Sample Exam Paper

MA103
Introduction to Abstract Mathematics

Suitable for all candidates

Instructions to candidates

This paper contains **8** questions, divided into two sections.
You should answer **3** questions from Section A and **3** questions from Section B.
If additional questions are answered, only your **best 3** answers from Section A and your **best 3** answers from Section B will count towards the final mark.
All questions carry equal numbers of marks.
Answers should be justified by showing work.
Please write your answers in dark ink (black or blue) only.

Time Allowed	**Reading Time:**	*None*
	Writing Time:	*3 hours*
You are supplied with:		*Answer booklets*
You may also use:		*No additional materials*
Calculators:		*Calculators are not allowed in this examination*

Section A

Question 1

(a) (i) Construct the truth table of the statement $(p \Rightarrow r) \vee (q \Rightarrow r)$.

(ii) Show that the statement $(p \Rightarrow r) \vee (q \Rightarrow r)$ is not logically equivalent to the statement $(p \vee q) \Rightarrow r$.

(b) The sequence $S_1, S_2, \ldots$ is defined as follows:

$$S_1 = 1, \quad S_2 = 2, \quad \text{and} \quad S_n = 2S_{n-1} + S_{n-2} - 2, \text{ for } n \geq 3.$$

For $n \in \mathbb{N}$, let $P(n)$ be the statement "S_n is even if and only if n is even".
Use induction to show that $P(n)$ is true for all $n \in \mathbb{N}$.

(c) (i) Using the exponential form $z = Re^{i\theta}$, find all solutions in $\mathbb{C}$ for the equation $z^2 = 2\overline{z}$.

(ii) Write the solutions you found in part (i) in the standard form $z = a + ib$, with $a, b \in \mathbb{R}$.

Question 2

(a) (i) Define what it means to say *d is a divisor of m*, for $m \in \mathbb{Z}$ and $d \in \mathbb{N}$.
Define the *greatest common divisor* $\gcd(m, n)$ of two integers m, n, not both zero.

(ii) Explain why the condition "m, n not both zero" is required for the definition of $\gcd(m, n)$.

(iii) Find $\gcd(-51, 141)$.

(b) Let x be the real number $x = 0.0\overline{119}$.

(i) Express x as a rational number p/q, with p and q integers.

(ii) Find a rational number r such that $0.0119 < r < 0.0\overline{119}$.

(iii) Find an irrational number z such that $0.0119 < z < 0.0\overline{119}$.
(*You may use any results from the course.*)

(c) Prove that for all sets A, B, C we have

$$(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C).$$

Question 3

(a) Let P be the following statement for a rational number p/q with $0 < p/q < 1$ and natural number $k \geq 2$:

"if p/q can be expressed as an Egyptian fraction with k terms, then p/q can be expressed as an Egyptian fraction with $k+1$ terms".

(i) Write down the *contrapositive* and the *converse* of the statement P.

(ii) Prove that for all natural numbers $a \geq 2$, there exist natural numbers b, c, $b \neq c$, such that $1/a = 1/b + 1/c$.
(*Hint: think "greedy" algorithm.*)

(iii) Prove that P is true for all $k \geq 2$ and all rational numbers p/q with $0 < p/q < 1$.
(*Hint: do not use induction.*)

(iv) Deduce whether or not the *contrapositive* of P is true for all $k \geq 2$ and all rational numbers p/q with $0 < p/q < 1$.

(b) Consider the following system of equations in $\mathbb{Z}_7$:

$$\begin{cases} 5x + 3y = 2, \\ cx + 2y = 1. \end{cases}$$

Here $c \in \mathbb{Z}_7$ is a constant.

(i) Show that this system has no solution if $c = 1$.

(ii) Find all solutions for all values of $c \neq 1$ (your answers should use expressions involving c).

Question 4

(a) (i) Define what it means for a function $f : X \to Y$ to be *surjective*, to be *injective*, and to be *bijective*.

(ii) Write down one of the forms of the *Pigeonhole Principle*.

(iii) Let X be a *finite* set and $f : X \to X$ a function.
Prove that if f is injective, then f is surjective.

(iv) Show that the statement you proved in part (iii) is not always true if X is allowed to be an *infinite* set.

(b) Let R be the following relation on the set $\mathbb{N}$ of natural numbers:

$$xRy \iff \gcd(x+1, y+1) \geq 2.$$

(i) Is R reflexive on $\mathbb{N}$?

(ii) Is R symmetric on $\mathbb{N}$?

(iii) Is R transitive on $\mathbb{N}$?

(iv) Is R an equivalence relation on $\mathbb{N}$?

Section B

Question 5

(a) (i) For A a set of real numbers, what does it mean to say that s is an *upper bound* for A? What does it mean to say that s is the *supremum* of A?

(ii) Let A and B be non-empty subsets of $\mathbb{R}$, bounded above.

Show that $\sup(A \cup B) \geq \sup A$.

(iii) For sets A and B of real numbers, we say that A *dominates* B if, for every $b \in B$, there is some $a \in A$ with $a \geq b$.

Suppose that A dominates B. Show that $s = \sup(A)$ is an upper bound for $A \cup B$, and deduce that $\sup(A \cup B) = s$.

(iv) Is it true that, if $\sup(A \cup B) = \sup A$, then A dominates B? Justify your answer by means of a proof or a counterexample.

(v) Is it true that, if $\sup A = \sup B = s$, and $s \notin B$, then A dominates B? Justify your answer by means of a proof or a counterexample.

(b) Let $f : [0, \infty) \to \mathbb{R}$ be a decreasing continuous function with $f(0) = 1$.

Show that there is exactly one $x > 0$ such that $f(x) = \sqrt{x}$.

(*You may use any results from the course, provided they are clearly stated.*)

Question 6

(a) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

(i) What does it mean to say that the sequence is *convergent*, with limit 1?

Suppose that $(a_n)_{n\in\mathbb{N}}$ is convergent, with limit 1.

(ii) Show, directly from the definition, that $(a_n^2)_{n\in\mathbb{N}}$ is convergent, with limit 1.

Let $b_n = \max(a_n, a_n^2)$, for $n \in \mathbb{N}$.

(iii) Show that $(b_n)_{n\in\mathbb{N}}$ is convergent, with limit 1.

(b) For each of the following sequences, find the limit of the sequence or show that the sequence does not converge. Justify your answers.

(i) $(a_n)_{n\in\mathbb{N}}$, where $a_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n-1})$;

(ii) $(b_n)_{n\in\mathbb{N}}$, where $b_n = (-1)^n 2^{1/n}$.

(*You may use results from the course.*)

Question 7

(a) Let $(G, *)$ and $(G', *')$ be groups, with identity elements e and e' respectively.

(i) What does it mean to say that a function $\phi : G \to G'$ is a *homomorphism*?

(ii) What is the *kernel* $\ker(\phi)$ of ϕ?

(iii) Show that $\ker(\phi)$ is a subgroup of $(G, *)$.

(*You may use without proof that, for a homomorphism ϕ, $\phi(e) = e'$ and $\phi(a^{-1}) = (\phi(a))^{-1}$ for every $a \in G$.*)

(b) Let $\phi : G \to G'$ be a homomorphism. Suppose that $h \in \mathrm{im}(\phi)$, and let g be an element of G with $\phi(g) = h$.

(i) Show that $S_h = \{g \in G \mid \phi(g) = h\}$ is equal to the left coset $g * \ker(\phi)$.

(ii) Deduce that, if G is a finite group and $\phi : G \to G'$ is a homomorphism, then each set S_h, for $h \in \mathrm{im}(\phi)$, has size $|\ker(\phi)|$, and so

$$|G| = |\ker(\phi)| \cdot |\mathrm{im}(\phi)|.$$

(c) Let $(G, *)$ be a group. Let $\theta : G \to G$ be defined by $\theta(g) = g * g$ for all $g \in G$. Show that θ is a homomorphism if and only if G is Abelian.

(d) If $(G, *)$ is a finite Abelian group, use parts (b) and (c) to show that

$$|G| = |\{a \in G \mid a = g * g \text{ for some } g \in G\}| \cdot |\{g \in G \mid g * g = e\}|.$$

Question 8

(a) (i) What is a *basis* of a vector space V?
For d a natural number, what does it mean to say that V has dimension d?

Let V be a vector space of dimension 3, and let U and W be subspaces of V, with $\dim(U) = \dim(W) = 2$.

(ii) Show that there is some non-zero vector in $U \cap W$.
(*Hint: Take a basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of U, and a basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ of W, and use the fact that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2$ is not linearly independent.*)

(b) Let X be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, with pointwise addition and scalar multiplication.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz* if there is some constant K_f such that $|f(x) - f(y)| \leq K_f|x - y|$ for all $x, y \in \mathbb{R}$.

Show that the set L of Lipschitz functions forms a subspace of X.

END OF PAPER