-trivial $I_\infty(x)$. If the model is not perfect there is no non-trivial set of indistinguishable states. In other words: if the model is structurally imperfect, then 'no state of the model has a trajectory consistent with observations of the system' (*ibid*. 228): the states in $I(x)$ become distinguishable as time evolves.

In physical situations as the ones considered in Sections 2 and 3, the probability distribution expresses the uncertainty about the system's initial condition. The conditions we are uncertain about are the ones we cannot distinguish by a measurement. So if we determine that, for all practical purposes the initial condition is $x$, we put a probability distribution over $I(x)$. The distribution in Figure 2a, for instance, is a distribution over the indistinguishable states of the position measurement. In keeping with the notation of Section 2, let us call this distribution $p_0(x)$; $p_t^T(x)$ and $p_t^A(x)$ are then defined as before. The above theorem then has the

[19] We use 'non-distinct' to avoid confusion because 'indistinguishable' is used in a technical – and different – sense in (Judd and Smith 2004).

consequence that the relative entropy of the two distributions increases,[20] and they may even come unstuck in the way seen in Figure 2.

The theorem holds true under very general assumptions. In particular it holds for Hamiltonian as well as for dissipative chaotic systems; and it holds true both for discrete time evolutions of the kind discussed here and for continuous flows. This drives the point home that the effects and problems described in Section 3 are not specific to the logistic map and indeed occur in a vast class of systems.

## 5. Imperfect Models in Action

As we have briefly indicated above, the qualification that one must know the true dynamics to make useful forecasts is more than academic hair-splitting. The scenario we have discussed is a close cousin of many real-world research projects. In most scientific scenarios the truth is beyond our reach and we have to rest content with an approximation – it is a well-rehearsed truism that all models are wrong. Real scientists are therefore often in the position of the freshman apprentice in that they produce predictions with a less than perfect model. Some of these predictions are then used to assess the risk of future outcomes, for instance when setting insurance policies or assessing policy option. So insurers and policy makers are often like the owner of the pond casino: they have to set premiums or make policies on the basis of imperfect model outcomes. Of course scientists and insurers know that their models aren't perfect, but they nevertheless make predictions and set premiums using these models. The tacit assumption behind this practice may be the closeness-to-goodness link: they believe that if the model is close enough to the target system (which the models are assumed to be), then its predictions are good enough inform the setting of insurance premiums and policy making.

[20] This is a direct consequence of the definition of the time-evolution of a distribution which we mentioned above: $p_t(x) := \phi_t[p_0(x)] = \Sigma_i p_0(z_i)$

Examples can be drawn from domains as different as load forecasting in power systems (Fan and Hyndman 2012), inventory demand management (Snyder, Ord, and Beaumonta 2012) and weather forecasting (Hagedorn and Smith 2009).

In these cases, and no doubt many others, probability distributions are moved forward in time in the manner described above and hence there are serious worries about whether these model outputs can be used as decision relevant probabilities. However, the case we would like to highlight especially is climate change. For one, predicting the world's future climate is a modelling exercise *par excellence*. This case particularly interesting also because unlike daily forecasting activities where one can experience the success or failure of a forecasting system day after day, there is little, arguably no, relevant out-of-sample verification for future climate change predictions. For another, climate change is one of the important challenges of our time and so it is important that forecasts on which many wide-ranging policy decisions are based be reliable.

There is now a widespread consensus that the earth's climate is warming up and that human activities, in particular the burning of fossil fuels, are the main cause (Oreskes 2007; Dessler and Parson 2011, Ch. 3). But knowing that *on the whole* (or *on average*) the climate is getting warmer is of limited use if we aim to design effective adaptation strategies.[21] The impact of climate change on humans (as well as other organisms) occurs at a local scale, and so ideally we would like to know what changes we have to expect in our immediate environment. For instance, how does the precipitation change in London by the end of this century? The answer to questions of this kind have significant implications, for instance, for the planning of water reservoirs, agriculture and flood defences, and so having a reliable answer would greatly aid policy makers (Smith and Stern 2011).

[21] It may well be enough for mitigation: knowing that it happens is enough for not wanting to go there. However, it is now widely acknowledged that the question we are facing can no longer be described as an either-or question with mitigation and adaptation as alternatives. Since we are already in the midst of climate change, some level of adaptation is unavoidable even if we still ought to aim at mitigating against worse things happening.

A recent government-funded research project called *United Kingdom Climate Projections* (UKCP) aims to answer exactly such questions by making high resolution forecasts of the climate out to 2100. UKCP predicts, for instance, that there is a 0.5 probability for a 20-30% reduction in precipitation in London in 2080.[22] How are such predictions generated and how trustworthy are they?

This is the point at which high resolution general circulation models enter the scene.[23] In the case of climate models $X$ consists of relevant weather variables (such as air temperature, precipitation, …), and $\phi_t$ tell us how they change over time. When described at that level of abstraction, one could be left under the impression that climate models are rather simple things. It is important to counter this impression before it gains traction. A full specification of the system's state space would involve giving the air temperature, precipitation, etc at *every point* on the surface of the earth! It is not only a practical impossibility to obtain these data; it is also an impossibility to store and process them with digital technology. So we discretise the state space, meaning that we put a grid with a finite number of cells on $X$ and represent the state of an *entire cell* by one set of values for the relevant variables. The grid size is the length of the sides of the cells. Typically the grid size used in a climate model is well over 100km. Covering the world with such a grid still leaves us an enormous amount of data! Yet it is important to emphasise that the volume of numbers notwithstanding, this is a rather coarse description. For instance, the weather in the entire city of London is now represented by one set of numbers (one number for temperature, one for precipitation, etc.).

The dynamics of the model raises further issues. The sheer scale and complexity of the task makes it unavoidable that models are imperfect. In order to specify $\phi_t$ we have to make a number of strongly idealising assumptions: we distort important aspects of the topography of the surface of the earth as the resolution of these models does not allow for realistic mountain ranges like the Andes, does not resolve the

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[22] See http://www.ukcip.org.uk/wordpress/wp-content/UKCP09/Summ_Pmean_med_2080s.png; retrieved on 12 October 2011.

[23] For a general introduction to climate modelling see (McGuffie and Henderson-Sellers 2005); a discussion of UKCP in particular can be found in (Frigg, Smith, and Stainforth 2012).

southern half of the state of Florida, many islands simply don't exist, including small volcanic islands chains easily visible in satellite photographs due to their interaction with clouds, and of course clouds fields themselves are not modelled realistically. Based on these idealising assumptions we can use basic physics (essentially fluid dynamics and thermodynamics) to formulate the equations of motion for the simplified earth's climate system. These equations are non-linear and we cannot solve them analytically. For this reason we resort to the most powerful computers available to compute solutions. The result of these computer simulations is $\phi_t$.

It is practically impossible to specify the exact state of the earth-system at some time $t_0$ because there is no measurement device that provides exactly true values and so every measurement result comes with a certain margin of error.[24] Climate models are then used to turn these probabilities into predictions for the future. These models (e.g. HadCM3) are *de facto* non-linear. So we are in the position that we have to use an imperfect dynamical law to move current uncertainties forward in time, and for this reason all the above worries arise. UKCP probabilities are formed in a more complicated manner than by simply applying the Default Position; they are calculated by combining outputs from multiple (imperfect) models using Bayesian methods.[25] However, it is unclear why combining the outputs of several structurally imperfect models in a complicated manner should make the problems we describe go away. In the very least the burden of proof in this matter lies with those who wish to maintain that this is the case. So there is a serious question whether these model outputs can be trusted. When calculating, say, monthly precipitation in the 2080s based on climate models we may well not fare better with our planning of flood provision and water systems than the freshman apprentice with his casino!

[24] Some would go even further and say that there is no exact initial condition because there is no such thing as *the* true wind speed in a model grid point corresponding to central London! Whatever number we settle on is an average of some kind or other; all we can truthfully say is something like 'the wind speed at a particular random location within that grid cell is likely to lie within a certain range'. For what follows nothing depends on the issues of whether imprecise initial conditions are the result of practical or in-principal limitations.

[25] http://ukclimateprojections.defra.gov.uk/23239 and http://ukclimateprojections.defra.gov.uk/23210.

## 6. Attempts at Exorcism

The main consequence of the above argument is that there are serious questions concerning a widely used modelling methodology. The first reaction of those trying to help policy makers is therefore to question the soundness of our argument. In this section we discuss several attempts to do so. We deal with them in ascending order of severity and conclude that they prove unsuccessful. We concentrate on the climate case; similar arguments can be made for other modelling context and the replies remain the same *mutatis mutandis*.

### 6.1 Get Rid of Non-Linearity

A quick and simple reply is to say that if non-linearity causes so many difficulties, then should just get rid of it and construct a linear model instead. This reply is confused. We don't construct non-linear models because we for some reason like them – the choice between linear and non-linear models is not like the one between strawberry and vanilla ice cream. Non-linearity is forced upon us by nature because many processes in nature are non-linear (phase transitions of water are but one example), or at any rate are better modelled by nonlinear mathematical equations than linear ones So one can't simply choose to have a linear rather than a non-linear model.

A more nuanced version of this objection would be that climate is not linear tout court, but the non-linearities in climate systems are small and to make predictions on the time scale of interest (50-100 years), the system can treated as linear (i.e. it can be linearised) without loss. This does not seem to be plausible either. To begin with, it would be incomprehensible why scientists put immense resources (in terms of finances and research time) towards programming and running super computers to numerically integrate nonlinear equations if these equations could effectively be linearised and hence dealt with much more easily. A look at actual examples soon reveals that there is nothing irrational about scientists use of resources. Essential variables of climate models are strongly non-linear and we cannot simply linearise a nonlinear model (the change in albedo due to the transition from snow to water as tenmperature passes through zero degrees Celsius is a case in point).

### 6.2 Climate is About Averages

The next line of defence to fall is that while it is true that weather models are non-linear (and hence suffer from the problems we describe, climate science is about averages and averages obey linear laws.

This objection is mistaken twice over. First, climate is not about averages. How exactly to define climate is an interesting question, but it has been pointed out variously that it ought not be equated with averages. As early as 1938, Kendrew insisted that there was more to climate than ‘mean conditions’; in 1959 the first edition of the Glossary of the American Meteorological Society defined the climate as ‘the long term manifestations of weather, however they may be expressed’; and in 1982 Lamb bemoaned that climate was ‘wrongly defined in the past as just “average weather”.[26]

Second, even if climate science dealt with averages, there would be no guarantee that averages are governed by linear equations. There is nothing in the notion of an average that makes it subject to linear laws. Indeed, there is no guarantee that averages are governed by any law at all! Usually many states of a system are compatible with a certain average and so there is need not be a dynamical law that governs averages as such.

There is, however, a more sophisticated argument along the same lines. The challenge now is that we are playing fast and loose with the notion of prediction. While the freshman apprentice wants to predict what happens exactly two months from now, the above-mentioned climate prediction is for the 2080s, an entire 10 year period. So we seem to be comparing apples and oranges because in the climate case we are interested in a decadal average and not a prediction for a specific instant of time as in the fish case. Once this is realised, so the objection concludes, our argument loses its bite.

[26] For references an a further discussion of the relation between weather and climate see (Smith and Stainforth 2012)

Implicit in this rejoinder is the assumption that averaging over a period makes the above problems go away. This need not be so. In a simple system like the logistic map is may well be the case that over a sufficiently long period the initial distribution traces the entire state space in a way that makes averages insensitive towards the actual trajectory taken (for instance, it would not matter whether the distribution peaks in $[0, 0.5]$ at $t = 4$ and in $(0.5, 1]$ at $t = 8$ or vice versa). In models with literally tens of thousands of dimensions (such as climate models), however, the distribution need not trace out the entire state space during the relevant time period, and so model averages may well differ from their real world counterparts.

Furthermore, the situation in UKCP predictions is not as good as the critic assumes. What looks like a decadal average is in fact an annual distribution. This is not so different from weekly predictions in the fish model. Other predictions made by UKCP are even more precise, e.g. the forecasts for the hottest day in August of a particular year. So what UKCP provides are not long term averages and hence an appeal to averages does not help circumventing the difficulties we describe.

### 6.3 Quibbles about time scales

We argue that structural model error leads to getting the distribution wrong, and that once this has occurred one will have averages and extremes wrong. This argument is as unassailable as it is simple. The only way out is to respond that the time scale for this to occur is much larger than the time scale of interest.

In some cases this seems to be the right response. In weather forecasting, for instance, we are mainly interested in predicting the immediate future and hence limiting model runs to the short term is the right thing to do. But this response does not seem to work in all cases. In both weather and climate modelling, for instance, we are also interested in the medium or long term behaviour and so we cannot limit predictions to short lead times. Of course what counts as short-term or long-term is relative to the model and it could be the case that by standards of the relevant climate models a prediction for 2080 is still a short term prediction.

We are doubtful that this is the case. Indeed, it would be surprising if predictions for 2080 would turn out to be short term even by the lights of a model used to make that prediction. Our scepticism is rooted in the fact that state of the art climate models differ in terms of their performance over the past century. The (empirically measured) change in global mean temperature over the last century was approximately 0.5 centigrade, but the systematic error in model simulations is around 3 centigrade (Smith 2012). Furthermore, currently available models differ significantly in their medium and long term predictions. Comparing predictions for the relative changes in precipitation for the period 2090–2099 (relative to 1980–1999) of different models shows that for several parts of the world – Spain, the southern part of the United States and substantial portions of Latin America and Africa – less than 66% of the models considered agree even on the sign of the change! Some models say it will rain more and some say it will rain less (IPCC2007 Figure SPM.7). So we know that the details of the models have a significant impact on expected results and hence there is no reason to assume that projections of 60 to 80 years are of a kind that is unproblematic.[27]

Another challenge along the same lines argues for the opposite conclusion: what we are interested in is long term behaviour and so we do not need detailed predictions at all and can just study the invariant measure of the dynamics (in this context often referred to as the *climatology*). The invariant measure reflects a system's long term behaviour because the initial distribution "washes out", and hence it is immaterial where we started. It then doesn't matter that for medium times the distributions look different because we are simply not interested in them. This view gains support from the fact that we seem to have revealed only half of the truth in Section 3. If we continue evolving the distribution forward to higher lead times rather than stopping at $t = 8$ we find that the two distributions start looking similar again and, moreover, that they start looking very much like the invariant measure of the dynamics (Lichtenberg and Liebermann 1992, 501). This is shown in Figure 6 below. Hence, it seems that in

[27] See (Smith 2002).

the long run all we need to make reliable predictions is the invariant measure and we can forget about the 'medium term aberrations' seen in Figure 2.

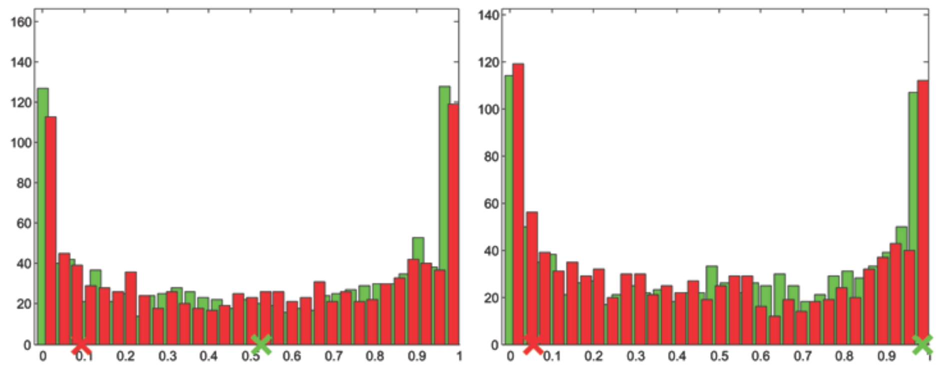


Figure 6 – The same scenario as the one seen in Figure 2 but for lead times $t = 16$ and $t = 32$.

Implicit in this proposal is the assumption that the invariant measures of similar dynamical laws are similar, because unless Equations 1 and 2 have similar invariant measures there is no reason to assume that adjusting beliefs according to the invariant measure is less misleading than adjusting them according to $p_t^m(x)$. However, it is at best unclear whether this is so. Even though Figure 6 suggests that this is so in the case of the logistic and the quartic map, there is no reason to assume (let alone a proof) that in general invariant measures have this property. Nonlinear systems are not expected to be structurally stable in general, and invariant measures of nearby systems are not expected to be similar.

Furthermore, unlike our pond, the world's climate system is not a stationary system. But transient systems do not have invariant measures, which forecloses a response along the above lines. A response to this is that we are overstating the transient character of the climate because, one of the most prominent models of the worlds climate, namely HadCM3, in fact has an attractor. There is an invariant measure on the attractor and so we can do exactly what was suggested above just with the qualification that we study the invariant measure on the attractor.

HadCM3's attractor is a red herring for several reasons. First, there is no proof that such an attractor exists. If we focus on the model *as implemented on a digital computer* we are bound to find recurring phenomena, but these need not be indicative of an attractor. A digital computer is finite machine with a finite number of states and hence sooner or later the same states are revisited – so what we find are periodic orbits of the machine. If we look at the full equations of the model, then we simply don't know whether there is an attractor. Second, even if there is an attractor, this is so only for fixed carbon dioxide levels. However, a core issue in any discussion of climate change is that carbon dioxide levels increase and hence whether there is an attractor for fixed levels is a somewhat moot point. Third, even if we focus on the scenario with fixed carbon dioxide levels, it will take the system thousands of years to reach the attractor (Smith 1987), which is too long to be of interest to humans. Lastly, HadCM3 is only one model in a class of climate models, and different models will have different attractors (if they have attractors at all), and predictions generated from studying these attractors need not coincide (recall the above point about model discrepancies for relative changes in precipitation).

### 6.4 Probabilism Reloaded

So far we have shown that making probabilistic forecasts with structurally imperfect models can be seriously misleading. But have we not just used probabilities in a bad way? An immediate response to the above problem would be to point out that by only using one particular model to make generate predictions we have implicitly assigned probability one to that model. Given that we have no reason to assume that this model is true – indeed, there are good reasons to assume that it's not! – this confidence is misplaced and one really ought to take uncertainty about the model into account. This can be done by using probabilities: put a probability measure on the space of all models which expresses our uncertainty about the true model, generate predictions with all those models, and take some kind of weighted aggregate of the result. This, so the argument goes, would avoid the above problem which is rooted in completely ignoring second order uncertainty about models.

Unfortunately this strategy does not work. Setting aside the fact that it is practically infeasible to generate predictions with an entire class of models, there are theoretical limitations that ground the project. The first problem is that it is not clear what the relevant model class would be. This class would contain all possible models of a target system (such at the earth's atmosphere). The nice phrase 'all models' masks the fact that mathematically this class is not defined, and indeed it's not clear whether it is definable at all. The second problem is that even if one could construct such a class in one way or another, there are both technical and conceptual problems with putting an uncertainty measure on this class. The technical problem is that the relevant class of models would be a class of functions and function spaces do not come equipped with measures. In fact, it is not clear how to put a measure on function spaces.[28] The conceptual issue is that even if the technical problem could be circumvented somehow, what measure would we chose? The model class will contain an infinity of models and it is at best unclear whether there is a non-arbitrary measure on such a set that reflects our uncertainty about model choice. For these reasons this response does not seem to be workable.[29]

## 7. Sustainable Odds

So far we have discussed problems with imperfect models and pointed out that there is no easy fix. One natural reaction would be to throw in the towel and conclude that the best would be not to use such models at all. This would be throwing out the baby with the bathwater. As we have seen above, *in some cases at least* the model provides insight (for instance for $t = 2$ and $t = 4$ in Figure 3). So the question is: how can we use the information in the model without being too dramatically misled in the cases in which the model goes wrong. This question has no easy answer because unlike in our thought experiment, in which we have access to the true dynamics, in every day science we don't know what the truth is and so we cannot simply compare our models with the true dynamics (indeed, if had access to the truth the model would be superfluous!). So in everyday science we are like the freshman apprentice without the Demon. So what should the apprentice do to improve his casino? In this section we propose the use of non-probability odds as one way around the problem. We introduce

[28] This is a well-known problem in the foundation of statistical mechanics; see (Frigg and Werndl).

[29] A suggestion somehow along the lines of the above is (Murphy et al. 2007); however, they conclude that all one can derive is a lower bound and not full probabilities (*ibid.*, 2011).

these in Section 7.1. In Section 7.2 we show that they indeed solve the above problem, and we end with a few cautionary remarks about decision theory (Section 7.3).

### 7.1 Non-Probability Odds

As we have seen above, the odds $o(E)$ the casino offers on an event $E$ is the ratio of total payout to stake (where 'total' indicated that the payout includes the initial stake). If there is a probability $p(E)$ for $E$, then the odds on $E$ are the reciprocals of the probabilities: $o(E) = 1/p(E)$. From this it follows, trivially, that the inverse of a complete set of odds adds up to one. In more detail, let $\alpha := \{E_1, ..., E_n\}$ be complete set of events (in the sense that any event that can possibly occur is in $\alpha$),[30] and let $o(E_i)$, $i = 1,...,n$, be the odds on all the events in $\alpha$. Then we have

$$\sum_{i=1}^{n} \frac{1}{o(E_i)} = 1. \tag{3}$$

So far we have taken probabilities as the starting point and talked about odds as if they were the derivative quantities. This need not be so: we can just as well take odds as out starting point, and say that the longer the odds for an event $A_i$, the more surprising it is if the event occurs. Odds thus understood do not necessarily have any connection to probabilities. Considering again the complete event set of events $\alpha$, and let $\omega := \{o(E_1), ..., o(E_n)\}$ be the complete set of odds for $\alpha$. The $o(E_i)$ simply reflect how surprising or unsurprising we consider certain events to be. We then say that the odds $\omega$ are *probability-odds* if, and only if, $\omega$ satisfies Equation 3; they are *non-probability odds* otherwise.[31]

Let us then call $\pi(E_i) := 1/o(E_i)$ the betting quotients on $E_i$. The $\pi$ are "probability-like" in that they are numbers between zero and one, with one indicating that the

[30] We only consider discrete and countable event spaces.

[31] Non-probability odds have been introduced in (Judd 2007) and (Smith 2007).

obtaining of an event is no surprise at all and zero representing complete surprise. If odds are probability-odds, then $\pi(E_i) = p(E_i)$.[32]

Non probabilistic odds are interesting. On the one hand they immediately induce cold sweat in everybody interested in rational decision because there are formal results for the conclusion that one is irrational (one faces guaranteed loss) if one accepts bets that do not respect probability calculus - we briefly come back to these results below in Section 7.3. On the other hand they are ubiquitous in every day situations. Real casinos, for instance, do not offer probability-odds. Assuming that they have true probabilities for simple gambling devices like dice and roulette wheels, offering probability-odds would result in them breaking even in the long run, but to run the casino sustainably also in the short and medium term, they shorten the odds. The American roulette wheel has 36 numbers plus 0 and 00. Let us assume that the wheel is perfect (and is spun so that no one can calculate outcomes by taking the initial condition and the speed into account). Then the probability for certain slot, say #23, is $1/38$ and hence $o(\#23) = [1/p(\#23)] = 38$. However, all bets are paid at odds $36$, i.e. odds that would be true fair the wheel only had 36 numbers. So we have $p(\#23) < \pi(\#23)$, and hence $\sum_{i=1}^{n} 1/o(E_i) > 1$ for the odds offered by the casino.

For a commercial casino shortening odds is simply a business decision. The main point of this section is the shortening odds can also be used as a tool to guard against the unquantified risk catastrophic loss (of the kind we have seen in Section 3), and the amount of shortening can be regarded as a measure for our uncertainty about the model – i.e. the apprentice's uncertainty about his model outputs.

### 7.2 Non-probability Odds in Action: Threshold and Damping

Let us continue our thought experiment. After the first casino fiasco the Apprentice wants to try again, but this time without going bust. From his last experience he

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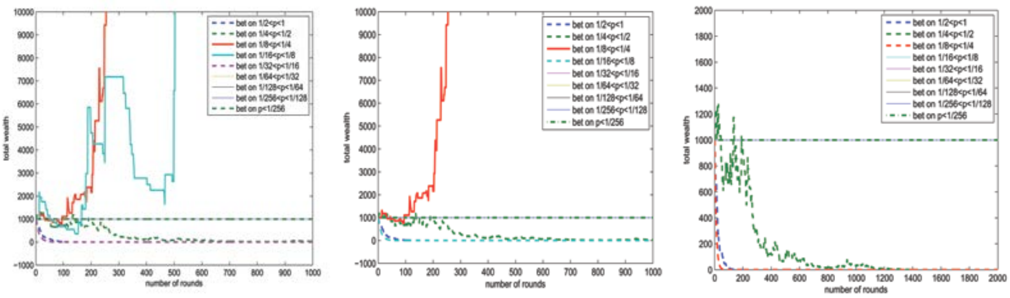
[32] Sometimes the $\pi$ are called *implied probabilities*. This nomenclature is misleading because, as we have just seen, not all $\pi$ are propbabilities.

knows that using probabilistic odds set according to $p_t^A(x)$ is recipe for disaster. So he decides to shorten his odds to guard against loss. Of course you can always guard against loss by not paying out any net gain at all and merely returning the stake to punters when they win (i.e. by setting all $o(E_i) = 1$). This, however, is not interesting to punters and they would not play in his new casino. So he aims to offer a game that is as attractive as possible by offering odds that are as long as possible, but only so long that he does not unexpectedly go bust.

There are different ways of shortening odds. Perhaps the simplest way is to impose a threshold $\tau$ on the $\pi_t(E_i)$: $\pi_t(E_i) = p_t^A(E_i)$ if $p_t^A(E_i) > \tau$ and $\pi_t(E_i) = \tau$ if $p_t^A(E_i) \leq \tau$, where $\tau$ can be any real number so that $0 \leq \tau \leq 1$. We call odds thus calculated *threshold-odds*. For the limiting case of $\tau = 0$ the $\pi_t(E_i)$ become probabilities and the respective odds probabilistic odds. It is important to emphasise the threshold rule applies to *all* possible events and not only the atoms of the partition, the idea being that one simply does not offer $\pi$'s smaller than $\tau$ no matter what the event under consideration is. In particular, the rule applies to events and their negation. If, for instance, we set $\tau = 0.2$ and have $p_t^A(E_i) = 0.95$ (and hence, by the axioms of probability $p_t^A(\neg E_i) = 0.05$), then $\pi_t(E_i) = 0.95$ and $\pi_t(\neg E_i) = 0.2$, where $\neg E_i$ is the negation of $E_i$ (i.e. the non-occurrence of $E_i$).

This move is motivated by the following observation. In Figure 2 we see that if the two distributions we are offering, based on $p_t^A$ very long odds on events that are in reality (i.e. according to $p_t^T$) very likely to happen; that is, we wrongly regard likely events as unlikely and therefore offer long odds on them. Putting a threshold on the $\pi_t(A_i)$ amounts to limiting the length of odds and so one limits the amount one amount one pays out for likely events that one wrongly regards as unlikely.

We now repeat the scenario of Figure 4 with one exception: the Freshman apprentice now offers non-probability odds with a thresholds of $\tau = 0.05$, $\tau = 0.1$ and $\tau = 0.2$. The result of these calculations is shown in Figure 7a, 7b and 7c respectively.



(a) (b) (c)

Figure 7 – Wealth of punters as a function of the number of rounds played with the casino offering non threshold-odds with thresholds of 0.05, 0.1 and 0.2.

In Figure 7 we see that this strategy is a success. Already a very low threshold of $\tau = 0.05$ undercuts the success of five out of seven punters, and only two still manage to take money off the casino. A slightly higher threshold of $\tau = 0.1$ brings the number of successful punters down to one. For . So for $\tau = 0.2$ the Freshman apprentice achieves his goal of running a sustainable casino.

The second way of shortening odds is damping. On this method the betting quotients are given by $\pi_t(E_i) = 1 - \beta[1 - p_t^A(E_i)]$, where the damping parameter $\beta$ is a real number $0 \le \beta \le 1$. We see that for $\beta = 1$ the $\pi_t$ are probabilities. We call odds thus calculated *damping-odds*. We now repeat the same calculations as above and the results are very similar indeed. For $\beta = 0.95$ only two punters succeed (indeed the same two as above). With a slightly stronger damping of $\beta = 0.9$ only one is still winning (again the same as above), and for $\beta = 0.8$ all punters are either losing or not playing at all (because no bets in their range are on offer).

The moral of this last part of our tale of the Demon and his apprentice is that by shortening odds either by introducing a threshold or by damping one can guard against unexpected losses.

Furthermore we can regard the amount of deviation of the 'shortening parameters' from their 'probability limits' (i.e. the deviation of $\tau$ from zero and of $\beta$ from one) as a measure of how the model uncertainty: the further away the parameters are from their probability limits the more uncertain we are about the model.

Before moving on we briefly want to continue the discussion in Section 4 and point out that also this last part of our take is closer to reality than it seems. The sustainable yet interesting casino is modelled on a cooperative insurance company. Rather than playing for gain, the ‘bets’ placed are insurance policies they buy to compensate for losses suffered should certain events happen. What makes our insurance a cooperative insurance is its attempt to offer has high a payout as possible (to compensate its clients as best as possible), but only to the point that it operates in a sustainable way (an insurance company that goes bust unexpectedly is of little use). So our non-probability odds casino has a close real-world cousin, and the morals drawn above are relevant beyond the tale about Laplace’s Demon.

### 7.3 Epistemology, Not Decision Theory

So far we have seen that one is bound to lose money when betting on imperfect model-probabilities. The conclusion of our argument might be seen as a decision theoretic one: that is, that it is pragmatically advantageous to adopt non probabilistic odds. This is not the interpretation we favour. We prefer to see it as an epistemological argument, albeit one that involves talk of betting. We are not making any decision theoretic assumptions in coming to our conclusions. Talk of casinos, betting and going bust, helps putting an epistemic problem into focus – the main point is that the pragmatic flaw (going bust) points to an epistemological flaw in the agent’s representation of belief.

Needless to say, the use of non-probability odds raises a host of issues. How exactly should non-probability odds inform decision making? Presented with non-probability odds (or equivalently: betting quotients), what decision rules should we apply? These important questions for decision theory and rational choice, but we cannot discuss these here.

An attempt to dismiss these issues quickly might be to try to bring these issues back into well charted territory by denying that non-probability odds are really *sui generis* items. Regarding them as such, so the argument goes, is a red herring because even if we have odds whose inverses don't add up to one it is trivial to renormalise them and we then retrieve the homely probabilities for which there are well worked out decision theories. Renormalisation amounts to multiplying all odds with $\upsilon := \sum 1 \backslash o(E_i)$, or, equivalently, all $\pi$ with $1/\upsilon$.

Unfortunately things are not as simple as that because the $\pi$ do not satisfy the axioms of probability even if they add up to one. The problem is that non-probability-odds do not respect the symmetry between betting for and betting against that is enshrined into probabilities. For probabilities we have $p(E) + p(\neg E) = 1$ for any event $E$ (where $\neg E$ is the non-obtaining of $E$). Non-probability odds need not add up to one: $\pi(E) + \pi(\neg E)$ can take in principle take value $x \geq 1$. This can be easily seen when calculating non-probability odds as in the first of the above examples. Consider three events with respective probabilities of 0.95, 0.01 and 0.04. It is easy to see that $\upsilon = 1.35$. We represent the values (all rounded to two digits) of these probabilities as well as the non-probability odds and the renormalized nonprobability-odds $\pi_R$ in the following chart:

| | $E_1$ | $E_2$ | $E_3$ | $\neg E_1$ | $\neg E_2$ | $\neg E_3$ |
|---|---|---|---|---|---|---|
| $p$ | 0.95 | 0.01 | 0.04 | 0.05 | 0.99 | 0.96 |
| $\pi$ | 0.95 | 0.2 | 0.2 | 0.2 | 0.99 | 0.96 |
| $\pi_R$ | 0.7 | 0.15 | 0.15 | 0.15 | 0.73 | 0.71 |

We see that the $\pi_R$ for $E_1$, $E_2$, and $E_3$ add up to unity as one would expect from probabilities, and yet $p(E_i) + p(\neg E_i) \neq 1$ for all $i$. For this reason the $\pi_R$ are not probabilities, and renormalising is not an easy route back into the territory of probabilism.

We would not like to leave the issue without a brief remark about Dutch books. One might worry that our freshman apprentice, with his non probabilistic odds, is subject to a Dutch book. That is, one might worry that a smarter bettor might be able to guarantee to make money out of the apprentice by buying a set of bets that guarantee the bettor a sure gain, whatever happens. This is not the case. This is for the same reason that casinos can't be Dutch booked. That is, you cannot bet against an event happening. Likewise, our apprentice is only offering bets on, not bets against events (Bradley 2012). This is not the place to discuss this point further.

## 8. Conclusion

We have argued that the combination of non-linear dynamics and model imperfection is a poison pill in that it shows that treating model outputs as probabilistic predictions can be seriously misleading. At some level, probabilistic forecasts are therefore unreliable and do not provide a good guide for action.

This raises two questions. The first concerns the premises of the argument. The model being non-linear has been an essential ingredient of our story. While this assumption is realistic in that many relevant models have this property, there is still a question whether the effects we describe are limited to non-linear models. Arguably, if the world was governed by linear equations, then imperfect linear models might perhaps not suffer from the effects we discuss. One might like to avoid the assumption that the world is governed by any equations, of course, but the relevant point here is the role of model imperfections: a linear model will suffer from these effects if its imperfections are were nonlinear. The model being linear does not remove the difficulty we note. And of course, in practice the best models are often not linear, nor are the relevant laws of physics.

The second question is what conclusion we are to draw from the insight into the unreliability of models. An extreme reaction would be to simply get rid of them. But this would often amount to throwing out the baby with the bathwater because, as we have seen, in the short run model results are roughly correct, and even in the medium term about one third of the cases the model indicates usefully. This raises several questions. The first is: how long is long? As we have indicated in Section 6, there probably will be no general answer, but there may be answers in particular cases. So future research may want to pay more attention than it has done so far to the issues of estimating the time span of reliable prediction. The second question is how we can use the model when it provides insight while guarding against damage when it does not. Finding a way of doing this is a challenge for future research. We have indicated that one way possible route could be to use non-probability odds, but more needs to be said about how these can be used to provide decision support, and there may be altogether different ways of avoiding the difficulties we sketch.

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