THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

**Summer 2016 examination**

# MA100

## Mathematical Methods

2015/2016 syllabus only — not for resit candidates

### Instructions to candidates

This examination counts 75% towards your final grade for MA100.

This paper contains **6** questions. Answer **all 6** questions. All questions carry equal numbers of marks.

Answers should be justified by showing work.

Please write your answers in dark ink (black or blue) only.

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| **Time Allowed** | **Reading Time:** | *None* |
| | **Writing Time:** | *3 hours* |
| **You are supplied with:** | | *Answer booklets* |
| **You may also use:** | | *No additional materials* |
| **Calculators:** | | *Calculators are not allowed in this examination* |

# Question 1

Consider the linear system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 & 0 \\ 2 & 10 & 0 & 2 \\ 4 & 20 & 1 & 3 \\ 1 & 5 & 0 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 8 \\ 15 \\ 4 \end{pmatrix}.$$

(a) Find the reduced row echelon form of the augmented matrix $(\mathbf{A}|\mathbf{b})$ and the general solution of the system $\mathbf{Ax} = \mathbf{b}$.

(b) Find a basis $B$ for the column space of $\mathbf{A}$ and obtain the coordinates $(\mathbf{c}_1)_B$, $(\mathbf{c}_2)_B$, $(\mathbf{c}_3)_B$, $(\mathbf{c}_4)_B$ of the columns $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$ of $\mathbf{A}$ with respect to the basis $B$.

(c) Find three different linear combinations of the columns $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$ that produce the vector $\mathbf{b}$.

(d) Find a basis $C$ for the null space of $\mathbf{A}^T$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$.

(e) Hence, or otherwise, obtain a Cartesian description in $\mathbb{R}^4$ for the column space of $\mathbf{A}$.

(f) Using your answer to part (e), find a set of equations that must be satisfied by the components $k, l, m, n$ of the vector

$$\mathbf{d} = \begin{pmatrix} k \\ l \\ m \\ n \end{pmatrix}$$

in order for the system $\mathbf{Ax} = \mathbf{d}$ to be consistent. You do **not** need to solve this set of equations.

## Question 2

The production function for a particular manufacturer has the Cobb-Douglas form

$$P(x, y) = 100x^{1/5}y^{4/5}$$

where the variables $x$ and $y$ represent labour and capital, respectively. The cost of labour is 200 pounds per unit and the cost of capital is 400 pounds per unit; i.e., the cost function is

$$C(x, y) = 200x + 400y.$$

(a) Sketch the feasible region $D \subset \mathbb{R}^2$ defined by $x \geq 0, y \geq 0$ and the requirement that the total cost of capital and labour cannot exceed 100,000 pounds. Also sketch roughly a few contours of the production function $P(x, y)$ in order to establish the existence of a point $M \in D$ corresponding to the constrained maximum of $P(x, y)$ on $D$.

(b) Write down a suitable Lagrangian for the maximisation of $P(x, y)$ on $D$ and use it to find the coordinates $(x^*, y^*)$ of $M$.

(c) Does the problem of minimising $P(x, y)$ on $D$ admit a solution? If your answer is yes, state where. If your answer is no, briefly explain why.

(d) On a separate graph, sketch roughly the feasible region $R \subset \mathbb{R}^2$ defined by $x \geq 0, y \geq 0$ and the requirement that the total production cannot be less than 40,000 product units. Also sketch a few contours of the cost function $C(x, y)$ and indicate on your graph the point $m \in R$ corresponding to the constrained minimum of $C(x, y)$ on $R$.

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = (x - 1)^2 + (y - 1)^3 + (z - 1)^4.$$

(e) Show that $f$ has a single stationary point.

(f) Is the matrix $f''$ evaluated at this point a positive definite, a positive semi-definite, a negative definite, a negative semi-definite or an indefinite matrix? You need to justify your answer.

(g) Is the stationary point of $f$ a local maximum, a local minimum or a saddle point? You need to justify your answer.

## Question 3

Consider the basis $B = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ of $\mathbb{R}^3$ consisting of the vectors

$$\mathbf{f}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{f}_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

(a) Use the Gram-Schmidt process to find an orthonormal basis $C = \{\mathbf{u}_1, \mathbf{u}_2\}$ for the two-dimensional subspace $\text{Lin}\{\mathbf{f}_1, \mathbf{f}_2\}$.

(b) Noting that $\mathbf{f}_3$ is orthogonal to both $\mathbf{f}_1$ and $\mathbf{f}_2$, extend your basis $C = \{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthonormal basis $K = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$.

Now let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$S(\mathbf{f}_1) = 2\mathbf{f}_1, \quad S(\mathbf{f}_2) = 2\mathbf{f}_2, \quad S(\mathbf{f}_3) = \mathbf{f}_3.$$

(c) Write down the matrix $\mathbf{A}_S^{B\to B}$ that represents $S$ with respect to the basis $B = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ and explain why

$$\mathbf{A}_S^{B\to B} = \mathbf{A}_S^{K\to K},$$

where $\mathbf{A}_S^{K\to K}$ is the matrix that represents $S$ with respect to the basis $K = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(d) Also explain why the matrix $\mathbf{A}_S$ that represents $S$ with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ must be a symmetric matrix.

(e) Hence write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{A}_S = \mathbf{P}\mathbf{D}\mathbf{P}^T.$$

You do **not** need to find $\mathbf{A}_S$.

(f) Find the first column $\mathbf{c}_1$ of $\mathbf{A}_S = (\mathbf{c}_1 \mathbf{c}_2 \mathbf{c}_3)$ using the relations $S(\mathbf{f}_1) = 2\mathbf{f}_1$ and $S(\mathbf{f}_3) = \mathbf{f}_3$.

## Question 4

For $t \in \{0, 1, 2, \dots\}$, consider the system of difference equations

$$\begin{cases} x_{t+1} &= x_t + y_t \\ y_{t+1} &= -2x_t + 4y_t \\ z_{t+1} &= 5z_t \end{cases}$$

satisfied by the sequences $\{x_t\}$, $\{y_t\}$, $\{z_t\}$.

(a) Express the particular solution of this system subject to the initial conditions $x_0 = 1$, $y_0 = 2$, $z_0 = 3$ in the form

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \mathbf{P}\mathbf{D}^t\mathbf{P}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for some suitable invertible matrix **P**, diagonal matrix **D** and column vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

You do **not** need to perform the matrix multiplications.

For $n \in \{0, 1, 2, \dots\}$, the sequence $\{w_n\}$ satisfies the difference equation

$$w_{n+2} - 5w_{n+1} + 6w_n = 4n.$$

(b) Find the general solution of this equation.

(c) Determine all values of the arbitrary constants appearing in your general solution for which

$$(i)\ \ w_n \to \infty \ \text{ as } \ n \to \infty \qquad (ii)\ \ w_n \to -\infty \ \text{ as } \ n \to \infty.$$

## Question 5

For $x > 0$, consider the homogeneous ordinary differential equation

$$2x^2 \frac{dy}{dx} = x^2 + y^2.$$

(a) Introduce a new dependent variable $z(x)$ by

$$z(x) = \frac{y(x)}{x}$$

and transform this homogeneous equation into a separable one.

(b) Obtain the general solution of this separable equation in the form $G(x, z)$=C for some function $G$ and arbitrary constant $C$.

(c) Hence, obtain in the explicit form $y = f(x)$ the particular solution of the homogeneous equation subject to the condition that $y = 9$ when $x = 1$.

(d) Noting that $x > 0$ and that $x = 1$ belongs to the domain of $f$, find the largest set $D \subset \mathbb{R}$ for which $f : D \to \mathbb{R}$ is continuous.

Suppose that the general solution of a first-order ordinary differential equation for a function $w(t)$ is given implicitly by

$$H(t, w) = k,$$

where $H$ is a given function and $k$ is an arbitrary constant.

(e) Use implicit differentiation to find an expression for the ordinary derivative $\frac{dw}{dt}$ in terms of the partial derivatives of $H$.

(f) Hence, obtain an exact ordinary differential equation of the form

$$M(t, w)dt + N(t, w)dw = 0$$

whose general solution is given implicitly by $H(t, w) = k$.

## Question 6

The set

$$V = \left\{ f : [-3, 3] \to \mathbb{R} \mid f(x) = a + bx + cx^2 \text{ where } a, b, c, \in \mathbb{R} \right\}$$

is a vector space under the standard operations of pointwise addition and scalar multiplication of functions; that is, under the operations

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \\ (\lambda f)(x) &= \lambda f(x), \end{aligned}$$

where $f, g \in V$ and $\lambda \in \mathbb{R}$.

(a) Identify which function $z : [-3, 3] \to \mathbb{R}$ is the zero vector in $V$.

Now consider the vectors $f_1, f_2, f_3 \in V$ given below:

$$f_1(x) = 2, \quad f_2(x) = 1 + x, \quad f_3(x) = x + x^2.$$

(b) Show that the set $B = \{f_1, f_2, f_3\}$ is a linearly independent set.

(c) Show that $B$ spans $V$ and state the dimension of $V$.

(d) Determine whether or not the subset $W$ of $V$ given by

$$W = \left\{ f : [-3, 3] \to \mathbb{R} \mid f(x) = a + ax + x^2 \text{ where } a \in \mathbb{R} \right\}$$

is a vector subspace of $V$.

The vector space $V$ is turned into an inner product space by introducing the inner product

$$\langle f, g \rangle = \int_{-3}^{3} f(x)g(x)dx.$$

(e) Considering the vectors $f_1, f_2 \in V$, determine whether or not these vectors are orthogonal to each other and show that their lengths $||f_1||$ and $||f_2||$ are equal.

END OF PAPER