MA100

Mathematical Methods

Solutions to summer 2016 Examination - Resit candidates

1. (a) A vector parametric equation for the line ℓ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

[2 marks]

(b) Eliminating the free parameter from the parametric equation, we get a Cartesian description for ℓ:

$$x_1 - 1 = x_3 - 3, \quad x_2 = 2, \quad x_4 = 4.$$

[2 marks]

(c) Performing the row reductions, we obtain

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 1 & -2 & 2 & 2 \\ 5 & -3 & a & b \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 7 & 0 & -3 \\ 0 & 7 & a-10 & b-10 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 7 & 0 & -3 \\ 0 & 0 & a-10 & b-7 \end{array}\right).$$

[2 marks]

(i) Therefore, if $a \neq 10$, the system admits exactly one solution for all values of b. **[2 marks]**

(ii) If $a = 10$ and $b \neq 7$, the system admits no solutions. **[2 marks]**

(iii) If $a = 10$ and $b = 7$, the system admits infinitely many solutions. **[2 marks]**

(d) A basis for the column space of $\mathbf{A}$ consists of the columns of $\mathbf{A}$ that correspond to the leading columns of $RRE(\mathbf{A})$:

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right\}.$$

A basis for the null space of $\mathbf{A}$ is obtained by inspecting $RRE(\mathbf{A})$. We get

$$B_2 = \left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

[3 marks]

(e) The linear system is consistent if $\mathbf{b}$ belongs to the column space of $\mathbf{A}$; i.e., if $\mathbf{b}$ can be written as a linear combination of the vectors in B_1. We have

$$\begin{pmatrix} 9 \\ 0 \\ k \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \qquad \text{for some } \lambda, \mu.$$

Solving this system, we find that

$$\lambda = 1, \quad \mu = 2, \quad k = 7.$$

[5 marks]

(f) Every vector in the basis B_2 of the null space of $\mathbf{A}$ gives a linear combination of the columns of $\mathbf{A}$ which is equal to the zero vector. Therefore

$$\mathbf{c}_1 - 3\mathbf{c}_2 + \mathbf{c}_3 = \mathbf{0} \quad \text{and} \quad -5\mathbf{c}_1 - 2\mathbf{c}_2 + \mathbf{c}_4 = \mathbf{0},$$

from which we find that

$$\mathbf{c}_3 = \begin{pmatrix} 11 \\ -5 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{c}_4 = \begin{pmatrix} 13 \\ 8 \\ 19 \end{pmatrix}.$$

[5 marks]

2. (a) The Taylor polynomial P_n is given by

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

We have

$$\begin{array}{lcl@{\qquad}lcl} f(x) & = & (1-x)^{-1} & f(0) & = & 1 \\ f'(x) & = & (1-x)^{-2} & f'(0) & = & 1 \\ f''(x) & = & 2(1-x)^{-3} & f''(0) & = & 2 \\ f'''(x) & = & 3!(1-x)^{-4} & f'''(0) & = & 3! \\ f^{(4)}(x) & = & 4!(1-x)^{-5} & f^{(4)}(0) & = & 4! \end{array}$$

and so on, so

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n.$$

[7 marks]

(b) We have

$$\begin{aligned}(1-x)P_n(x) &= (1-x)(1+x+x^2+x^3+\cdots+x^n)\\ &= 1+x+x^2+x^3+\cdots+x^n\\ &\quad -x-x^2-\cdots-x^n-x^{n+1}\\ &= 1-x^{n+1}.\end{aligned}$$

[2 marks]

(c) It follows from the above that

$$P_n(x) = \frac{1-x^{n+1}}{1-x}.$$

Comparing this expression with

$$f(x) = \frac{1}{1-x}$$

and using the fact that

$$P_\infty(x) = \lim_{n\to\infty} P_n(x),$$

we deduce that $P_\infty(x)$ converges to $f(x)$ only if $\lim_{n\to\infty} x^{n+1} = 0$. This happens only if $|x| < 1$.

[4 marks]

(d) We let $y = g(x) = \arcsin(x)$, which implies that $x = \sin(y)$. Therefore, the derivative of g is given by

$$g'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos(y)} = \pm\frac{1}{\sqrt{1-\sin^2(y)}}.$$

We now use the fact that whenever $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have that $\cos(y) > 0$, from which we deduce that $\cos(y) = +\sqrt{1-\sin^2(y)}$. Finally, replacing $\sin^2(y)$ by x^2, we obtain

$$g'(x) = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}},$$

as required.

[7 marks]

(e) The quadratic expression inside the root can be written as

$$-x^2 - 6x - 5 = -(x^2 + 6x + 5) = -\left[(x+3)^2 - 4\right] = 4 - (x+3)^2,$$

so the integral becomes

$$\int \frac{dx}{\sqrt{4-(x+3)^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{x+3}{2})^2}} = \int \frac{d(\frac{x+3}{2})}{\sqrt{1-(\frac{x+3}{2})^2}} = \arcsin\left(\frac{x+3}{2}\right) + C,$$

where the last step follows from part (d).
[5 marks]

3. (a) We row reduce $(\mathbf{A}|\mathbf{b})$:

$$\left(\begin{array}{cccc|c} 1 & 5 & 1 & 0 & 3 \\ 2 & 10 & 0 & 2 & 8 \\ 4 & 20 & 1 & 3 & 15 \\ 1 & 5 & 0 & 1 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 5 & 1 & 0 & 3 \\ 0 & 0 & -2 & 2 & 2 \\ 0 & 0 & -3 & 3 & 3 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 5 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -3 & 3 & 3 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 5 & 0 & 1 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

[6 marks]

(b) A basis B for $CS(\mathbf{A})$ is $B = \{\mathbf{c}_1, \mathbf{c}_3\} = \left\{\begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ since $RRE(\mathbf{A})$ has leading ones in the 1st and 3rd column. Further, we can read from the part of the general solution corresponding to the null space of $\mathbf{A}$ that

$$-5\mathbf{c}_1 + \mathbf{c}_2 = \mathbf{0} \quad \text{and} \quad -\mathbf{c}_1 + \mathbf{c}_3 + \mathbf{c}_4 = \mathbf{0}.$$

Therefore,

$$(\mathbf{c}_1)_B = \begin{pmatrix}1\\0\end{pmatrix}_B, \quad (\mathbf{c}_2)_B = \begin{pmatrix}5\\0\end{pmatrix}_B, \quad (\mathbf{c}_3)_B = \begin{pmatrix}0\\1\end{pmatrix}_B, \quad (\mathbf{c}_4)_B = \begin{pmatrix}1\\-1\end{pmatrix}_B.$$

[6 marks]

(c) Every solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ gives $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$. Choosing $s = 0, t = 0$ in the general solution obtained in part (a), we find that

$$\mathbf{b} = 4\mathbf{c}_1 - \mathbf{c}_3.$$

Choosing $s = 1, t = 0$ we find that

$$\mathbf{b} = -\mathbf{c}_1 + \mathbf{c}_2 - \mathbf{c}_3,$$

and choosing $s = 0, t = 1$ we find that

$$\mathbf{b} = 3\mathbf{c}_1 + \mathbf{c}_4.$$

[4 marks]

(d) We row reduce $\mathbf{A}^T$

$$\begin{pmatrix}1 & 2 & 4 & 1\\5 & 10 & 20 & 5\\1 & 0 & 1 & 0\\0 & 2 & 3 & 1\end{pmatrix} \to \begin{pmatrix}1 & 2 & 4 & 1\\0 & 0 & 0 & 0\\0 & -2 & -3 & -1\\0 & 2 & 3 & 1\end{pmatrix} \to \begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 3/2 & 1/2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

so a basis for the null space of $\mathbf{A}^T$ is

$$\left\{ \begin{pmatrix}-1\\-\frac{3}{2}\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-\frac{1}{2}\\0\\1\end{pmatrix} \right\}$$

or simply

$$C = \left\{ \begin{pmatrix}-2\\-3\\2\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\2\end{pmatrix} \right\}.$$

[4 marks]

(e) Since $N(\mathbf{A}^T) \perp RS(\mathbf{A}^T) = CS(\mathbf{A})$, a Cartesian description in $\mathbb{R}^4$ for the column space of $\mathbf{A}$ is given by

$$\begin{pmatrix} -2 & -3 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is,

$$\begin{cases} -2x_1 - 3x_2 + 2x_3 &= 0 \\ -x_2 + 2x_4 &= 0 \end{cases}$$

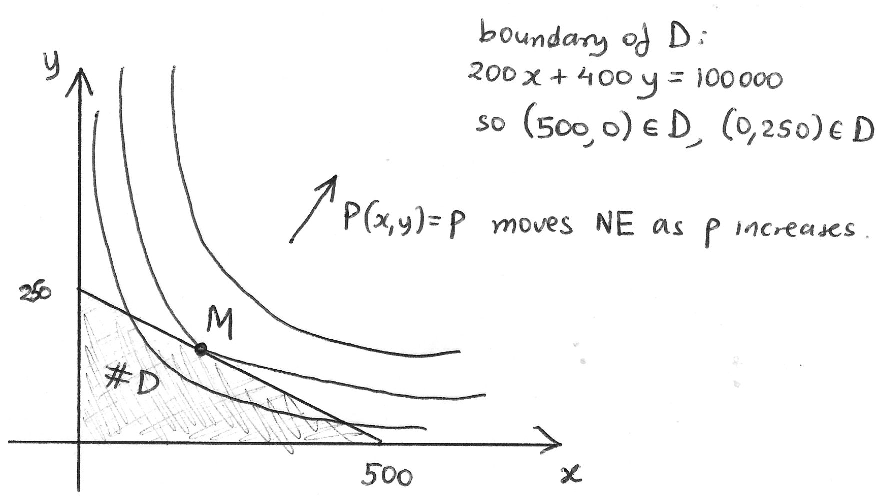
[3 marks]

(f) For the system $\mathbf{Ax} = \mathbf{d}$ to be consistent, we must have $\mathbf{d} \in CS(\mathbf{A})$; i.e., $\mathbf{d}$ must satisfy the Cartesian description for $CS(\mathbf{A})$ found in part (e). So k, l, m, n must satisfy

$$\begin{cases} -2k - 3l + 2m &= 0 \\ -l + 2n &= 0. \end{cases}$$

[2 marks]

4. (a) The relevant sketch is shown below:



[3 marks]

(b) We have

$$L(x, y, \lambda) = 100x^{1/5}y^{4/5} + \lambda(100000 - 200x - 400y)$$

and

$$\begin{cases} L_x = 20x^{-4/5}y^{4/5} - 200\lambda = 0 \\ L_y = 80x^{1/5}y^{-1/5} - 400\lambda = 0 \\ L_\lambda = 100000 - 200x - 400y = 0 \end{cases}$$

Eliminating λ from the first two equations, we find that

$$y = 2x.$$

Substituting this equation into the constraint, we find that

$$x^* = 100 \qquad \text{and} \qquad y^* = 200,$$

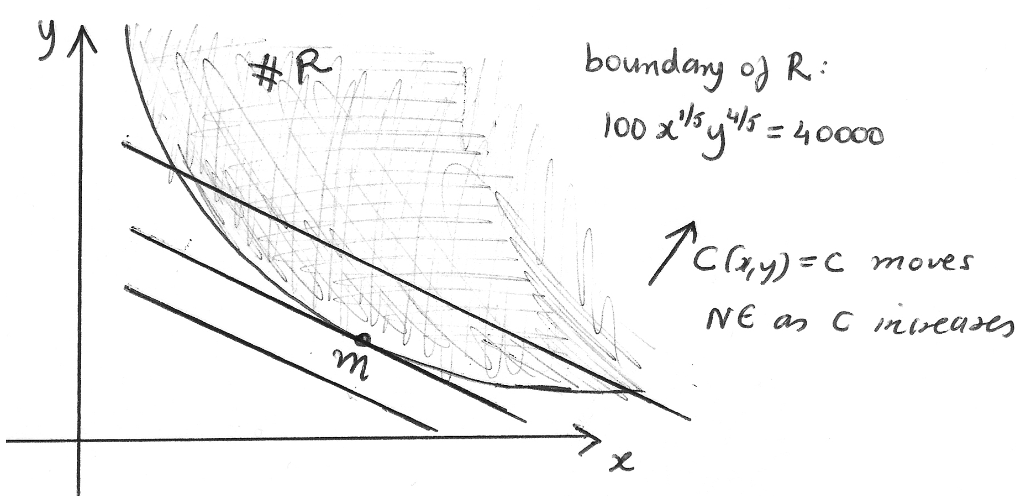
which are the coordinates of the point M.

[6 marks]

(c) Yes, any point of the form $(x, 0)$ where $0 \le x \le 500$ and any point of the form $(0, y)$ where $0 \le y \le 250$ is a constrained minimum of $P(x, y)$ on D.

[2 marks]

(d) The relevant sketch is shown below:



[3 marks]

(e) Setting the derivatives of f to zero, we have

$$f_x = 2(x-1) = 0, \quad f_y = 3(y-1)^2 = 0, \quad f_z = 4(z-1)^3 = 0,$$

so f has a single stationary point at $(1, 1, 1)$.

[2 marks]

(f) The matrix

$$f''(x,y,z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6(y-1) & 0 \\ 0 & 0 & 12(z-1)^2 \end{pmatrix}$$

evaluated at the stationary point becomes

$$f''(1,1,1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The principal minors test fails, but the eigenvalue test is conclusive. Since the eigenvalues of $f''(1,1,1)$ are $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 0$, the symmetric matrix $f''(1,1,1)$ is positive semi-definite.

[**4 marks**]

(g) The classification test to determine the nature of the stationary point based on $f''(1,1,1)$ is still inconclusive since the latter is semi-definite. However, inspecting $f(x,y,z)$ we see that

$$f(1, 1+\epsilon, 1) = \epsilon^3,$$

which implies that $f(1,1+\epsilon,1) > f(1,1,1) = 0$ if $\epsilon > 0$ and $f(1,1+\epsilon,1) < f(1,1,1) = 0$ if $\epsilon < 0$. Therefore the point $(1,1,1)$ is a saddle point.

[**5 marks**]

5. (a) Following the Gram-Schmidt process, we obtain

$$\mathbf{u}_1 = \frac{\mathbf{f}_1}{||\mathbf{f}_1||} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{f}_2 - \langle \mathbf{f}_2, \mathbf{u}_1 \rangle \, \mathbf{u}_1 \\ &= \begin{pmatrix}1\\2\\0\end{pmatrix} - \left\langle \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix} \right\rangle \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix} \\ &= \begin{pmatrix}0\\1\\-1\end{pmatrix}, \end{aligned}$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{||\mathbf{w}_2||} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

So an orthonormal basis for $\text{Lin}\{\mathbf{f}_1, \mathbf{f}_2\}$ is

$$C = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\}.$$

[6 marks]

(b) Since $\mathbf{f}_3$ is orthogonal to both $\mathbf{f}_1$ and $\mathbf{f}_2$, we just rescale it to unit length:

$$\mathbf{u}_3 = \frac{\mathbf{f}_3}{||\mathbf{f}_3||} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

So an orthonormal basis K for $\mathbb{R}^3$ is

$$K = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} \right\}.$$

[2 marks]

(c) We have $A_S^{B\to B} = (\ (S\mathbf{f}_1)_B(S\mathbf{f}_2)_B(S\mathbf{f}_3)_B\ ) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We also have $A_S^{K\to K} = (\ (S\mathbf{u}_1)_K(S\mathbf{u}_2)_K(S\mathbf{u}_3)_K\ ) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ because $\mathbf{u}_1$ and $\mathbf{u}_2$ belong to $\text{Lin}\{\mathbf{f_1}, \mathbf{f}_2\}$ and $\mathbf{u}_3$ belongs to $\text{Lin}\{\mathbf{f}_3\}$. So $\mathbf{u}_1$ and $\mathbf{u}_2$ are stretched by S by a factor of 2 and $\mathbf{u}_3$ is stretched by S by a factor of 1.

[5 marks]

(d) The matrix $\mathbf{A}_S$ that represents S with respect to the standard basis must be symmetric because the eigenspaces corresponding to distinct eigenvalues are orthogonal; i.e., $\mathbf{A}$ is orthogonally diagonalisable and hence symmetric.

[4 marks]

(e) Letting $\mathbf{P}$ be the transition matrix $\mathbf{P}_E$ from E-coordinates to standard coordinates,

$$\mathbf{P} = (\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix},$$

and letting $\mathbf{D} = A_S^{K\to K}$,

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we know that $\mathbf{A}_S = \mathbf{P}_K \mathbf{A}_S^{K\to K} \mathbf{P}_K^T = \mathbf{P}\mathbf{D}\mathbf{P}^T$.

[3 marks]

(f) Expressing the relations $S(\mathbf{f}_1) = 2\mathbf{f}_1$ and $S(\mathbf{f}_3) = \mathbf{f}_3$ in standard coordinates, we obtain the matrix equations

$$(\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad (\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3)\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

The first equation implies that $\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and the second equation implies that $2\mathbf{c}_1 - \mathbf{c}_2 - \mathbf{c}_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$. Adding these equations together, we obtain

$$3\mathbf{c}_1 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}; \text{ i.e., } \quad \mathbf{c}_1 = \frac{1}{3}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

[5 marks]

6. (a) We write the system of equations as $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ yields the eigenvalues

$$\lambda_1 = 2, \qquad \lambda_2 = 3 \qquad \text{and} \qquad \lambda_3 = 5.$$

The corresponding eigenspaces are

$$N(\mathbf{A} - 2\mathbf{I}) = N\begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \text{Lin}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$N(\mathbf{A} - 3\mathbf{I}) = N \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \text{Lin} \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

$$N(\mathbf{A} - 5\mathbf{I}) = N \begin{pmatrix} -4 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Lin} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Hence

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Therefore, the particular solution of the system subject to the initial conditions is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \mathbf{P}\mathbf{D}^t\mathbf{P}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

[10 marks]

(b) The auxiliary equation is

$$m^2 - 5m + 6 = 0,$$

which yields $m_1 = 2$ and $m_2 = 3$.

The complementary sequence is therefore

$$(CS)_n = A\,2^n + B\,3^n,$$

where A and B are arbitrary constants.

For a particular sequence we try

$$(PS)_n = an + b$$

for some a, b to be determined. We have

$$(PS)_{n+1} = an + a + b, \qquad (PS)_{n+2} = an + 2a + b,$$

so substituting these expressions into the non-homogeneous equation we find

$$an + 2a + b - 5(an + a + b) + 6(an + b) = 4n.$$

This equation must be satisfied identically in n, so, comparing coefficients, we find that

$$2a = 4 \quad \text{and} \quad -3a + 2b = 0$$

and hence that

$$a = 2 \qquad \text{and} \qquad b = 3.$$

The general solution of the difference equation is therefore

$$w_n = 2n + 3 + A\,2^n + B\,3^n.$$

[9 marks]

(c) (i) Inspecting the form of the general solution found in (a), we see that $w_n \to \infty$ as $n \to \infty$ if

- either $B > 0$ and A is arbitrary; or
- $B = 0$ and $A \geq 0$.

[3 marks]

(ii) Similarly, we see that $w_n \to -\infty$ as $n \to \infty$ if

- either $B < 0$ and A is arbitrary; or
- $B = 0$ and $A < 0$.

[3 marks]

7. (a) Using the relations

$$y = xz \quad \text{and} \quad \frac{dy}{dx} = z + x\frac{dz}{dx}$$

we obtain an ordinary differential equation for the function $z(x)$:

$$2x^2\left(z + x\frac{dz}{dx}\right) = x^2 + x^2z^2.$$

We eliminate the factor x^2,

$$2\left(z + x\frac{dz}{dx}\right) = 1 + z^2,$$

and send the term $2z$ to the right hand side. The resulting equation is clearly separable:

$$2x\frac{dz}{dx} = z^2 - 2z + 1.$$

[6 marks]

(b) We separate the variables and integrate:

$$2\int \frac{dz}{z^2 - 2z + 1} = \int \frac{dx}{x}.$$

The denominator of the integrand on the left hand side is a complete square, so we have

$$2\int \frac{dz}{(z-1)^2} = \int \frac{dx}{x}$$

which yields the general solution for $z(x)$ in implicit form:

$$\ln(x) + \frac{2}{z-1} = C.$$

[4 marks]

(c) Before we apply the condition $(x, y) = (1, 9)$ let us find the corresponding solution for the function $y(x)$. First we make $z(x)$ the subject of the above equation to find that

$$z = 1 - \frac{2}{\ln(x) - C}$$

and then replace z by $\frac{y}{x}$ to obtain the general solution for $y(x)$:

$$y = x\left(1 - \frac{2}{\ln(x) - C}\right).$$

Finally, using the condition that y is equal to 9 when x is equal to 1, we find that

$$9 = 1 + \frac{2}{C},$$

which implies that

$$C = \frac{1}{4}.$$

Hence, the particular solution for $y(x)$ is

$$y = x\left(1 - \frac{2}{\ln(x) - \frac{1}{4}}\right).$$

[5 marks]

(d) We see that

$$f(x) = x\left(1 - \frac{2}{\ln(x) - \frac{1}{4}}\right)$$

has a vertical asymptote when $\ln(x) - \frac{1}{4} = 0$; i.e. when $x = e^{1/4}$. It follows that the largest set $D \subset \mathbb{R}$ for which $f : D \to \mathbb{R}$ is continuous is

$$D = (0, e^{1/4}),$$

noting that $x = 1$ belongs to this interval.

[3 marks]

(e) Regarding w as a function of t and applying the chain rule of differentiation, we find

$$H_t + H_w \frac{dw}{dt} = 0,$$

so

$$\frac{dw}{dt} = -\frac{H_t}{H_w}.$$

[4 marks]

(f) The equation

$$H_t dt + H_w dw = 0$$

has the required form $M(t, w)dt + N(t, w)dw = 0$ and its general solution is given implicitly by $H(t, w) = k$ for some arbitrary constant k. Moreover, the equation is exact, since

$$\frac{\partial}{\partial w} H_t = \frac{\partial}{\partial t} H_w.$$

[3 marks]

8. (a) The zero vector $z \in V$ is defined by the property that for any $f \in V$, we have that $f + z = f$. This means that for all $x \in [-3, 3]$ we have that

$$(f + z)(x) = f(x); \quad \text{i.e.} \quad f(x) + z(x) = f(x); \quad \text{i.e.} \quad z(x) = 0.$$

In other words, $z(x)$ is the identically zero function on $[-3, 3]$.

[2 marks]

(b) To prove linear independence, we assume that for all $x \in [-3, 3]$ we have

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) = z(x).$$

This implies the following identity in x:

$$2\alpha_1 + \alpha_2(1+x) + \alpha_3(x+x^2) = 0.$$

Expanding and equating coefficients we get the linear system

$$\begin{aligned} 2\alpha_1 + \alpha_2 &= 0 \\ \alpha_2 + \alpha_3 &= 0 \\ \alpha_3 &= 0 \end{aligned}$$

which has the unique solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$. It follows that f_1, f_2, f_3 are linearly independent.

[5 marks]

(c) To show that B spans V, let a general vector $f \in V$ be $f(x) = k + lx + mx^2$ for some $k, l, m \in \mathbb{R}$. We need to show that there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that the equation

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) = f(x)$$

is identically satisfied for all x; that is

$$(2\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3)x + \alpha_3 x^2 = k + lx + mx^2.$$

Equating coefficients, we get the linear system

$$\begin{aligned} 2\alpha_1 + \alpha_2 &= k \\ \alpha_2 + \alpha_3 &= l \\ \alpha_3 &= m \end{aligned}$$

which implies that

$$\alpha_3 = m, \quad \alpha_2 = l - m \quad \text{and} \quad \alpha_1 = \frac{k - l + m}{2}.$$

So any vector $f \in V$ can be written as a linear combination of the vectors in B and hence B spans V. Moreover, since B is a linearly independent set by part (b), we deduce that B is a basis for V and hence $\dim(V) = |B| = 3$.

[7 marks]

(d) We see that W is not a subspace of V because the zero vector $z(x)$ identified in part (a) is not in W. Alternatively, W is not closed under addition or scalar multiplication.

[2 marks]

(e) We calculate the inner product

$$\langle f_1, f_2 \rangle = \int_{-3}^{3} 2(1+x)dx = \left[2x + x^2\right]_{-3}^{3} = 12.$$

Since $\langle f_1, f_2 \rangle \neq 0$, the vectors f_1 and f_2 are not orthogonal with respect to the given inner product.

Furthermore, we have

$$||f_1|| = \sqrt{\langle f_1, f_1 \rangle} = \sqrt{\int_{-3}^{3} (2)(2)dx} = \sqrt{[4x]_{-3}^{3}} = \sqrt{24}$$

and

$$\begin{aligned} ||f_2|| &= \sqrt{\langle f_2, f_2 \rangle} = \sqrt{\int_{-3}^{3} (1+x)^2 dx} = \sqrt{\int_{-3}^{3} (x^2 + 2x + 1)dx} \\ &= \sqrt{\left[\frac{x^3}{3} + x^2 + x\right]_{-3}^{3}} = \sqrt{24}, \end{aligned}$$

so their lengths are equal.

[9 marks]