

Intangible capital, non-rivalry, and growth *

Nicolas Crouzet¹, Janice Eberly², Andrea Eisfeldt³, and Dimitris Papanikolaou⁴

¹Kellogg School of Management, Northwestern University

²MIT, Golub Center for Finance and Policy, Kellogg School of Management, Northwestern University, and NBER

³Anderson School of Management, UCLA, and NBER

⁴Kellogg School of Management, Northwestern University, and NBER

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Abstract

We show how growth may slow even when technology is improving, due to the role of intangible assets. Intangible assets are distinct from physical capital; intangibles represent information that, lacking a physical presence, requires a storage medium. When easily replicated, intangibles can be non-rival. They need not be perfectly non-rival, however, if there are losses in replicating or re-using the intangible. Increasing non-rivalry stimulates growth as new technologies can be used more broadly, more readily. However, non-rivalry also undermines the incentive to invest in intangibles. Innovators must be able to capture a return to their efforts. Hence, improving technologies may drive growth by increasing non-rivalry, but with sufficiently weak property rights protections, these advances also undermine excludability and the incentive to invest. We show conditions under which improving technologies can lower intangible investment and lower growth.

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Over the past four decades, the importance of intangibles in firms' capital stock has grown substantially (Corrado, Hulten, and Sichel, 2005; Eisfeldt and Papanikolaou, 2014; Crouzet and Eberly, 2019). A key economic feature of intangibles — one that distinguishes them from physical capital — is *non-rivalry*. When a firm uses an intangible asset for a particular purpose, the asset does not necessarily become unavailable for other uses within the firm. When a firm stamps its brand on a new retail location, the brand remains available to other retail locations. Likewise, when a firm uses one of its patents in a new production facility, the patent remains available to other production facilities. More generally, intangible assets represent information that needs to be stored in order to be used in production (Crouzet, Eberly, Eisfeldt, and Papanikolaou, 2022). Non-rivalry may arise when the technology used to store intangibles makes it easy for the asset to be replicated at low cost within the firm.

Growth theory has long emphasized the fact that the production of non-rival capital goods is a core engine of long-run growth (Romer, 1986, 1990; Jones, 1995). Intuition suggests that the rapid expansion of the intangible capital stock should then have led to a rise in growth overall. In this paper, we argue that this intuition may not be correct.

Our starting point is the two main differences between ideas, as they are generally treated in endogenous growth models, and intangible assets. First, the intangibles a firm owns may not be *fully* non-rival. The Fourier transform, or the recipe for ketchup (two examples of the type of ideas that are the focus of endogenous growth models) are fully non-rival goods. On the other hand, implementing an inventory management system in a firm's new warehouse requires relocating technical experts there and training employees and management, which might reduce the efficiency with which existing inventory systems are run elsewhere in the company. In other words, given a total stock of intangible capital, increasing the intangibles used in a particular task does not require reducing their use in other tasks one-for-one (as would be the case for physical capital), but it is not entirely costless, either (as would be the case for pure ideas).

The second difference between ideas, as they are generally treated in endogenous growth models, and intangibles, is that intangibles are partly *excludable*; that is, they generally are not *public* capital goods. In principle, control and cash-flow rights over intangibles can be assigned to particular firms and enforced, as for physical assets. However, in practice, excludability is often imperfect. Property rights of certain intangibles, like patents, are well-defined and enforced in many countries, making them effectively excludable capital goods in countries with strong institutions. For other types of intangibles, like managerial methods, organization capital, or even databases and software, excludability is often more difficult to enforce, even in an environment with otherwise well-defined property rights.

The nature of intangibles as information that needs to be stored leads to a unique interplay between technological improvements and institutions that protect property rights. The technological frontier for information storage has evolved from speech to writing to punch cards and now to electronically-coded software and databases. New innovations continue. Speech allowed for intangibles to be alienable from individuals as ideas were communicated to others and to teams. The innovation of writing allowed for intangibles to be stored independently of individuals, increasing their scope and the opportunities for non-rival use. Modern-day electronic data storage, software, and communication allows the same information and processes to be used world-wide at the same time and at large scale. At the same time, the ease with which intangibles can be replicated at high quality and low cost also opens the door to imitation and expropriation. Property-rights institutions are designed to enforce excludability and assign ownership of intangibles so that they can become appropriable assets. Excludability enhances the incentive to innovate. However, by limiting the use of ideas, excludability intrinsically comes at a cost of lower spillovers.

Our main contribution is to explore the implications of partial non-rivalry and limited excludability for growth. We formalize these features and then embed them into an endogenous growth model. We show that, contrary to existing intuition from endogenous growth theory, a higher degree of non-rivalry does not necessarily increase long-run growth. More specifically, the relationship between non-rivalry and growth follows an inverse-U shape. When intangibles are close to rival — and thus resemble physical capital — there are no spillovers from the creation of new intangibles, and thus no growth. Introducing non-rivalry generates growth, but the highest rate of growth in the model is generally *not* achieved when intangibles are fully non-rival, as in the [Romer \(1990\)](#) model. Rather, it generally occurs for an intermediate degree of non-rivalry.

Two main forces shape equilibrium growth in the model: spillovers and incentives. The spillover channel, described above, is the idea that replication of existing intangibles by new entrants can enhance growth. This is the source of growth generally analyzed in work on endogenous growth, in the case of ideas. In the model, non-rivalry allows for spillovers. When intangibles are fully excludable, there are no spillovers, and therefore no growth. However, if intangibles are not fully excludable, then greater non-rivalry can *accelerate* spillovers, by allowing new entrants to replicate the intangible stock of incumbents at a faster rate. This raises returns to the creation of new firms, and thus leads to higher growth. Thus the spillover channel creates a positive relationship between growth and non-rivalry.

The second force is the incentive to create new firms. We model a firm as a collection of product streams (and which can be thought of as retail locations, production facilities, or establishments that the firm has control over), and call the number of product streams the span of the firm. When

intangibles are more non-rival, they can be deployed simultaneously across more product streams. Thus, a higher span raises the marginal return to intangibles, and more so when intangibles are more non-rival. However, by increasing its span, the firm also exposes itself to more competitive risk. In the model, we capture this by assuming that a greater span exposes the firm to a higher risk of expropriation of its product streams by a fringe of imitators. Firms optimally choose span to trade off this competitive risk against the higher returns to intangible capital. As non-rivalry increases, the benefits of higher span grow, so the firm is willing to tolerate a higher degree of competitive risk. Ex-ante, however, the greater competitive risk makes the creation of new firms less attractive. This disincentive effect eventually reduces growth.

Equilibrium growth reflects the balance of these two forces. We show that in general, spillover effects will dominate when intangibles are close to being rival — similar to physical capital —, because in these situations the optimal span of new firms is small and competitive risk is limited. When intangibles are closer to being fully non-rival — similar to ideas in endogenous growth theory —, the optimal span of new firms is high, competitive risk is strong, and disincentive effects are dominant. Nevertheless, we also highlight certain conditions under which spillover effects can strictly dominate, so that the relationship between growth and non-rivalry is increasing. These tend to be situations in which excludability is very weak, so that new firms optimally choose a small span to limit competitive risk.

A broad implication of the comparative statics of the model is that changes in the cost of storing intangibles, which in our model are equivalent to a change in the degree of non-rivalry of intangibles within the firm, have ambiguous effects on equilibrium growth. Thus, the model highlights the potential downside of, say, improvements in information technology that make the storage of data-related intangible assets easier.

Relation to the literature Our work complements the literature on endogenous growth that starts with [Romer \(1990\)](#). Our contribution is to provide a new model of production using intangible capital and explore its implications for long-run growth. Our model explicitly allows for the degree of non-rivalry and the degree of enforcement of intellectual property protection to be structural parameters that could vary with the specific characteristics of the intangible and the technologies for transmitting ideas and enforcing property rights. As such, it essentially nests the standard [Solow \(1956\)](#) model where capital is fully rival and there is no long-run growth and the [Romer \(1990\)](#) model in which ideas are fully non-rival and long-run growth obtains. Specifically, we show that the degree of non-rivalry of intangibles maps the continuum between the two types of models and that, surprisingly, it has a non-monotonic effect on growth.

We should emphasize, however, our notion of intangibles is distinct, and likely broader, than the notion of intangibles in existing models of endogenous growth along two dimensions. First, we model intangibles as a direct (partly) non-rival input in production with the firm choosing the scale of implementation subject to concerns about excludability. By contrast, intangibles in [Romer \(1990\)](#) essentially represent the exclusive right to produce a particular good—or a technology vintage that can be used to produce an existing good at lower marginal cost as in [Aghion and Howitt \(1992\)](#) or [Klette and Kortum \(2004\)](#).¹ As such, our notion encompasses not only the idea of a new product or specific production process, but also concepts such as organization capital that can be deployed at a broader scale or scope. Second, our model features an explicit tradeoff between the scope (or scale) of deployment of the intangible and the likelihood of expropriation by imitators. Indeed, the presence of imitators introduces a further wedge between private and social returns in the creation of new intangibles—by contrast, in [Romer \(1990\)](#) spillovers occur only in the creation of new projects which build on the existing frontier. This tradeoff that limits the scale of deployment of intangibles is conceptually distinct from monopolistic competition in [Romer \(1990\)](#) or the need for complementary inputs that are in scarce supply ([Atkeson and Kehoe, 2005](#)).

Our modeling of intangible assets is perhaps most similar to [McGrattan and Prescott \(2009, 2010a\)](#), in that the same intangible can be used (in our language, replicated) in multiple locations (countries in their setting). A key difference with their work is that, rather than restricting the number of possible locations, we introduce a cost of expanding scope: the possibility of expropriation. Along these lines our work also complements the literature that models intangibles as a factor of production that allows firms to reduce the cost of entering new markets ([Argente, Moreira, Oberfield, and Venkateswaran, 2020](#); [Hsieh and Rossi-Hansberg, 2022](#)). Relative to these papers, we allow for the replicability of intangibles to be imperfect; allow for concerns about expropriation to endogenously limit the deployment of intangibles and focus on the implications of the model for long-term growth.

Our work also complements the literature on the diffusion of technology through imitation and spillovers ([Lucas and Moll, 2014](#); [Perla and Tonetti, 2014](#); [Stokey, 2015](#); [Perla, Tonetti, and Waugh, 2021](#); [Akcigit and Ates, 2019](#)). [Lucas and Moll \(2014\)](#) describe their technology as “partially rival”, since agents do not all have access to the frontier technology, which they acquire over time by randomly meeting other agents. Relative to these papers, we model ideas (intangibles) as a partially non-rival input, analyze the impact of the degree of non-rivalry on growth, and emphasize the “dark side” of spillovers, namely the fact that innovators cannot fully appropriate the value of their creations.

¹Additional recent work in this area includes [De Ridder \(2022\)](#); [Aghion, Bergeaud, Boppart, Klenow, and Li \(2022\)](#).

Our model is also related to quality ladder models, which have a different notion of technological diffusion. In these models, such as [Aghion and Howitt \(1992\)](#), [Segerstrom \(1998\)](#), and [Klette and Kortum \(2004\)](#) the frontier technology diffuses immediately because the leading firm takes over the entire market, including all the productive factors of potential rivals. The technology rises in "drastic" steps that compensate the leading firm for the cost of innovation, even though it only holds the lead position until the next step occurs. This allows firms to earn profits that justify investing in innovation. Aggregate productivity is always equal to the frontier, not because of non-rival technology diffusion but because the frontier innovator becomes a monopolist. Innovations are thus perfectly excludable and accrue solely to the innovating firm. This strong excludability eliminates spillovers across firms, but quality ladders instead rely on the monopoly firm to take the frontier technology economy-wide.

The non-monotonic effect of non-rivalry on long-run growth may be reminiscent of the non-monotonic relation between the degree of competition in the product market and the level of innovation and growth emphasized in [Aghion, Bloom, Blundell, Griffith, and Howitt \(2005\)](#). These two mechanisms have some similarities but are conceptually distinct: the main comparative static we focus on is changes in the degree of non-rivalry within the firm, a notion which is absent in their model. That said, our notion of expropriation by imitators has some conceptual similarities to the loss of rents to competitors. What drives the positive relation between the level of competition and growth in [Aghion et al. \(2005\)](#) is that investing in innovation allows firms to lower the risk of losing monopoly rents—what they term the 'escape the competition effect'. By contrast, in our setting, greater investment in creating intangibles, or deploying them at a greater scale, always increases the risk of expropriation.

Recent advances in this literature have explored the importance of certain non-rival types of capital goods, and in particular data ([Farboodi and Veldkamp, 2020](#); [Jones and Tonetti, 2020](#)). For instance, [Jones and Tonetti \(2020\)](#) model the increasing returns that can be achieved through the use of customer data to improve productivity: the more people consume a given product the more data the firm has to improve productivity which leads to higher output and hence more data. Our main contribution relative to this work is to study the role of non-rivalry of intangibles more generally, and to allow for innovations to be expropriated, depending on the degree of non-rivalry of intangibles.

Our model of production using intangible capital implies a strong association between the size of the firm and its scope of operations. In our model, a higher degree of non-rivalry implies a greater complementarity between the stock of intangibles and firm scope. Consistent with this prediction, [Ding, Fort, Redding, and Schott \(2022\)](#) find that firms with greater in-house production of auxiliary

services (a measure of intangibles) grow faster and are more likely to enter new industries than other firms.

More broadly, a large literature in macroeconomics and finance has highlighted the macroeconomic implications of the rise in intangible capital for various topics, including investment and asset prices, labor markets, and growth (McGrattan and Prescott, 2010a,b; Haskel and Westlake, 2017; Crouzet and Eberly, 2019, 2021). Our key distinction relative to these papers lies in how we model the production process using intangibles. Existing work models intangible capital as a factor of production that is qualitatively similar to physical capital—intangible capital is yet another factor of production that can be accumulated subject to adjustment costs. The main difference between tangible and intangible capital in these models is simply due to parameter choices—intangibles are fully excludable and rival capital inputs, but may differ in their rate of depreciation, price, or riskiness. We examine these issues using a different model of production that emphasizes the distinction between physical and intangible capital: their degree of non-rivalry, and their degree of excludability.

Our work also highlights the challenges in measuring intangible assets in the data. Existing work has focused on measuring specific types of intangible capital, including patented innovations (Hall, Jaffe, and Trajtenberg, 2005; Kogan, Papanikolaou, Seru, and Stoffman, 2017; Kelly, Papanikolaou, Seru, and Taddy, 2021); management practices (Bloom and Van Reenen, 2007); software and data-related assets (Bresnahan, Brynjolfsson, and Hitt, 2002); brands and customer capital (Gourio and Rudanko, 2014); or organization capital (Eisfeldt and Papanikolaou, 2013; Bhandari and McGrattan, 2021). Our model shows that the private value of an intangible asset, its acquisition cost, its contribution to aggregate output, and its social value are in general distinct objects that need not coincide.

1 Economic environment

This section describes the model. Section 1.1 outlines the main elements, starting with its production side. Section 1.2 discusses important assumptions in more detail, and Section 1.3 describes potential extensions. For brevity, proofs are relegated to Appendix A.1.

1.1 Model

Figure 1 provides an overview of the model. Time t is continuous. Production takes place within projects, which are indexed by i . $\tau(i)$ is the date at which project i is created, so that $t - \tau(i)$ is the age of the project.

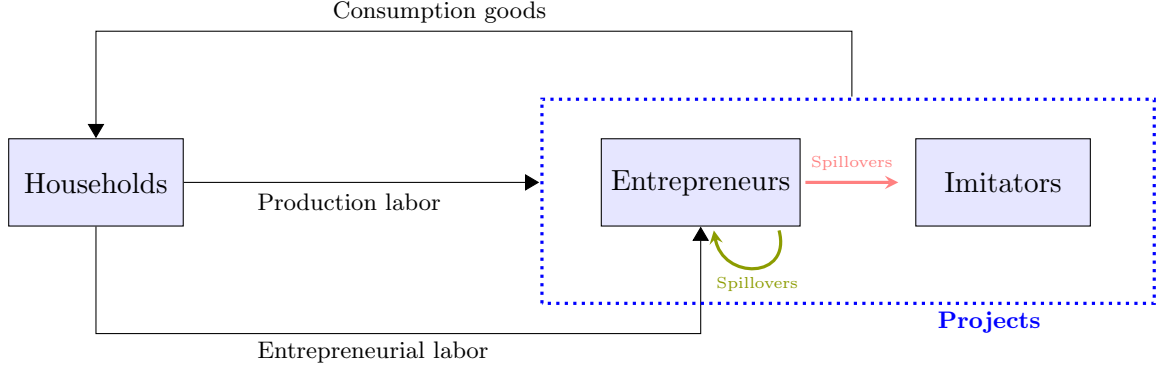


Figure 1: Overview of the model.

We refer to the initial owner and creator of the project as the ‘entrepreneur’, though this term could encompass any stakeholder in the project that has contributed to the creation of its intangible capital, including founders, key employees, or managers. Over time, ownership of the project is reallocated from entrepreneurs to other groups of agents, including a fringe of imitators.

The supply of entrepreneurs is not fixed, but rather there is free entry: households are indifferent between working in production and becoming entrepreneurs and creating new projects.

We first describe the problem of an incumbent entrepreneur. We then turn to how expropriation risk and spillovers benefit imitators. Finally, we describe entry and equilibrium.

Incumbent entrepreneurs

At each point in time, an entrepreneur has ownership over a collection of product streams, indexed by $s \in [0, x_{i,t}]$, where $x_{i,t}$ is the total number of streams still under control of the entrepreneur at date t . Within each project stream, intangible capital, $N_{i,t}(s)$, and labor, $L_{i,t}(s)$, are combined according to a Cobb-Douglas production function with labor share ζ . We start by describing the allocation of labor and intangible capital to different streams. We then discuss how the total stock of intangible capital, as well as the value of $x_{i,t}$, evolve over time.

Static allocation decision The entrepreneur makes static two choices at each point in time: how much total production labor to hire; and how to allocate production labor and intangible capital across product streams. Specifically, they solve:

$$\Pi(x_{i,t}, N_{i,t}; W_t, \rho_i) = \max_{\{N_{i,t}(s), L_{i,t}(s)\}_{s \in [0, x_{i,t}]}, L_{i,t}} \int_0^{x_{i,t}} N_{i,t}(s)^{1-\zeta} L_{i,t}(s)^\zeta ds - W_t L_{i,t} \quad (1)$$

$$\text{s.t.} \quad \int_0^{x_{i,t}} L_{i,t}(s) ds \leq L_{i,t} \quad (2)$$

$$\left(\int_0^{x_{i,t}} N_{i,t}(s)^{\frac{1}{1-\rho_i}} ds \right)^{1-\rho_i} \leq N_{i,t} \quad (3)$$

Here, $N_{i,t}$ denotes the total intangible capital stock of the project, and W_t the wage rate. We make the following central assumption about the parameter ρ_i .

Assumption 1 (Non-rivalry). *The parameter ρ_i satisfies $0 \leq \rho_i \leq 1$.*

To understand the role of this assumption, first consider how labor is allocated across production streams. Given a total labor input $L_{i,t}$, adding more labor to a stream requires removing it from another. Thus, the resource constraint (2) implies that labor is perfectly rival within project.

Assumption 1 allows intangibles to be partially non-rival within a project. When $\rho_i = 0$, the resource constraints for labor (2) and intangibles (3) are the same, as they are both rival inputs; given the total stock of intangibles, adding more intangible capital to a product stream requires taking it away from another one. By contrast, in the special case where intangibles are fully non-rival ($\rho_i = 1$) the constraint (3) becomes

$$\max_{s \in [0, x_{i,t}]} N_{i,t}(s) \leq N_{i,t}. \quad (4)$$

In this case, adding intangibles to a particular stream does not require reducing the input in any other stream: intangible capital is fully non-rival within project.

More broadly, when $\rho_i \in (0, 1)$, intangibles are partially non-rival within the project. The marginal rate of technical substitution between streams is less than 1 (as when $\rho_i = 0$, or for labor) but greater than 0 (as $\rho_i = 1$, or fully rival intangibles). Thus we associate the parameter ρ_i with the degree of non-rivalry of intangibles within project i .

We allow the parameter ρ_i to vary across cohorts of projects, but we assume that its value is fixed within project, because it reflects the particular features of the technology used to store and reallocate intangibles within projects.

The solution to the allocation problem (1) is symmetric across product streams:

$$\forall s \in [0, x_{i,t}], \quad N_{i,t}(s) = x_{i,t}^{-(1-\rho_i)} N_{i,t}, \quad (5)$$

$$L_{i,t}(s) = \left(\frac{\Lambda_t}{1-\zeta} \right)^{\frac{1}{\zeta}} N_{i,t}(s), \quad (6)$$

leading to the following expressions for the total level of profits and labor demand for the project

$$L_{i,t} = x_{i,t}^{\rho_i} \left(\frac{\Lambda_t}{1-\zeta} \right)^{\frac{1}{\zeta}} N_{i,t}, \quad (7)$$

$$\Pi_{i,t} = x_{i,t}^{\rho_i} \Lambda_t N_{i,t}, \quad (8)$$

where we defined:

$$\Lambda_t \equiv (1 - \zeta) \left(\frac{\zeta}{W_t} \right)^{\frac{\zeta}{1-\zeta}}. \quad (9)$$

Expropriation risk After the initial creation of the project, the entrepreneur faces a risk of expropriation. Expropriation affects product streams individually, as opposed to the project as a whole. If the entrepreneur is expropriated from a production stream, ownership of the production stream is transferred to an imitator fringe. Upon expropriation, the entrepreneur also loses ownership of intangibles that had been used in the expropriated stream. Both the loss of the stream and the loss of the associated intangibles entail a transfer of value from the entrepreneur to the imitator fringe, as well as to future entrepreneurs, through spillovers. We come back to this value transfer below, after describing the expropriation process.

Assumption 2 (Limits to excludability). *Within each stream, expropriation follows a Poisson process with intensity $\lambda_i = \gamma(x_{i,\tau(i)})$, where $x_{i,\tau(i)}$ is the initial span of the project. Moreover, expropriation is independent across streams. Finally, the function $\gamma(\cdot)$, mapping $[1, +\infty)$ to $[0, +\infty)$, satisfies:*

$$\gamma(1) \geq 0, \quad \lim_{x \rightarrow +\infty} \gamma(x) = +\infty, \quad \gamma'(1) = 0, \quad \gamma'(x) \geq 0, \quad \text{and} \quad \gamma''(x) > 0.$$

We associate expropriation risk, λ_i , and the function γ that determines it, with the degree of excludability of intangibles. The case $\gamma(\cdot) = 0$ would correspond to perfect excludability, as no product stream is ever expropriated. On the other hand, $\gamma(\cdot) = +\infty$ corresponds to no excludability: immediately after the project is created, all its product streams are expropriated.

Assumption 2 implies that, conditional on not becoming obsolete by time $t \geq \tau(i)$, the number of streams still under the control of the entrepreneur follows:

$$x_{i,t} = e^{-\lambda_i(t-\tau(i))} x_{i,\tau(i)}. \quad (10)$$

Additionally, Appendix A.1.1 shows that the total stock of intangibles owned by the entrepreneur follows:

$$N_{i,t} = e^{-\lambda_i(1-\rho_i)(t-\tau(i))} N_{i,\tau(i)}. \quad (11)$$

Equation (11) highlights the interaction between excludability and non-rivalry. In the case of perfect non-rivalry ($\rho_i = 1$), expropriation does not affect the total stock of intangibles that the entrepreneur

owns at all: $N_{i,t} = N_{i,\tau(i)}$. In this case, the entrepreneur uses the total stock of intangibles of the project in *each* stream. Even if the entrepreneur has been expropriated from one its product streams, expropriation doesn't reduce the amount of intangibles available for the remaining streams. By contrast, in the case in which intangibles are fully rival ($\rho_i = 0$), the total stock of intangibles shrinks at the rate λ_i , which is increasing in the number of product streams. In this case expropriation reduces one-for-one the total stock of intangibles available for the remaining streams. We discuss in more detail in Section 1.2 what Equation (10)-(11) implies for how much intangibles are invested in each stream over time.

As a direct result of equations (11) and (10), conditional on no obsolescence, the cashflows to the entrepreneur at $t \geq \tau(i)$ are given by

$$\Pi_{i,t} = e^{-\lambda_i(t-\tau(i))} \Lambda_t x_{i,\tau(i)}^{\rho_i} N_{i,\tau(i)} = e^{-\lambda_i(t-\tau(i))} \frac{\Lambda_t}{\Lambda_{\tau(i)}} \Pi_{i,\tau(i)} \quad (12)$$

Examining equation (12), we note that these cashflows decrease over time for two reasons. First, the market price of labor changes as newly created projects compete for workers with existing projects; this effect is captured by the evolution of the term Λ_t , which will have a negative drift in equilibrium, because the supply of labor is fixed. Second, the entrepreneur progressively loses control of the project to imitators; from her perspective, this is equivalent to the cashflows depreciating at a rate λ_i .

Spillovers *Spillover intangibles* from project i determine the stock of knowledge that can be freely built on to create new intangibles. We define spillovers $S_{i,t}$ associated with project i up to time t , conditional on no obsolescence, by the following implicit relationship:

$$N_{i,\tau(i)} \equiv \left(S_{i,t}^{\frac{1}{1-\rho_i}} + N_{i,t}^{\frac{1}{1-\rho_i}} \right)^{1-\rho_i}. \quad (13)$$

Equation (13) defines the stock of spillover intangibles $S_{i,t}$ so that the total intangibles associated with project i (those still used by the entrepreneur, and those that have been expropriated), cannot exceed the initial quantity that the entrepreneur created, $N_{i,\tau(i)}$. Thus the total intangibles initially invested in the project are conserved over time, though their ownership is reallocated over time from the entrepreneur to the rest of the economy (imitators and future entrepreneurs).

Using the law of motion for $N_{i,t}$, we obtain:

$$S_{i,t} = \left(1 - e^{-\lambda_i(t-\tau(i))} \right)^{1-\rho_i} N_{i,\tau(i)}. \quad (14)$$

Note that the degree of non-rivalry ρ affects the *speed* at which spillovers from the project are generated. Specifically:

$$S_{i,t} = \begin{cases} N_{i,\tau(i)} - N_{i,t} & \text{if } \rho_i = 0 \\ N_{i,\tau(i)} & \text{if } \rho_i = 1 \end{cases}$$

When $\rho_i = 1$, a new project immediately generates the maximum level of spillovers, since $S_{i,t} = N_{i,\tau(i)}$, regardless of the degree of excludability λ_i . Recall that when $\rho_i = 1$, each stream contains the entire intangible stock of the project. Expropriation of any stream immediately makes the whole intangible stock of the project available to outsiders. By contrast, when $\rho_i = 0$, as the project's stock of intangibles decreases through expropriation, spillover intangibles increase one-for-one. In this case, spillovers happen gradually, at rate λ_i .

Last, we define the aggregate stock of spillover intangibles S_t as:

$$S_t = \int_{\tau(i) \leq t} S_{i,t} a_{i,t} di = \int_{\tau(i) \leq t} (1 - e^{-\lambda_i(t-\tau(i))})^{1-\rho_i} N_{i,\tau(i)} a_{i,t} di, \quad (15)$$

where $a_{i,t}$ is an indicator for whether the intangible capital of project i has become obsolete by time t . As we now discuss, the total amount of spillover intangibles in the economy, Equation (15), will be a key input in the creation of new intangibles.

The value of entry for entrepreneurs The first group of agents that benefit from spillovers from existing projects are the entrepreneurs that create new projects. Entrepreneurs discount future cash flows at rate r . Given an initial span of x and an initial intangible capital input of N , an entrepreneur's value from project i at the time of its creation is, $t = \tau(i)$, is:

$$V_{i,t}^{(e)}(x, N) = \mathbb{E}_t \left[\int_t^{+\infty} e^{-r(s-t)} \Pi_{i,s} dt \right] = \Lambda_t x^{\rho_t} N \tilde{v}_t^{(e)}(x) \quad (16)$$

where

$$\tilde{v}_t^{(e)}(x) \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-(r+\gamma(x))(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right]. \quad (17)$$

In examining equation (17), note that the dependence on the initial size of the project x is purely through the resulting risk of expropriation $\lambda = \gamma(x)$.

Starting a new project requires that an entrepreneur spend a unit of labor. The labor spent leads to the same stock of intangibles for all projects that are created at time $t = \tau(i)$, so there is no idiosyncratic risk in the creation of new projects. All projects that are started at the same date have the same value, so we omit the subscript i in what follows when the context is clear.

Spillover intangibles facilitate the creation of new intangibles. As such, the initial stock of

intangibles for a new project created at time t is given by:

$$N_t^e = \nu S_t. \quad (18)$$

Here, the parameter ν , which is fixed and exogenous, governs the size of new projects relative to the overall stock of spillover intangibles. The fact that ν is constant helps ensure the existence of a balanced growth path. One can think of this equation as analogous to the equation in a standard neoclassical growth model stating that the investment rate at the intensive margin is constant, since it assumes that the ratio between the existing capital stock (in this case, the part of the capital stock that generates spillovers, S_t), and new intangibles (in this case, N_t^e) is constant.

The entrepreneur chooses the initial span of the project, x , to maximize her value from the project, described in Equation (16). Thus the optimal choice of initial project span is given by:

$$x_t = \arg \max_x x^{\rho_t} \tilde{v}_t^{(e)}(x), \quad (19)$$

while the value of a new project is:

$$V_{i,t}^{(e)} = \Lambda_t x_t^{\rho_t} (\nu S_t) \tilde{v}_t^{(e)}(x_t), \quad (20)$$

In choosing span in (19), the entrepreneur faces a trade-off between two forces. A higher span raises the marginal revenue product of intangibles (assuming some degree of non-rivalry, $\rho_t > 0$), which is reflected in the term x^{ρ_t} in the equation above. At the same time, a higher span x also increases the likelihood of expropriation within each product stream, $\gamma(x)$, thus lowering the marginal revenue product of intangibles. This cost is reflected in the term $\tilde{v}_t(x)$ in the expression above. The trade-off between scale and the risk of expropriation will ensure that entrepreneurs choose a finite span.

Since the optimal span is the same for all newly-created projects at time t , the risk of expropriation is also identical. To simplify notation, in what follows we will denote:

$$\lambda_t \equiv \gamma(x_t) \quad \text{and} \quad v_t^{(e)} \equiv \tilde{v}_t^{(e)}(x_t). \quad (21)$$

Imitators

The second group of agents that benefit from spillovers are the “imitators”. When the product line of a project is expropriated, we assume that it is taken over by a fringe of imitators. These imitators then produce output in the expropriated streams, using spillover intangibles from those streams. Moreover, they do not face any expropriation risk themselves, though they face the same

obsolescence risk as the entrepreneurs.

The span of the imitator fringe associated with the project is the complement of the streams that remain under control of the entrepreneur in $[0, x_{i,\tau(i)}]$. Imitators produce using the stock of spillover intangibles from the project, $S_{i,t}$. Simple computation shows that total flow profits to imitators in project i are:

$$\Pi_{i,t}^{(m)} = \Lambda_t x_{\tau(i)}^{\rho_i} N_{\tau(i)} - \Pi_{i,t}. \quad (22)$$

Given the above, we can write the total value of the project when it is created as

$$V_{i,t} = V_{i,t}^{(m)} + V_{i,t}^{(e)} = \Lambda_t x_t^{\rho_i} (\nu S_t) v_t \quad (23)$$

where

$$v_t \equiv \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right]. \quad (24)$$

Here, the total value of the project equals the sum of the value of the project to the entrepreneur that creates it, and to imitators that eventually benefit from it. Note that this is not the social value of the project, since it does not include the value created by raising the stock of spillover intangibles on which future entrepreneurs will eventually build. Comparing equation (23) to (16), we see that the entrepreneur can appropriate a fraction

$$\theta_t(x) = \frac{\tilde{v}_t(x)}{v_t} \quad (25)$$

of the total value of her project given her choice of span x . Equivalently, the span chosen by the entrepreneur satisfies

$$x_t = \arg \max_x x^{\rho_i} \theta_t(x). \quad (26)$$

This re-formulation of the entrepreneur's optimal span choice highlights the fact that by choosing a higher span, and therefore raising expropriation risk and redistributing surplus to imitators, leads the entrepreneur to retain a *smaller* fraction of the project's overall value.

Labor markets

Labor can be allocated to two activities: creating new projects, and producing output. We denote by $L_{e,t}$ the amount of labor allocated to the creation of projects. We assume that labor is perfectly substitutable across the two activities, and we normalize total labor supply to 1 throughout.

There is free entry in the creation of new projects. Since the creation of each project requires

spending one unit of labor, free-entry implies that

$$W_t \geq \Lambda_t x_t^{\rho_t} \nu S_t v_t^{(e)}, \text{ with equality if } L_{e,t} > 0. \quad (27)$$

Total demand for production labor is the sum of demand for labor from each project:

$$L_{p,t} = \left(\frac{\Lambda_t}{1 - \zeta} \right)^{\frac{1}{\zeta}} \int_{\tau \leq t} x_{\tau}^{\rho_{\tau}} (\nu S_{\tau}) L_{e,\tau} d\tau. \quad (28)$$

In deriving (28), we exploit the fact that newly-created projects at time t start with the same stock of intangibles νS_t ; that each entrepreneurial unit of labor creates one project; and that both entrepreneurs and imitators produce, and therefore have demand for production labor. Labor market clearing then requires that

$$L_{p,t} + L_{e,t} = 1. \quad (29)$$

Finally, note that given that new projects in a cohort are identical, the law of motion for the stock of spillover intangibles can be rewritten as:

$$S_t = \int_{\tau \leq t} (1 - e^{-\lambda_{\tau}(t-\tau)})^{1-\rho_{\tau}} (\nu S_{\tau}) L_{e,\tau} d\tau. \quad (30)$$

Equilibrium

We define equilibrium in a standard way. Appendix A.1.4 provides the list of equilibrium conditions.

Definition 1 (Equilibrium). *Given an exogenous stochastic process for the degree of non-rivalry, $\{\rho_t\}$, an equilibrium of this model is a set of stochastic processes $\{L_{e,t}, L_{p,t}, \Lambda_t, W_t, v_t^{(e)}, x_t, S_t\}$ that satisfy Equation (9), the definition of the marginal revenue product of intangibles within each stream; Equations (19) and (21), the value of a new project to entrepreneurs; Equation (27), the free-entry condition; Equation (28), the aggregate demand for production labor; Equation (29), the labor market clearing condition; and Equation (30), the law of motion for spillover intangibles.*

Note that we have left the consumption side of the model unspecified. Given the assumption that the interest rate r is fixed, the model can be interpreted as representing an economy with an infinite elasticity of intertemporal substitution and a time discount rate of r . It would be straightforward to extend the model to allow for a finite intertemporal elasticity of substitution.

1.2 Discussion of key assumptions

Non-rivalry

We view intangible capital as information that must be stored in a particular medium in order to be used in production. The technology for storing this information makes intangible capital *partially* non-rival within a firm. Our assumption of partial non-rivalry (Assumption 1) captures this idea. The two edge cases $\rho = 0$ and $\rho = 1$ correspond, respectively, to completely rival and completely non-rival intangibles. The case $\rho = 0$ makes intangible capital exactly analogous to physical capital, while the case $\rho = 1$ makes intangible capital analogous to total factor productivity, or ideas in endogenous growth model — which are generally treated as a non-rival within firm.

The current state of storage technology for a specific intangible asset determines ρ . As a first example of intangible assets, consider past data on the purchasing behavior of a firm’s customers, which are used in the firm’s call centers to sell additional products. In the past, customer data was stored using pen and paper. Currently, customer data is stored in computer memory, which makes it significantly easier to replicate the data and make it available across all call centers in the firm. As such, we argue that this corresponds to the case of an intangible asset with relatively high ρ ; further, the availability of computers has likely increased ρ for customer data over time as they can be more easily stored and disseminated.

As a second example, consider a particular method for organizing production, which is used within a firm’s production plants. In this case, the intangible is stored in key employees, since it is part of the human capital of assembly-line workers and production managers. Replicating the production method and making it available to other plants within the firm will require supervision and training, which may be costly or imperfect. Even if the firm could hire and train a sufficient number of production managers, something may still be lost in translation. In our view, this corresponds to the case of an intangible asset with lower ρ . Further, advances in the technology for communicating instructions on the new production method to workers and supervising them will likely increase the intangible’s ρ .

In general, we view the degree of non-rivalry of the intangible asset as an intrinsic feature of the storage technology used to create it. Since intangibles are fixed within project, we assume that ρ is also constant within a project. However, we allow it to potentially change over time as the technology for storing intangibles improves.

Last, we have implicitly assumed in Equation (30) that the total amount of spillovers in the economy, the variable S_t , is the sum of spillovers across existing projects—or stated differently, spillover intangibles are substitutes across projects. We make this assumption because ρ may differ

across projects; that is, the storage technology used to create the intangible assets may change. As a result, spillovers from later cohorts may not necessarily enhance those of prior cohorts.

Limits to excludability

Our second key assumption (Assumption 2) is that the institutions and contracts that assign control and cash flow rights to intangible assets may be imperfect.² Importantly, the effects of contractual incompleteness can be exacerbated by non-rivalry.

To see this interaction, let us revisit the expression for the spillovers generated by an individual project (14). Consider two hypothetical projects, $i = 1, 2$, created at the same date that face the same degree of excludability λ but have different levels of non-rivalry $\rho_{\tau(2)} = \rho_2 > \rho_1 = \rho_{\tau(1)}$. The ratio of the two spillovers generated from these projects equals

$$\frac{S_{2,t}}{S_{1,t}} = \left(\frac{1}{1 - e^{-\lambda t}} \right)^{\rho_2 - \rho_1} > 1. \quad (31)$$

Equation (31) indicates that, for a given level of λ , expropriation risk, spillovers build up more rapidly when non-rivalry is higher.³ Put differently, for any degree of excludability, λ , the ease in which intangibles can be stored, ρ , determines the *speed* at which spillovers occurs. When ρ is close to 1, spillovers happen very quickly, while they happen more slowly when ρ is closer to zero.

Assumption 2 also states that the entrepreneur loses ownership of intangibles invested in streams that are expropriated. This results in a decline in the total stock of intangibles that remains under ownership of the entrepreneur, as described by Equation (11). In turn, using Equation (5), this implies that the intangible capital invested in *each* of the project's streams remains constant over time:

$$N_{i,t}(s) = x_{i,t}^{-(1-\rho_i)} N_{i,t} = x_{i,\tau(i)}^{-(1-\rho_i)} N_{i,\tau(i)}. \quad (32)$$

This assumption captures the idea that specific intangibles are tied to specific product streams. For instance, managers attached to a particular production facility may be those whose organization capital is most adapted to that facility.⁴ On the flipside, under alternative assumptions whereby the entrepreneur retains ownership over *some* of the intangibles that were invested in an expropriated

²Incomplete or unenforceable contracts may also affect ownership of physical capital; imperfect excludability is not specific to intangibles. In fact, the model allows for imperfect property rights over physical capital, which would correspond to a case in which the function γ is strictly positive, but $\rho = 0$. In this case, the evolution of $N_{i,t}$ within a project would represent the progressive expropriation of entrepreneurs from their physical assets.

³This example is only meant to illustrate the interaction between the function λ and the parameter ρ . In the full model, the rate of expropriation $\lambda_{\tau(i)}$ is a function of the optimal span of the project, which depends on the parameter ρ , so that the comparative static described in this example is more complex. But the basic insight from the example—that for a given level of expropriation risk, the rate at which spillovers build up increases with ρ —would remain.

⁴Formally, because under Assumption 2 the stock of intangibles invested in each stream is constant, one could equivalently model the intangible allocation as occurring only at date 0, immediately after the project is created.

stream, the amount invested in each stream would *rise* over time, making the marginal product of labor in each remaining stream higher. Thus expropriation would have positive spillovers between streams of the project, at odds with the idea that expropriation poses a risk to the entrepreneur. Relatedly, the definition of spillover intangibles in Equation (14) implies that the amount of intangible capital invested in a stream taken over by imitators is the same as what the entrepreneur had been investing. Other assumptions would imply that the per-stream and project-wide amount of intangible capital would instead increase.

Our formulation for $\gamma(x)$ assumes that expropriation risk is constant within a project and increases with the initial span of the project. Our assumption that $\gamma'(x) > 0$ captures the idea that there are increasing returns to imitation: imitators might find it easier to copy a project's intangibles if they observe more product streams. The assumption that expropriation risk within project is constant is primarily for tractability; time-dependence in expropriation risk within a project would complicate the analysis but deliver no further insights. That said, since ρ_t may vary over time, different cohorts of projects will face different expropriation risk.

Last, an important assumption regarding limited excludability is that the intangibles that create spillovers for new entrants are *only* those that have already been expropriated, as opposed to the *entire* stock of intangibles, including intangibles used by original project streams that have not yet been expropriated. This assumption sharply distinguishes our model from typical models of endogenous growth, where the entire stock of ideas is a public capital input. Clearly, this may be viewed as an extreme assumption; however, it is meant to capture the idea that there is a downside to strong intellectual property rights enforcement in that it may stifle innovation. In Section 2, we discuss a variant of the model in which the entire stock of intangibles is a public capital input — as in endogenous growth models —, but where entrepreneurs only retain ownership over a fraction of the total value of the project they create. Non-rivalry then tends to depress equilibrium growth. Assuming that the entire stock of intangibles creates spillovers for new projects, but spillover intangibles help more (perhaps because they are in the public domain) would deliver similar insights at the cost of less analytical tractability.

Other assumptions

We have made several additional assumptions, two of which are worth discussing in detail. First, we have introduced a fringe of imitators. The imitator fringe does not add to the stock of intangibles, so that they do not have a direct effect on growth. However, the imitators take over the expropriated product streams, and these streams continue generating output rather than being destroyed. Given that these streams employ labor, the existence of these imitators has an indirect (negative) effect on

growth: they reduce the amount of labor available for the creation of new projects. The assumption that these projects continue to generate output distinguishes expropriation from capital depreciation and implies that the entrepreneur captures less than the full value of her project. That said, a model in which these production streams were destroyed would deliver similar comparative statics—the fact that imitators and entrepreneurs compete for workers is not a key driver of our results.

Second, we have assumed that intangible capital is fixed within a project, and in particular, cannot be accumulated after date 0. Aggregate intangible capital grows over time, as a result of new cohorts of projects being created, so that the model can still speak to the behavior of aggregate investment. Moreover, the assumption of no exogenous depreciation is without loss of generality, as we show in the next section. The assumption that all new investment happens in the extensive margin is in line with models of putty-clay technology or models with vintage capital (see, e.g. [Kogan, Papanikolaou, and Stoffman, 2020](#), for an example of a model with a similar structure). We view this as a natural assumption in which the technology for storing intangibles is allowed to vary over time but is specific within a project. Drawing the analogy with models with vintage-specific technical change, an increase in ρ in our model has a similar effect: improvements in the technology for storing intangibles displaces existing projects in favor of newly-created and future projects. That said, we should emphasize the distinction between projects and firms. The model as described has no firms per se. Firms can be arbitrary collections of projects; with additional assumptions on the boundary of the firm, we could define firms in the model, but this need not affect the aggregate implications of the model.

1.3 Model extensions

Obsolescence

The baseline model assumes that, after its creation by an entrepreneur, a project never exits — though the surplus it generates is progressively reallocated from entrepreneurs to imitators. To relax this assumption, in [Appendix A.1.5](#), we describe a version of the model in which each project faces obsolescence risk.

Formally, we assume that in each period, the entire stock of intangibles used in a project — either by entrepreneurs or by imitators — becomes obsolete with Poisson intensity δ . This obsolescence risk is independent over time and across projects, and also independent of the expropriation risk which entrepreneurs face. In order to capture the idea that the intangibles associated with the project ceases to have productive value, we also assume that spillovers from a project that has become obsolete stop contributing to the aggregate spillover stock, S_t .

[Appendix A.1.5](#) discusses how these assumptions change the model. There are two main

differences. First, the value of entry for prospective entrepreneurs now reflects obsolescence risk:

$$\tilde{v}_t^{(e)}(x) \equiv \mathbb{E}_t \left[\int_t^\infty e^{-(r+\delta+\gamma(x))(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right]. \quad (33)$$

Thus compared to the baseline model, obsolescence risk reduces the incentives to enter. Second, the aggregate spillover stock now evolves according to:

$$S_t = \int_{\tau \leq t} (1 - e^{-\lambda_\tau(t-\tau)})^{1-\rho_\tau} (\nu S_\tau) e^{-\delta(t-\tau)} L_{e,\tau} d\tau, \quad (34)$$

reflecting the fact that the measure of projects from each cohort that still contribute to current spillovers now declines at the geometric rate δ .

Intuitively, obsolescence risk could capture the idea that some of the storage media used for intangibles may decay over time. This applies both to intangibles stored using physical media — such as data, software, or patents —, but perhaps most importantly to intangibles primarily stored using firm-specific human capital, such as managerial know-how or firm-specific processes and production systems. Section 2.3 discusses in more detail how this modification of the model affects equilibrium growth.

2 Balanced growth path

In this section, we discuss the properties of the balanced growth path of the model. We assume that the parameter ρ that governs the storage technology for intangibles is constant,

$$\forall t, \quad \rho_t = \rho, \quad \rho \in [0, 1].$$

To keep the exposition concise, we relegate all proofs to Appendix A.2.

2.1 Equilibrium growth in the baseline model

We start by clarifying the forces that determine equilibrium growth, and discuss their dependence on the degree of non-rivalry (ρ) and the limits to excludability (the shape of the function $\gamma(x)$).

Lemma 1 (Balanced growth path). *A balanced growth path (BGP) is an equilibrium in which $L_{e,t}, L_{p,t}, x_t$, and $v_t^{(e)}$ are constant while Λ_t, W_t , and S_t are growing at the constant rates. For all $\rho \in [0, 1]$, if $\nu \geq \frac{\zeta}{1-\zeta} (r + \underline{\lambda}(\rho))$, then there exists a unique BGP, in which Λ_t, W_t and S_t are growing at constant rates $\zeta g, (1-\zeta)g$, and g , respectively, where $g > 0$.*

The proof of this lemma is reported in Appendix A.2.1, which also gives the expression for the reduced-form parameter $\underline{\lambda}(\rho)$ as a function of other structural parameters.⁵ In what follows we give a heuristic description of the forces that help determine growth on the BGP.

We start by characterizing the law of motion for the change in the aggregate stock of spillovers. Appendix A.1.2 contains the proof of the following Lemma.

Lemma 2 (Evolution of spillovers). *Assume that $\rho_t < 1$ for all t . Then the change in the stock of spillover intangibles is given by:*

$$dS_t = (dt) \int_{\tau \leq t} \left(1 - e^{-\lambda_\tau(t-\tau)}\right)^{-\rho_\tau} (1 - \rho_\tau) \lambda_\tau e^{-\lambda_\tau(t-\tau)} (\nu S_\tau) L_{e,\tau} d\tau, \quad (35)$$

Note that this lemma holds even when ρ_t is time-varying. It states that the change in S_t is a weighted average, across cohorts, of the rate of change in the spillovers created by each cohort. The weights in the average depend on $L_{e,\tau}$, which is the number of projects in cohort τ .⁶

Using Equation (35), and the fact that in a BGP entrepreneurial labor $L_{e,t}$ is constant, we see that, given a value for L_e , g is the (unique) solution to the following equation

$$g = \nu \underbrace{\frac{g}{\lambda} \mathcal{B}\left(\frac{g}{\lambda}, 2 - \rho\right)}_{n(g;\rho,\lambda)} L_e, \quad (36)$$

where, with some abuse of notation, we used $\lambda = \gamma(x)$ as the short-hand for the constant expropriation risk on the BGP, and where \mathcal{B} is the Beta function. Here, the function $n(g; \rho, \lambda)$ captures the return to entrepreneurial labor. Alternatively, since entrepreneurial labor builds on spillovers from existing projects, we can also think of the function n as capturing the intensity of spillovers on the BGP. For a given level of spillover intensity, n , Equation (36) can also be thought of as a demand curve for entrepreneurial labor: a higher rate of spillovers n reduces entrepreneurial labor needed to achieve a particular growth rate,

$$L_e^{(d)}(g; \underline{n}) = \frac{g}{n}. \quad (37)$$

Combining the free-entry condition (27) with the definition of the marginal revenue product of

⁵ $\underline{\lambda}(\rho)$ is the expropriation risk corresponding to zero growth, $\underline{\lambda}(\rho) = \gamma(\underline{x}(\rho))$, where $\underline{x}(\rho)\gamma'(\underline{x}(\rho)) - \rho\gamma(\underline{x}(\rho)) = \rho r$. In particular, $\underline{\lambda}(\rho)$ is independent of the value of ν , the upper bound on spillover intensity. In Appendix A.2, we show that $\underline{\lambda}(\rho)$ is strictly increasing with ρ , so that it is sufficient for the inequality in Lemma 1 to hold for $\rho = 1$.

⁶An additional implication of this lemma is that regardless of the process governing ρ_t , but so long as $\rho_t < 1$, then the change in the stock of spillovers is of order dt . By contrast, if $\rho_t = 1$, S_t need not have continuous sample paths but can potentially jump, because newly created projects immediately generated spillovers. We use this in Section 4 to construct the transitional dynamics of the model between steady-states.

intangibles within stream (9), we see that in a BGP:

$$d\Lambda_t = -\zeta g \Lambda_t dt. \quad (38)$$

Intuitively, for each project, the marginal revenue product of intangibles in each stream is falling because wages are rising. As newer projects with more intangibles are created, the demand for labor increases, while its supply is constant. In turn, this implies that the normalized value of a project is given by:

$$v = \frac{1}{r + \zeta g}. \quad (39)$$

Similarly, the normalized value of a project of size x to the entrepreneur equals

$$\tilde{v}^{(e)}(x) \equiv \frac{1}{r + \gamma(x) + \zeta g}. \quad (40)$$

The entrepreneur chooses the optimal span to maximize (19), which implies the following first-order condition on the BGP,

$$x \gamma'(x) \tilde{v}^{(e)}(x) = \rho. \quad (41)$$

The left-hand side of this expression captures the costs of higher span — more expropriation risk — while the right-hand side captures the benefits — a higher marginal revenue product of intangible capital, which increases with ρ .

The share of project value retained by the entrepreneur on the BGP equals

$$\theta = \frac{r + \zeta g}{r + \gamma(x) + \zeta g} = 1 - \rho \left(\frac{x \gamma'(x)}{\gamma(x)} \right)^{-1}. \quad (42)$$

This expression helps illustrate the effects of ρ and $\gamma(\cdot)$ on the optimal share of project retained by the entrepreneur. Keeping the elasticity of $\gamma(x)$ fixed, a higher degree of non-rivalry, ρ will tend to reduce the share of total project value accruing to the entrepreneur. This is because when ρ is higher, the entrepreneur generally chooses to operate at a larger span — which increases the total project value — but to keep a smaller share of that larger pie. This is optimal from the standpoint of a committed entrepreneur, but it changes the *relative* returns to entrepreneurial vs. production labor, and thus lowers the supply of entrepreneurial labor. Conversely, any factor that raises the elasticity of γ with respect to x will tend to increase the optimal share of the project retained by the entrepreneur. With a higher elasticity of γ with respect to x , the entrepreneur generally chooses a lower span, in order to avoid expropriation risk.

The supply of entrepreneurial labor is determined from the combination of the free-entry

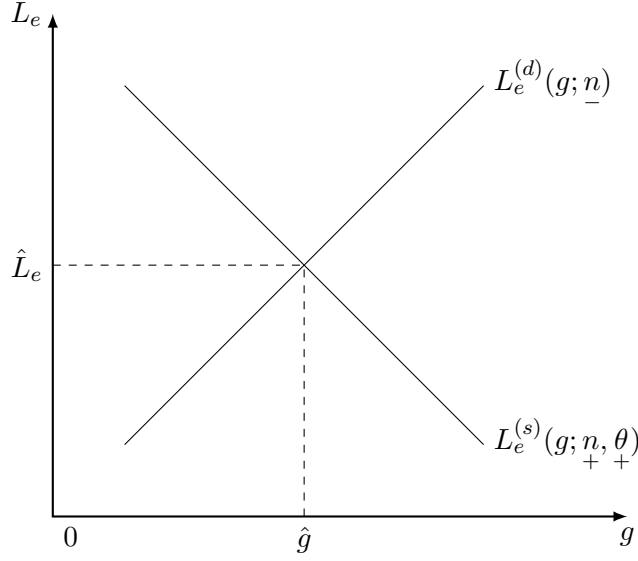


Figure 2: Determination of equilibrium growth rates. The entrepreneurial labor supply curve is described in Equation (43), and the labor demand curve is described in Equation (37).

condition, equation (27) — which pins down the opportunity cost of using labor in production — with the total labor used in production, equation (28). Together, in the BGP, these two conditions can be written as:

$$L_e^{(s)}(g; \underline{n}, \underline{\theta}) = 1 - \frac{\zeta}{1 - \zeta} \int_{\tau \leq t} \frac{S_\tau}{v S_t} d\tau = 1 - \frac{\zeta}{1 - \zeta} \frac{r + \zeta g}{n \theta}. \quad (43)$$

This equation captures the two forces determining the supply of entrepreneurial labor, specifically the ratio of the total stock of intangibles used in production (which determines its opportunity cost) and the average size of new projects (which determines the marginal return to entrepreneurship). The spillover intensity n increases the supply of entrepreneurial labor because, by making the stock of intangibles in the newest (marginal) project higher than in existing projects, it raises the relative returns to entrepreneurial labor.

Equation (43) describes the amount of entrepreneurial labor that is supplied at any given growth rate g . This labor supply schedule declines with growth rates, because higher growth rates imply higher wage growth, which in turn depresses the returns to projects. It increases with spillover intensity n , which raises the productivity of new projects relative to existing ones, and therefore the productivity of entrepreneurial relative to production labor. Finally, and importantly, the labor supply schedule depends positively on the share of project value that is retained by the entrepreneur. Intuitively, when the entrepreneur is able to retain a larger share of overall project value at inception, there is a stronger incentive to create new projects.

Figure 2 illustrates the determination of the equilibrium growth rate \hat{g} as the unique point at which the two schedules (37) and (43) intersect. The graph highlights that two main economic forces drive equilibrium growth in the model: the intensity of spillovers, n , and the fraction of overall project value that each entrepreneur retains, θ . Each has an opposing effect on equilibrium growth. Holding θ constant, higher spillovers reduce entrepreneurial labor demand and increase labor supply, unambiguously leading to higher equilibrium growth. By contrast, a lower share of project value retained by the entrepreneur θ reduces the supply of entrepreneurial labor, unambiguously leading to lower equilibrium growth rates.⁷

Both the intensity of spillovers n and the fraction of project value retained by the entrepreneur θ are related to the degree of non-rivalry ρ and the limits of excludability, as captured by the function $\lambda = \gamma(x)$. To illustrate the interaction between ρ and λ and how they shape our growth, we next consider two simple variants of the model. In each variant of the model only one of the two key mechanisms—spillover intensity n and the share of project value retained θ —depend on ρ and λ .

A version with $\theta = 1$

Consider first a model in which free-riding by the imitator fringe is eliminated. Assume that in order to enter, the imitator fringe must also spend a fixed amount of labor. Let the cost of entry in labor units be $1 - \kappa$ for imitators, and κ for entrepreneurs (they sum to 1 so as to keep this variant comparable to our baseline model). Eliminating the asymmetry between entrepreneurs and imitators leads to two respective free-entry conditions,

$$(1 - \kappa) W_t = \Lambda_t x^\rho \nu S_t (v - v^{(e)}), \quad \kappa W_t = \Lambda_t x^\rho \nu S_t v^{(e)}, \quad (44)$$

which combined imply the following labor supply schedule for total labor devoted to new projects (by both imitators and entrepreneurs)

$$L_n^{(s)}(g; n) = 1 - \frac{\zeta}{1 - \zeta} \frac{r + \zeta g}{n}. \quad (45)$$

This is identical to the supply equation (43), except that $\theta = 1$. By contrast, the labor demand equation is unchanged from (37). Thus equilibrium growth would be the same as in the baseline model if θ were constrained to be 1, that is, if entrepreneurs retained all project value.

⁷Note that this description is somewhat heuristic, since both n and θ are functions of the equilibrium growth rate, g . However, inspecting Equations (37) and (43), we see that so long as n and $n\theta$ are weakly decreasing functions of g , then the demand and supply schedules will be monotonic. The proof in Appendix A.2.2 establishes this.

Appendix A.2 shows that the function $n(g; \rho, \lambda)$ satisfies:

$$\frac{\partial n}{\partial \rho} \geq 0 \quad \text{and} \quad \frac{\partial n}{\partial \lambda} \geq 0. \quad (46)$$

Intuitively, as non-rivalry increases (higher ρ), spillovers accelerate; upon expropriation, imitators capture more of the stock of intangibles of existing projects. Likewise, when limits to excludability increase (higher λ), expropriation risk increases, so that spillover intensity rises. Thus, in this version of the model, higher ρ and higher λ are unambiguously positive for growth, as they only accelerate spillovers from old to new projects.

A version with constant spillover intensity

Next, consider a variant of the model in which the stock of intangible capital of new projects is instead given by:

$$N_t^e = \nu \int_{\tau \leq t} N_\tau L_{e,\tau} d\tau. \quad (47)$$

In this model, new projects, when they are born, have a size equal to a fraction ν of the *total* intangible capital invested in existing projects, and currently used by both imitators and entrepreneurs. Thus in this version of the model, intangibles are made freely available to new projects, so that non-rivalry and limits to excludability, while they continue to impact the relationship between existing entrepreneurs and imitators, have no bearing on spillovers to new entrepreneurs. In a balanced growth path, letting g now denote the growth rate of N_t^e , we have

$$n = \nu. \quad (48)$$

Thus in this variant of the model, spillover intensity n is independent of ρ and λ . Their only impact on equilibrium growth is through the effect on the distribution of project value between entrepreneurs and imitators, θ . Recalling the discussion around Figure 2, we can now conclude that any change in ρ or λ that raises θ will increase equilibrium growth, by making the incentive for entrepreneurs to enter stronger; conversely, any change in ρ or λ that lower θ will depress growth, by weakening the incentive for entrepreneurs to enter.

2.2 Comparative statics

The discussion above reveals that the comparative statics of spillover intensity n (which is increasing in ρ), and those of the share of project value retained by the entrepreneur (which is decreasing in ρ) suggest potentially opposing effects of non-rivalry ρ and degree of excludability λ on equilibrium

growth. To assess the relative strength of these effects, we next turn to explicit comparative statics of the model with respect to ρ and the shape of the function governing limits to excludability λ .

We begin by assuming a specific functional form for the limits to excludability,

$$\gamma(x) = \frac{(x-1)^{1+\xi}}{\eta(1+\xi)} \quad \text{where } \eta, \xi > 0. \quad (49)$$

Given the above, the first-order condition from the entrepreneur's optimal choice of span (41) implies that on the BGP,

$$\theta = 1 - \frac{\rho}{1+\xi} \frac{\hat{x}-1}{\hat{x}}. \quad (50)$$

The two equations above, together with (37) and (43) fully characterize the BGP.

The degree of non-rivalry ρ

We first consider the comparative statics with respect to the degree of non-rivalry, ρ . These comparative statics are reported in Figure 3. We use values for the remaining structural parameters, $(r, \zeta, \nu, \eta, \xi)$ that are set as follows. The values of $r = 0.07$ for the discount rate and $\zeta = 0.7$ for the labor share are standard. The value of $\nu = 0.7$ is chosen so that the growth rate in the version of the model with fully non-rival intangible, $\rho = 1$, is approximately 5%. Finally, we choose $\xi = 0.30$ and $\eta = 1/0.03$, but explore the impact of the parameters governing the shape of $\gamma(x)$ below.⁸

It is useful to start with two extreme cases: $\rho = 0$ and $\rho = 1$. When $\rho = 0$, there is no growth: $g = 0$. To see this, note that if $\rho = 0$ then there is no benefit to increasing span. Thus, entrepreneurs always choose the smallest possible value of $x = 1$, so as to retain as much of project value as possible. Given (49), the implied expropriation risk is $\lambda = \gamma(1) = 0$. So there are no spillovers, new projects do not enter, and there is no growth.

In the case of $\rho = 1$, the model closely resembles the Romer (1990) growth model. Any intangibles created by entrepreneurs immediately become available to imitators and future entrepreneurs, so that intangible capital is effectively a public capital good. The spillover intensity becomes constant, $n(g; 1, \lambda) = \nu$, regardless of the value of λ . This model is also similar to the second variant of the model discussed in Section 2.1, where the size of new projects is equal to a constant fraction ν of the total existing stock of intangibles. All of the comparative statics with respect to excludability are similar: higher excludability improves entry incentives and hence leads to higher growth.

⁸The functional form that we chose for $\gamma(x)$ implicitly normalizes the minimum span of a project to be $x = 1$, which will be chosen in equilibrium only if $\rho = 0$. A more general expropriation risk function with arbitrary minimum span is $(x - x_m)^\xi$ with $x_m > 0$; x_m will then be the span chosen by projects when $\rho = 0$.

In between the two extremes of $\rho = 0$ and $\rho = 1$, the top left panel of Figure 3 shows that the relationship between growth and non-rivalry ρ can be non-monotonic. The BGP growth rate peaks at degree of non-rivalry of approximately $\rho = 0.7$. Recall that:

$$g = nL_e. \tag{51}$$

The top middle and left panel report the comparative statics for entrepreneurial labor L_e and spillover intensity n . While the latter is increasing strictly with ρ , the former is non-monotonic.

Intuitively, higher ρ exacerbates the effects of expropriation risk. Recall that spillover intangibles from a project are given by:

$$S_{i,t} = \left(1 - e^{-\lambda(t-\tau(i))}\right)^{1-\rho} (\nu S_{\tau(i)}).$$

All else equal, spillovers are higher for larger values of ρ . When non-rivalry is higher, at any given rate of expropriation, imitators appropriate more of the overall stock of intangible of the project each time they expropriate a product stream, because more of the project's total intangibles are invested in each product stream. When spillover intensity is higher, the *relative* appeal of choosing entrepreneurship rises, because new projects will have a larger size relative to existing ones. Thus, a higher spillover intensity encourages prospective entrepreneurs to supply more labor.

The middle panels shows the opposing force. As non-rivalry increases, the incentive for entrepreneurs to choose a high span, x , increases. But, while they operate at a larger scale, expropriation risk λ also rises, so they also retain a small share of overall project value, θ . This is optimal from their perspective, but it lowers the *relative* returns to entrepreneurial labor, compared to production labor. Thus it lower the incentive for entrepreneurs to enter.

The bottom right panel summarizes the combination of the two effects — the stronger spillovers, and the weaker share of project value retained. The panel reports the product $nv^{(e)} = n\theta w$. This summarizes the total *relative* value of entry for a prospective entrepreneur. The value is non-monotonic, peaking at intermediate values of ρ .

Limits to excludability

We now turn to the comparative statics of the model with respect to η . Figure 4 reports the details of these comparative statics, for a specific degree of non-rivalry of intangibles of $\rho = 0.7$.

The middle panel on the second row of Figure 4 shows the relationship between η and the equilibrium risk of expropriation. When η is higher, the probability of expropriation for the project

risers.⁹ The parameter η thus captures limits to excludability.

Given the functional form in Equation (49) — which indicates that, for a given span x , expropriation risk $\lambda = \gamma(x)$ should *decline* as η increases —, this result might be surprising. The other panels in the second row of Figure 4 explain why. As η increases, entrepreneurs respond by choosing a larger span x . In turn, this leads to an increase in $\lambda = \gamma(x)$, despite the increase in η .

As for non-rivalry, the relationship between excludability and growth is non-monotonic: lower excludability initially increases BGP growth, but eventually reduces it. However, the mechanisms behind these comparative statics are somewhat different. In particular, the supply of entrepreneurial labor is strictly declining with the degree of excludability.

The main driver of this result is the optimal choice of span by the entrepreneur. As η increases, entrepreneurs respond by choosing a larger firm span but retaining a smaller overall share of the project. The smaller relative share of the project retained by committed entrepreneurs lowers the *relative* attractiveness of entrepreneurship, and thus leads to lower entrepreneurial labor supply. This is the effect that dominates the behavior of equilibrium growth when η is high.

An increase in η also intensifies spillovers across projects, by raising expropriation risk. This is shown in the top right panel of the top row of Figure 4. A higher spillover intensity is the effect that dominates the behavior of equilibrium growth rates when η is low.

Note that this non-monotonicity may not hold for all values of $\rho \in]0, 1[$. For instance, as $\rho \rightarrow 1$, spillover intensity becomes independent of expropriation risk, λ . In this case, the negative effects of η on entrepreneurial labor supply, and therefore growth, would dominate. On the other hand, as $\rho \rightarrow 0$, the choice of span becomes insensitive to ρ — there are no benefits to increasing span, so the entrepreneur seeks to minimize it, and would end up retaining a large share of the project's value. Thus any negative incentive effects would be small, and a higher value of η (that is, a lower degree of excludability) would likely increase equilibrium growth.

When is there an inverse-U shaped relationship between non-rivalry and growth?

Finally, we discuss whether the non-monotonic relationship between non-rivalry ρ and growth is robust to changes in parameter values. We focus on two specific parameters: ν , which captures the strength of the spillover channel; and η , which acts as a shifter in the degree of expropriation risk.

In Appendix A.2, we establish that, for any values of η and ν such that the BGP exists, $\frac{\partial g}{\partial \rho} \Big|_{\rho=0} > 0$. Thus a sufficient condition for non-rivalry ρ and growth to be non-monotonically related is: $\frac{\partial g}{\partial \rho} \Big|_{\rho=1} < 0$. In Appendix A.2, we show that this condition is also necessary, and moreover, characterize the values of η and ν for which it holds.

⁹By contrast, ξ captures the incremental expropriation risk associated with adding an additional product stream.

Figure 5 describes the corresponding partition of the parameter (ν, η) into three regions. The first region, labeled "No BGP", corresponds to parameter values for which the BGP does not exist at $\rho = 1$.¹⁰ The second region, labeled " $\hat{\rho} \in (0, 1)$ " shows the parameter region in which a BGP exists for all values of $\rho \in [0, 1]$, and where the value of ρ for which growth is maximized, $\hat{\rho}$, is interior. This corresponds to the baseline case discussed above. Finally, a third region, label $\hat{\rho} = 1$, corresponds to parameter values such that a BGP exists for all values of $\rho \in [0, 1]$, and BGP growth is strictly increasing with respect to ρ .

This graph highlights two main points. First, there is a region of values for η such that growth is *always* non-monotonically related to non-rivalry, regardless of the value of the spillover parameter ν . This regions corresponds to the values of η above the lower bound $\underline{\eta}$ marked on Figure 5. In this region, negative incentive effects always dominate spillover effects when $\rho = 1$. This corresponds to high values of η . When η is high, entrepreneurs choose a larger project span, x , at the expense of more expropriation risk. This choice is optimal conditional on becoming an entrepreneur, but since it also implies that the entrepreneur retains a lower overall share of the project, it makes entrepreneurship less attractive. When η is sufficiently high, the latter effect strictly dominates.

The second point is that if η is sufficiently low, the relationship between growth and non-rivalry can become strictly increasing. The intuition for this is the converse of the one described above. When η is sufficiently low, the entrepreneur optimally retains a larger share of the project (while operating with a lower span), making entrepreneurship more attractive, and thus raising entrepreneurial labor supply relative to production labor. However, the figure also shows that this effect dominates only when spillovers are sufficiently weak. Above a certain threshold for ν , spillovers become sufficiently strong to make entrepreneurship relatively unattractive.

Thus, the non-monotonic relationship between growth and non-rivalry holds for a wide range of parameter values for spillover intensity (ν) and limits to excludability (η), and in particular when excludability is weak (η is high) or when spillover intensity is high (ν is high).

2.3 Model extensions

Obsolescence risk The following lemma describes the conditions under which a balanced growth path with strictly positive growth exists in the model with obsolescence risk.

Lemma 3 (Balanced growth path with obsolescence risk). *Define the function*

$$H(\rho) \equiv n \left(\frac{\delta}{\underline{\lambda}(\rho)}, \rho \right) - \frac{\zeta}{1-\zeta} (r + \delta + \underline{\lambda}(\rho)) - \delta, \quad (52)$$

¹⁰Relative to Lemma 1, these are parameter values for which the condition $\nu \geq (\zeta/(1-\zeta))(r + \underline{\lambda}(1))$ is violated.

where, as in Equation (36), the function n is given by $n(y, z) = \nu y \mathcal{B}(y, 2 - z)$, with \mathcal{B} the Beta function. Then $\forall \rho \in [0, 1]$, there exists a unique balanced growth path of the model, if and only if,

$$H(\rho) \geq 0. \tag{53}$$

The proof is reported in Appendix A.2, which also gives the expression for the reduced-form parameter $\underline{\lambda}(\rho)$ as a function of other structural parameters in the model with obsolescence risk.¹¹ The main difference with the baseline model is that in this case, a BGP with strictly positive growth only exists when ρ is sufficiently large. Appendix Figure A1 reports a numerical example of the shape of the function $H(\rho)$, under the same calibration as the one used in Figure 3 for the baseline model. In this case, $H(\rho) > 0$ for all values of ρ above a strictly positive threshold satisfying $H(\bar{\rho})$. By contrast, no BGP with strictly positive growth exists when $\rho = 0$, for instance.

Intuitively, the reason is that, all other things equal, obsolescence risk reduces labor supply compared to the baseline model. In terms of the earlier discussion of this section, obsolescence risk lowers the present value of entry for prospective entrepreneurs, and therefore acts as a negative shift in the supply curve $L_e^{(s)}$. This effect can be sufficiently strong to make entry unprofitable at any positive level of growth. This will occur when the opposing force — the strength of spillovers, which governs the size of initial projects — is sufficiently weak, which in turn occurs when ρ is low.

Appendix Figure A2 compares the predictions of the model for equilibrium growth, to those of the baseline model. Besides the fact that a BGP with strictly positive growth only exists for sufficiently large values of ρ , the main difference is that overall growth (net) is lower in this model than in the baseline. In part, this reflects lower entry (that is, lower equilibrium values of L_e); but it also reflects weaker spillovers (that is, lower equilibrium values of n), which themselves arise from the fact that the aggregate stock of spillovers, S_t , grows more slowly, reflecting the progressive obsolescence of incumbent projects.

3 Implications For Macro Trends

This section discusses the implications of the model for the measurement of productivity growth, factor income shares and valuations, and concentration.

¹¹ $\underline{\lambda}(\rho)$ can still be interpreted as the expropriation risk corresponding to zero growth, which in this version of the model is given by $\underline{\lambda}(\rho) = \gamma(\underline{x}(\rho))$, where $\underline{x}(\rho)\gamma'(\underline{x}(\rho)) - \rho\gamma(\underline{x}(\rho)) = \rho(r + \delta)$.

3.1 Growth in Measured Productivity

We first ask what the implications of the model are for how to interpret the Solow residual, the most commonly used measure of total factor productivity. Since labor and intangibles are the only two factors of production in the model, we define the Solow residual as:

$$d \log(Z_t) \equiv d \log(Y_t) - s_{L,t} d \log(L_{p,t}) - (1 - s_{L,t}) d \log(K_{N,t}),$$

where Y_t is aggregate output, $L_{p,t}$ is aggregate labor used in production, $s_{L,t} \equiv W_t L_{p,t} / Y_t$ is the labor share, and $K_{N,t}$ is a measure of the productive intangible stock, to be defined below. Note that $L_{p,t}$ is constant on the BGP, so:

$$d \log(Z_t) = d \log(Y_t) - (1 - s_{L,t}) d \log(K_{N,t}).$$

Appendix A.1 shows that on the BGP, total output and the total wage bill are given by:

$$Y_t = \frac{\Lambda_t}{1 - \zeta} \int_{\tau \leq t} L_{e,\tau} x_\tau^{\rho_\tau} (\nu S_\tau) d\tau = \frac{\nu L_e x^\rho}{g(1 - \zeta)} \Lambda_t S_t, \quad W_t L_{p,t} = \zeta Y_t.$$

The labor share is therefore $s_{L,t} = \zeta$. Moreover, on the BGP, Y_t grows at rate $(1 - \zeta)g$.

Next, we define a replacement cost estimate of the productive intangible capital stock that is analog to measures used in national accounts. On the BGP, aggregate expenditure on intangibles, expressed in output units, are given by:

$$I_{N,t} = W_t L_e.$$

$I_{N,t}$, like W_t , grows at rate $(1 - \zeta)g$ on the BGP. We define $K_{N,t}$ as the perpetual inventory estimate of the capital stock associated with these expenditures. Let $\delta_N \geq 0$ be the rate used to capitalize these expenditures, and $\tilde{p}_{N,t}$ be the price index used to deflate them. We assume that on the BGP, $\tilde{p}_{N,t}$ is growing at a constant rate:

$$\frac{d\tilde{p}_{N,t}}{\tilde{p}_{N,t}} = g_{\tilde{p}}.$$

The law of motion for the estimate $K_{N,t}$ is:

$$dK_{N,t} = (1 - \delta_N dt) K_{N,t} + \frac{I_{N,t}}{\tilde{p}_{N,t}} dt.$$

Let the aggregate gross investment rate and the net growth rate of $K_{N,t}$ be defined as:

$$\iota_{N,t} \equiv \frac{I_{N,t}}{\tilde{p}_{N,t}K_{N,t}}, \quad g_{K,t} \equiv \frac{I_{N,t}}{\tilde{p}_{N,t}K_{N,t}}$$

Additionally, assume that $\iota_{N,t}$ and $g_{K,t}$ are constant on the BGP. Since $I_{N,t}$ grows at rate $(1 - \zeta)g$, we have:

$$(1 - \zeta)g = g_{\tilde{p}} + g_K \implies g_K = (1 - \zeta)g - g_{\tilde{p}}.$$

From the law of motion for $K_{N,t}$, the aggregate gross investment rate is then given by $\iota_N = (1 - \zeta)g - g_{\tilde{p}} + \delta_N$. Moreover, the Solow residual is given by:

$$\begin{aligned} d \log(Z_t) &\equiv d \log(Y_t) - (1 - \zeta)d \log(K_{N,t}) \\ &= (1 - \zeta)g - (1 - \zeta)((1 - \zeta)g - g_{\tilde{p}}) \\ &= (1 - \zeta)(\zeta g + g_{\tilde{p}}). \end{aligned}$$

Within the model, the correct measure of the price of intangibles is its replacement cost in output units, which is given by:

$$p_{N,t} = \frac{W_t}{\nu S_t} = \Lambda_t x^\rho v.$$

The first equality says that an entrepreneur must spend 1 unit of labor to create νS_t units of intangibles. The second equality follows from the free-entry condition. Since Λ_t is growing at rate $-\zeta g$, we have:

$$\tilde{p}_{N,t} = p_{N,t} \implies g_{\tilde{p}} = -\zeta g \implies d \log(Z_t) = 0.$$

Thus if $\tilde{p}_{N,t} = p_{N,t}$, the Solow residual will be exactly zero on the BGP.

On the other hand, if the measured price of intangibles, $\tilde{p}_{N,t}$, grows at a different rate from $p_{N,t}$, the Solow residual will not be zero. In this case, any change in structural parameters that affect trends growth, g , will pass through to the Solow residual. For instance, an increase in the degree of non-rivalry of intangibles would impact the Solow residual. The impact may be non-monotonic, with the Solow residual declining for sufficiently high degrees of non-rivalry, as in Section 2.

3.2 Tobin's Q for Intangibles and the profit share

The total value of existing projects (to both entrepreneurs and imitators) is given by:

$$V_{agg,t} = \Lambda_t x^\rho w \int_{\tau \leq t} L_{e,\tau}(\nu S_\tau) d\tau = \frac{p_{N,t}}{\theta} \frac{\nu L_e}{g} S_t.$$

Moreover, the total stock of intangibles is given by:

$$N_t = \int_{\tau \leq t} L_{e,\tau}(\nu S_\tau) d\tau = \frac{\nu L_e}{g} S_t.$$

Thus, aggregate Q is given by:

$$Q_t \equiv \frac{V_{agg,t}}{p_{N,t} N_t} = \frac{1}{\theta}$$

Aggregate Q is above 1 because imitators earn rents from expropriation. Note that, from the entrepreneurs' free-entry condition, we have:

$$Q_t^{(e)} \equiv \frac{V_{agg,t}^{(e)}}{p_{N,t} N_t} = 1,$$

where $V_{agg,t}^{(e)}$ is the value of existing projects retained by entrepreneurs:

$$V_{agg,t}^{(e)} = \Lambda_t x^\rho v \int_{\tau \leq t} L_{e,\tau} (e^{-\lambda(t-\tau)} \nu S_\tau) d\tau = p_{N,t} \frac{\nu L_e}{g} S_t.$$

For entrepreneurs only, $Q_t^{(e)}$ is one; all rents from entry flow to imitators.

This can also be seen by looking at factor income shares in the model, rather than valuation ratios. Define the user cost of intangible capital as:

$$R_{N,t} = r - \dot{p}_{N,t} = r + \zeta g = \frac{\theta}{v}.$$

(Note that this definition is consistent with the fact that intangibles do not depreciate, in this model.) Then on the BGP, we have:

$$\begin{aligned} Y_t - W_t L_{p,t} - R_{N,t} (p_{N,t} N_t) &= \frac{\nu L_e x^\rho}{(1-\zeta)g} \Lambda_t S_t - \frac{\nu L_e x^\rho \zeta}{(1-\zeta)g} \Lambda_t S_t - \frac{\theta}{v} (\Lambda_t x^\rho v) \frac{\nu L_e}{g} S_t \\ &= \frac{\nu L_e x^\rho}{g} \Lambda_t S_t (1-\theta). \end{aligned}$$

In other words, the pure profit share (that is, the share of total value added earned by owners of projects after competitive payments to capital) is exactly equal to $(1-\zeta)(1-\theta)$, and thus varies one-for-one with Total Q for intangibles, for a given value of ζ . Figure 3 shows the comparative statics of Total Q with respect to ρ . As non-rivalry increases, entrepreneurs choose to retain a smaller fraction of the overall project, increasing the share of rents per unit of capital. Finally,

Figure 3 reports imitators' share of total operating revenue, which is given by:

$$s_c = \frac{\int_{\tau \leq t} L_{e,\tau} \Pi_{i^c,\tau} d\tau}{\int_{\tau \leq t} L_{e,\tau} \Pi_{i,tot,\tau} d\tau} = \frac{1}{1 + \frac{g}{\lambda}}.$$

Consistent with the fact that ρ exacerbates the effects of limits to excludability, imitators' share of total operating revenue is increasing with the degree of non-rivalry of intangibles.

3.3 Concentration

Finally, we discuss the implications of the model for concentration. As discussed in Section 2, boundaries of the firm are not endogenously determined in this model. We use projects as our notion of firm boundaries to measure concentration.¹²

On the BGP, sales from a particular project, and aggregate sales, are given by:

$$Y_{i,t} = \frac{\Lambda_t}{1 - \zeta} x^\rho \nu S_{\tau(i)},$$

$$Y_t = \frac{\Lambda_t}{1 - \zeta} x^\rho \nu \frac{S_t}{n}.$$

Thus the sales share of a project is:

$$s_{i,t} = n e^{-g(t-\tau(i))}.$$

From this, the Herfindhal index of projects is:

$$H = \int_{\tau(i) \leq t} s_{i,t}^2 di = n^2 L_e \int_{\tau \leq t} e^{-2g(t-\tau)} d\tau = \frac{n}{2}.$$

Thus in the baseline model, the Herfindhal index of projects is exactly proportional to spillovers. The intuitive reason for this result is that when spillover intensity is higher, projects are born at a bigger size *relative to* previous cohorts. This tends to accentuate concentration.

One concern with this measure of concentration is that entrepreneurs and imitators may not directly compete in the same markets; indeed, imitation effectively expels entrepreneurs from specific product streams, to the benefit of competitors. Similar steps as above show that the Herfindhal

¹²Additionally, we do not distinguish between sales and value added in our model, so we use the two terms interchangeably in this section.

index of sales *among entrepreneurs* is given by:

$$H_e = \frac{n}{2} \left(1 + \frac{\lambda}{g} \right).$$

The additional term in the Herfindhal index for entrepreneurs captures the fact that the market share will also depend upon the rate at which entrepreneurs loose market share to the imitators. When equilibrium expropriation risk λ is low, entrepreneurs retain large market shares for a longer while, and so concentration within entrepreneurs is lower.

Figure 3 reports the two measures of concentration together, as a function of the degree of non-rivalry. Recall that we established in Section 2 established that spillover intensity is strictly increasing as a function of non-rivalry, ρ ; thus concentration among projects must monotonically increase with non-rivalry. Among entrepreneurs, the effect of non-rivalry is potentially ambiguous, since at low levels of non-rivalry, both λ and g are increasing. The example of Figure 3 suggests that the effects of rising expropriation risk tend to dominate, and that concentration increases monotonically with non-rivalry even among entrepreneurs.

Overall, we see how changes in the degree of non-rivalry in the model will produce a non-monotonic relationship between growth and concentration: the two grow in tandem when non-rivalry is low, but eventually diverge as intangibles approach complete non-rivalry.

4 Transitional dynamics

In this section, we study the transitional dynamics of the model between two balanced growth paths with different degrees of non-rivalry of intangible capital ρ . We show that after a change in ρ , the response of growth can be non-monotonic, analogously to the non-monotone comparative statics highlighted in Section 2.

4.1 Definition

Consider an exogenous and deterministic process for the degree of non-rivalry, $\{\rho_t\}$, that satisfies:

$$\forall t < 0, \quad \rho_t = \underline{\rho} \quad \text{and} \quad \lim_{t \rightarrow +\infty} \rho_t = \bar{\rho}, \quad (54)$$

where $0 \leq \underline{\rho}, \bar{\rho} < 1$, and, for all t ,

$$\nu \geq \frac{\zeta}{1 - \zeta} (r + \lambda(\rho_t)). \quad (55)$$

We define the transition path associated with this process for ρ as follows.

Definition 2 (Transition path). *The transition path associated with $\{\rho_t\}$ is the set of processes $\{L_{e,t}, L_{p,t}, \Lambda_t, W_t, v_t^{(e)}, x_t, S_t\}$ such that:*

1. $\forall t < 0, \{L_{e,t}, L_{p,t}, \Lambda_t, W_t, v_t^{(e)}, x_t, S_t\}$ are consistent with the balanced growth path associated with $\underline{\rho}$, with the normalization $S_0 = 1$;
2. $\forall t \geq 0, \{L_{e,t}, L_{p,t}, \Lambda_t, W_t, v_t^{(e)}, x_t, S_t\}$ satisfy equations (9) - (30).

In words, when $t < 0$, the economy is on the balanced growth path associated with $\underline{\rho}$; in particular, agents do not expect the degree of non-rivalry to change. At time $t = 0$, ρ_t can then jump, but its path for $t \geq 0$ is deterministic and known to all agents in the economy. The economy adjusts according to equations (9) - (30), but taking as given the existing history of spillovers $\{S_t\}_{t \leq 0}$ generated by the initial BGP.

Details of the computation of transitional dynamics are reported in Appendix A.3. The model contains both backward-looking equations (such as the law of motion for spillovers) and forward-looking equations (such as the value of new projects), and moreover, the state-space of the model consists of the entire history of spillovers across projects. We approximate the transitional dynamics using a combination of backward iteration and forward propagation for key model equations, and truncating the corresponding integral beyond a fixed horizon T , which we choose to be large relative to the horizon at which we analyze the transition path.

4.2 Baseline model

Figure 6 reports a numerical example of transitional dynamics in the baseline model. The calibration is identical to the one used for the comparative statics of Figure 3, except that we assume that the process for ρ_t is:

$$\forall t < 0, \quad \rho_t = \underline{\rho} \quad \text{and} \quad \forall t \geq 0, \quad \rho - t = \bar{\rho}, \quad (56)$$

and choose values of $\underline{\rho} = 0.70$ and $\bar{\rho} = 0.72$.

The top left panel of Figure 6 shows the path of the growth rate of the economy after this change. After an initial upward jump, growth peaks at around 2 years after the change in ρ_t . After that it declines, reaching the pre-shock level of growth about 11 years after the shock, and continuing to decline after that. Thus while the increase in non-rivalry initially boosts growth, this effect eventually fades, and growth declines below the pre-BGP level, indicated with a hollow gray marker on Figure (6).

The top right panel shows the amount of labor spent on the creation of new projects (or equivalently, the number of new projects created) along the transition path. Following a short-run

increase (immediately after the shock), entry declines, eventually to a value that is below the pre-shock level.

The other panels in the figure help clarify the forces at play behind the non-monotonic response of overall growth and the declining response of entry. As ρ jumps from $\underline{\rho} = 0.70$ to $\bar{\rho} = 0.72$, new projects enter with a higher span, all other things equal, than they had been on the pre-shock BGP, as shown on the right panel of the third row of Figure 6. As a result, future profits increase, and so the value of new projects increases, as shown on the right panel of the second row of Figure 6. Entry temporarily becomes more attractive, relative to production labor, as the short-run increase in entrepreneurial labor on the top right panel of Figure 6 shows. This temporarily pushes production labor wages down, as shown on the left panel of the third row of Figure 6.¹³ However, as more new projects with bigger span are progressively created, demand for production labor progressively increases. This eventually pushes up wages. Higher wages then erode the value of new projects, leading to reduced entry and, eventually, a rate of growth that falls below its pre-shock value.

It is important to note that the negative long-run growth effects of the increase in ρ happen despite the fact that higher ρ accelerates spillovers along the transition path, at least initially. This is visible in the right panel on the third row of Figure 6. This panel reports the deviation from trend of N_t , the size of new projects, along the transition path.¹⁴ N_t increases relative to trend, and persistently so: in fact, it remains above trend for the entirety of the twenty-year window reported on Figure 6. Thus, the decline in growth reflects the fact that in this calibration, the increase in production labor demand and the erosion of entry surplus through higher wages is stronger than the increase in spillovers.

This need not be true for all calibrations. Figure 7 reports the transitional dynamics associated with an increase in ρ from $\underline{\rho} = 0.50$ to $\bar{\rho} = 0.52$. In this case, the growth rate of the economy increases permanently above its pre-shock level, as shown on the top left panel of the Figure. There is a short-term overshoot in growth, driven by temporarily elevated entrepreneurial labor supply. But new projects add less to the overall demand for production labor than in the calibration reported on Figure 6, and so the eventual increase in wages relative to trend, while still present, is insufficiently large to offset the increased strength of spillovers.

The bottom left panel of Figure 6 reports the path of aggregate Tobin's Q , which is given by the aggregate value of existing projects divided by the replacement cost of the existing stock of

¹³This panel reports deviations of wages relative to their pre-shock trend; on the transition path, wages do not actually decline, but for the first 3 years after the shock, they increase less rapidly than along the BGP.

¹⁴Note that there is no immediate jump in N_t because, as explained in Lemma 35, the change in S_t is of order dt so long as $\rho_t < 1$.

intangibles.¹⁵ Contrary to Tobin’s Q for new projects, which is reported on the right panel of the second line of Figure, aggregate Tobin’s Q , after an initial decline, increases throughout the transition path. The initial decline is caused by the fact that the future expected rise in wages depresses the value of existing projects; the subsequent increase is caused by the progressive entry of new, higher-value projects in the economy. Thus, along the transition path, the model can simultaneously produce a decline in growth rates along with an increase in aggregate Tobin’s Q in the wake of a technology shock that affects the ease with which intangibles can be replicated within and across firms. Relatedly, the bottom right panel of Figure 6 reports the dynamics of the pure profit share along the transition path. The pure profit share progressively increases after the shock, as the distribution of surplus between entrepreneurs and imitators is fixed for existing projects, but shifts towards imitators for new projects.

5 Conclusion

Intangibles are particular types of capital inputs that differ from physical assets in one important way: they may be partly non-rival within the firm. The main contribution of this paper is to show the degree of non-rivalry of intangibles has an ambiguous effect on growth. While non-rivalry increases spillovers from existing to new firms, it also increases the risk that firms’ intangibles will be copied or appropriated by competitor. The former effect stimulates entry, but the latter inhibits it. The model thus sheds light on the macroeconomic implications of the technological changes in the way that intangible assets are codified, and which determines their degree of non-rivalry. Concrete examples include innovations in information technology used to store data-related intangibles, or managerial innovations used to create organization capital and communicate it to employees.

The paper leaves at least three questions unanswered. First, we have only compared steady-states, but the model can also speak to transitional dynamics between balanced growth paths, as the technological or legal environment around intangibles changes. The transitional dynamics may be non-monotonic, with spillovers dominating in the short-run but disincentive effects dominating in the long-run. Second, equilibria in this economy are generically inefficient because current entrants do not internalize the effect of their investment decisions on future ones; thus the model could speak to questions of industrial policy. Third, some of the first-order conditions of the model suggest ways of estimating directly the value of ρ , the degree of non-rivalry, and the value of λ , the equilibrium degree of competitive risk, in the data. We leave these to future research.

¹⁵Aggregate and project-level Tobin’s Q are defined in more detail in Section 3.

References

- Aghion, P., A. Bergeaud, T. Boppart, P. J. Klenow, and H. Li (2022). A Theory of Falling Growth and Rising Rents.
- Aghion, P., N. Bloom, R. Blundell, R. Griffith, and P. Howitt (2005, 05). Competition and Innovation: an Inverted-U Relationship. The Quarterly Journal of Economics 120(2), 701–728.
- Aghion, P. and P. Howitt (1992). A Model of Growth Through Creative Destruction. Econometrica 60(2), 323–351.
- Akcigit, U. and S. T. Ates (2019). What Happened to U.S. Business Dynamism?
- Argente, D., S. Moreira, E. Oberfield, and V. Venkateswaran (2020). Scalable Expertise.
- Atkeson, A. and P. J. Kehoe (2005). Modeling and Measuring Organization Capital. Journal of Political Economy 113(5), 1026–1053.
- Bhandari, A. and E. R. McGrattan (2021). Sweat Equity in US Private Business. The Quarterly Journal of Economics 136(2), 727–781.
- Bloom, N. and J. Van Reenen (2007). Measuring and Explaining Management Practices Across Firms and Countries. The Quarterly Journal of Economics 122(4), 1351–1408.
- Bresnahan, T. F., E. Brynjolfsson, and L. M. Hitt (2002). Information Technology, Workplace Organization, and the Demand for Skilled Labor: Firm-Level Evidence. The Quarterly Journal of Economics 117(1), 339–376.
- Corrado, C., C. Hulten, and D. Sichel (2005). Measuring Capital and Technology: an Expanded Framework. In Measuring Capital in the New Economy, pp. 11–46. University of Chicago Press.
- Crouzet, N. and J. Eberly (2019). Understanding Weak Capital Investment: the Role of Market Power and Intangibles. 2018 Jackson Hole Symposium, Federal Reserve Bank of Kansas City.
- Crouzet, N. and J. Eberly (2021). Rents and Intangible Capital: A Q+ Framework. The Journal of Finance.
- Crouzet, N., J. C. Eberly, A. L. Eisfeldt, and D. Papanikolaou (2022). The Economics of Intangible Capital. Journal of Economic Perspectives 36(3), 29–52.
- De Ridder, M. (2022). Market Power and Innovation in the Intangible Economy.

- Ding, X., T. C. Fort, S. J. Redding, and P. K. Schott (2022). Structural Change Within Versus Across Firms: Evidence from the United States.
- Eisfeldt, A. L. and D. Papanikolaou (2013). Organization Capital and the Cross-Section of Expected Returns. The Journal of Finance 68(4), 1365–1406.
- Eisfeldt, A. L. and D. Papanikolaou (2014). The Value and Ownership of Intangible Capital. American Economic Review 104(5), 189–94.
- Farboodi, M. and L. Veldkamp (2020). Long-run Growth of Financial Data Technology. American Economic Review 110(8), 2485–2523.
- Gourio, F. and L. Rudanko (2014). Customer Capital. Review of Economic Studies 81(3), 1102–1136.
- Hall, B. H., A. Jaffe, and M. Trajtenberg (2005). Market Value and Patent Citations. The RAND Journal of Economics 36(1), 16–38.
- Haskel, J. and S. Westlake (2017). Capitalism Without Capital. Princeton University Press.
- Hsieh, C.-T. and E. Rossi-Hansberg (2022). The Industrial Revolution in Services.
- Jones, C. I. (1995). R&D-based Models of Economic Growth. Journal of Political Economy 103(4), 759–784.
- Jones, C. I. and C. Tonetti (2020). Nonrivalry and the Economics of Data. The American Economic Review, 2819–58.
- Kelly, B., D. Papanikolaou, A. Seru, and M. Taddy (2021). Measuring Technological Innovation Over the Long Run. American Economic Review: Insights 3(3), 303–20.
- Klette, T. and S. Kortum (2004). Innovating Firms and Aggregate Innovation. Journal of Political Economy 112(5), 986–1018.
- Kogan, L., D. Papanikolaou, A. Seru, and N. Stoffman (2017, 03). Technological Innovation, Resource Allocation, and Growth. The Quarterly Journal of Economics 132(2), 665–712.
- Kogan, L., D. Papanikolaou, and N. Stoffman (2020). Left Behind: Creative Destruction, Inequality, and the Stock Market. Journal of Political Economy 128(3), 855–906.
- Lucas, R. E. and B. Moll (2014). Knowledge Growth and the Allocation of Time. Journal of Political Economy 122(1), 1–51.

- McGrattan, E. R. and E. C. Prescott (2009). Openness, Technology Capital, and Development. Journal of Economic Theory 144(6), 2454–2476.
- McGrattan, E. R. and E. C. Prescott (2010a). Technology Capital and the US Current Account. American Economic Review 100(4), 1493–1522.
- McGrattan, E. R. and E. C. Prescott (2010b). Unmeasured Investment and the Puzzling US Boom in the 1990s. American Economic Journal: Macroeconomics 2(4), 88–123.
- Perla, J. and C. Tonetti (2014). Equilibrium Imitation and Growth. Journal of Political Economy 122(1), 52–76.
- Perla, J., C. Tonetti, and M. E. Waugh (2021, January). Equilibrium Technology Diffusion, Trade, and Growth. American Economic Review 111(1), 73–128.
- Romer, P. M. (1986). Increasing Returns and Long-run Growth. Journal of Political economy 94(5), 1002–1037.
- Romer, P. M. (1990). Endogenous Technological Change. Journal of Political Economy 98(5, Part 2), S71–S102.
- Seegerstrom, P. (1998, December). Endogenous Growth without Scale Effects. American Economic Review 88(5), 1290–1310.
- Solow, R. (1956). A Contribution to the Theory of Economic Growth. The Quarterly Journal of Economics 70(1), 65–94.
- Stokey, N. L. (2015). Catching up and falling behind. Journal of Economic Growth 20(1), 1–36.

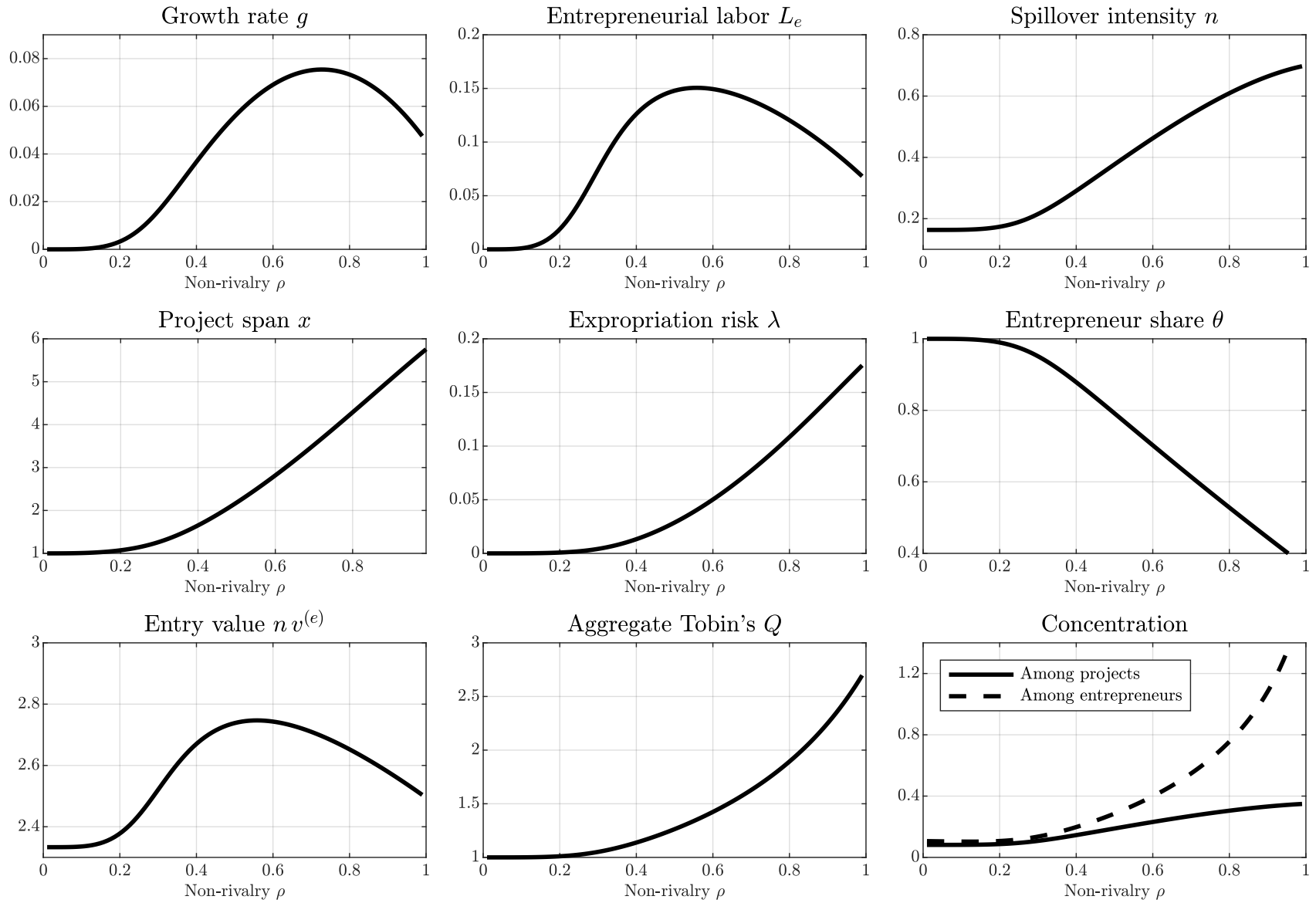


Figure 3: Comparative statics of the balanced growth path with respect to the degree of non-rivalry, ρ . The expropriation risk function is assumed to be $\gamma(x) = \frac{(x-1)^{1+\xi}}{\eta(1+\xi)}$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\eta = 1/0.03$, and $\xi = 0.3$.

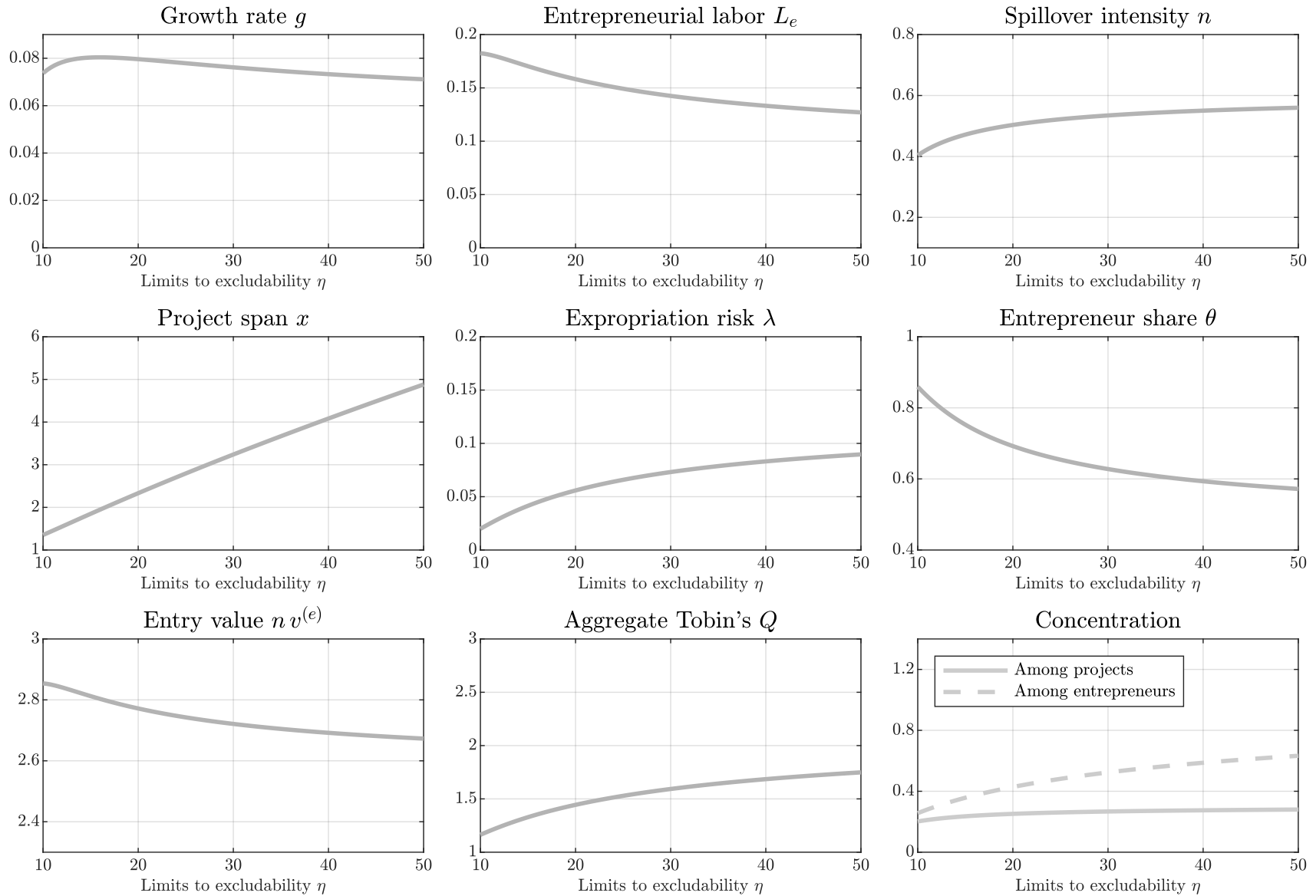


Figure 4: Comparative statics of the balanced growth path with respect to limits to excludability, η . The expropriation risk function is assumed to be $\gamma(x) = \frac{(x-1)^{1+\xi}}{\eta(1+\xi)}$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\rho = 0.7$, and $\xi = 0.3$.

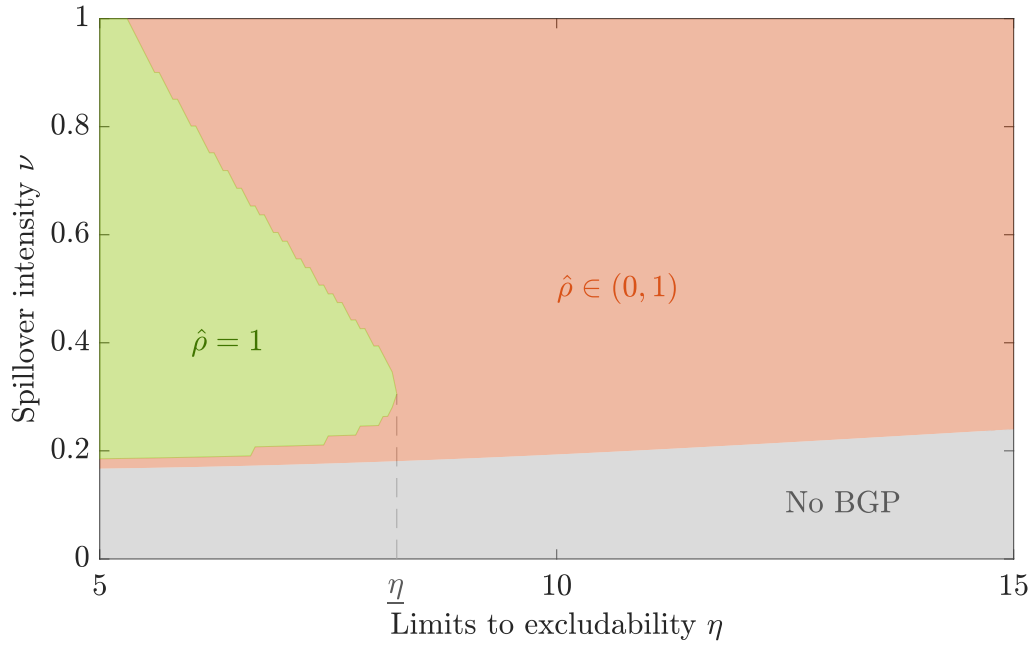


Figure 5: Parameter values for which the relationship between growth and non-rivalry is non-monotonic. The horizontal axis has values of η , the limits to excludability, with higher η corresponding to higher equilibrium expropriation risk. The vertical axis has value of ν , the maximum spillover intensity, which is also the spillover intensity when $\rho = 1$. The region labeled "No BGP" corresponds to values of (ν, η) for which the model has no BGP when $\rho = 1$. The region labeled " $\hat{\rho} \in (0, 1)$ " corresponds to values of (ν, η) for which the relationship between growth and non-rivalry is non-monotonic. The region labeled " $\hat{\rho} = 1$ " corresponds to values of (ν, η) for which relationship between growth and non-rivalry is strictly increasing.

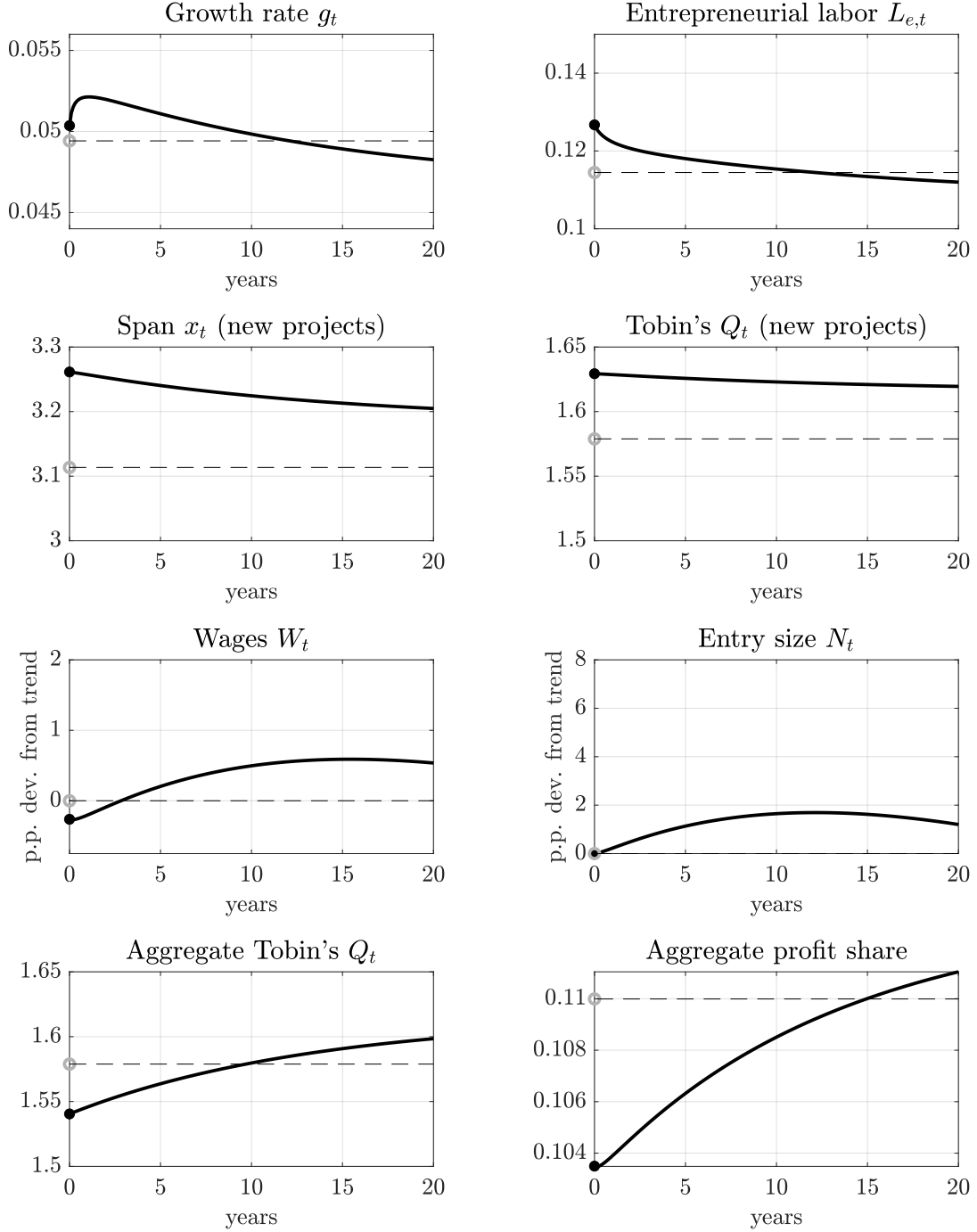


Figure 6: Transitional dynamics after a shock to the degree of non-rivalry, ρ . At time $t < 0$, the economy on the balanced growth path with $\rho_t = \underline{\rho} = 0.70$. At time $t = 0$, ρ_t unexpectedly jumps to $\rho_t = \bar{\rho} = 0.72$, and remains at this value afterwards. We use the model with no obsolescence risk, $\delta = 0$. The expropriation risk function is assumed to be $\gamma(x) = ((x - 1)^{1+\xi})/(\eta(1 + \xi))$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\eta = 1/0.03$, and $\xi = 0.3$. In all graphs, the horizontal lines indicate pre-shock values. The graph for wages is expressed in log-point deviation from the pre-shock balanced growth path, that is, $100 \times \log(W_t/\tilde{W}_t)$, with $\tilde{W}_t = W_0 e^{\underline{g}(1-\zeta)t}$, and \underline{g} the pre-shock BGP growth rate. The graph for the initial size of projects is constructed similarly. Computational details are described in Appendix A.3; we use a grid with a time step of $\Delta = 1/10$ of a year. Dashed lines indicate pre-shock values.

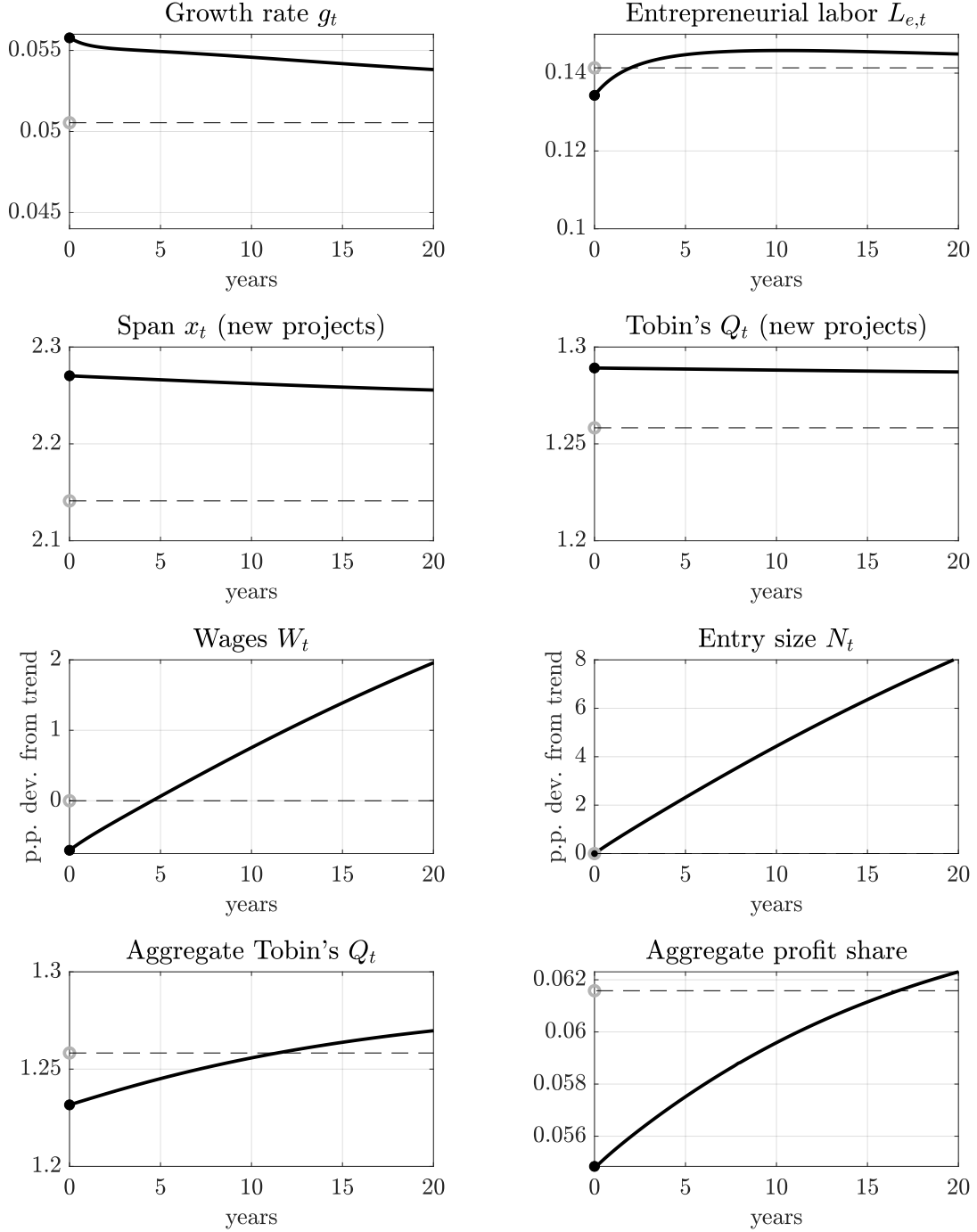


Figure 7: Transitional dynamics after a shock to the degree of non-rivalry, ρ . At time $t < 0$, the economy on the balanced growth path with $\rho_t = \underline{\rho} = 0.50$. At time $t = 0$, ρ_t unexpectedly jumps to $\rho_t = \bar{\rho} = 0.52$, and remains at this value afterwards. We use the model with no obsolescence risk, $\delta = 0$. The expropriation risk function is assumed to be $\gamma(x) = ((x - 1)^{1+\xi})/(\eta(1 + \xi))$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\eta = 1/0.03$, and $\xi = 0.3$. In all graphs, the horizontal lines indicate pre-shock values. The graph for wages is expressed in log-point deviation from the pre-shock balanced growth path, that is, $100 \times \log(W_t/\tilde{W}_t)$, with $\tilde{W}_t = W_0 e^{\underline{g}(1-\zeta)t}$, and \underline{g} the pre-shock BGP growth rate. The graph for the initial size of projects is constructed similarly. Computational details are described in Appendix A.3; we use a grid with a time step of $\Delta = 1/10$ of a year. Dashed lines indicate pre-shock values.

Appendix

A.1 Appendix to Section 1

For brevity, in this appendix we use the notation:

$$x_i = x_{i,\tau(i)} \quad (57)$$

to refer to the initial value of the span of project i , which is the same for all projects born at the same date $\tau(i)$.

A.1.1 Additional derivations

Law of motion for $N_{i,t}$. Given that the probability of expropriation is independent across streams and constant over time, the number of streams follows:

$$x_{i,t} = x_i e^{-\lambda_i(t-\tau(i))},$$

Aggregating across streams, total intangibles in use in the project follow:

$$\begin{aligned} N_{i,t+dt} &= \left(\int_0^{x_{i,t}+dx_{i,t}} N_{i,t}(s)^{\frac{1}{1-\rho_i}} di \right)^{1-\rho_i} \\ &= \left(\int_0^{x_{i,t}+dx_{i,t}} x_{i,t}^{-1} N_{i,t}^{\frac{1}{1-\rho_i}} di \right)^{1-\rho_i} \\ &= N_{i,t} \left(1 + \frac{dx_{i,t}}{x_{i,t}} \right)^{1-\rho_i}. \end{aligned}$$

Letting $\hat{N}_{i,t} = N_{i,t}^{\frac{1}{1-\rho}}$, the equation above implies that:

$$\frac{d\hat{N}_{i,t}}{\hat{N}_{i,t}} = \frac{dx_{i,t}}{x_{i,t}}.$$

Thus,

$$\frac{\hat{N}_{i,t}}{\hat{N}_{i,\tau(i)}} = \frac{x_{i,t}}{x_i}$$

Substituting $\hat{N}_{i,t}$ for $N_{i,t}$ and using the law of motion for $x_{i,t}$ above gives the result. ■

A.1.2 Proofs

Proof of Lemma 2. Assume that $\rho_t < 1$ for all t . Then:

$$S_{t+dt} = \underbrace{0}_{\text{spillovers from projects started at } t+dt} + \underbrace{\nu \int_{\tau(i) \leq t} S_{\tau(i)} \left(1 - e^{-\lambda_i(t+dt-\tau(i))}\right)^{1-\rho_i} di}_{\text{spillovers from existing projects}}$$

To see why there are no immediate spillovers from new projects, recall that spillovers from an individual project are given by:

$$S_{i,t} = \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} \nu S_{\tau(i)},$$

which is zero when $t = \tau(i)$, i.e. when the project is just created, except if $\rho_i = 1$. Using the expression above, we get:

$$\begin{aligned} dS_t &= \nu \int_{\tau(i) \leq t} S_{\tau(i)} \left[\left(1 - e^{-\lambda_i(t+dt-\tau(i))}\right)^{1-\rho_i} - \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} \right] di \\ &= \nu \int_{\tau(i) \leq t} S_{\tau(i)} \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} \left[\left(\frac{1 - e^{-\lambda_i(t+dt-\tau(i))}}{1 - e^{-\lambda_i(t-\tau(i))}}\right)^{1-\rho_i} - 1 \right] di \\ &= \nu \int_{\tau(i) \leq t} S_{\tau(i)} \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} \left[\left(1 + \frac{e^{-\lambda_i(t-\tau(i))}}{1 - e^{-\lambda_i(t-\tau(i))}} \lambda_i dt\right)^{1-\rho_i} - 1 \right] di \\ &= (\nu dt) \int_{\tau(i) \leq t} S_{\tau(i)} \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{-\rho_i} e^{-\lambda_i(t-\tau(i))} (1 - \rho_i) \lambda_i di \\ &= (\nu dt) \int_{\tau \leq t} S_{\tau} L_{e,\tau} (1 - \rho_{\tau}) \lambda_{\tau} \left(1 - e^{-\lambda_{\tau}(t-\tau)}\right)^{-\rho_{\tau}} e^{-\lambda_{\tau}(t-\tau)} d\tau. \end{aligned}$$

In the last line, we used the fact that $L_{e,\tau}$ projects get created in each cohort τ . We also used the fact that all project starting at the same date choose the same span x_t , and are therefore exposed to the same degree of expropriation risk $\lambda_t = \gamma(x_t)$. This establishes the result. \blacksquare

A.1.3 List of equilibrium conditions

An equilibrium of this model is a set of processes for $\{\Lambda_t, W_t, x_t, v_t^{(e)}, L_{e,t}, S_t, L_{p,t}, \lambda_t\}$ that satisfy:

$$\Lambda_t = (1 - \zeta) \left(\frac{\zeta}{W_t}\right)^{\frac{\zeta}{1-\zeta}} \quad (58)$$

$$x_t = \arg \max_{x \geq 1} x^{\rho_t} \tilde{v}_t^{(e)}(x) \quad (59)$$

$$v_t^{(e)} = \tilde{v}_t^{(e)}(x_t) \quad (60)$$

$$0 = L_{e,t} \left(W_t - \Lambda_t x_t^{\rho_t} (\nu S_t) v_t^{(e)} \right) \quad (61)$$

$$0 \leq L_{e,t} \quad (62)$$

$$0 \leq W_t - \Lambda_t x_t^{\rho_t} (\nu S_t) v_t^{(e)} \quad (63)$$

$$L_{p,t} = \left(\frac{\Lambda_t}{1 - \zeta} \right)^{\frac{1}{\zeta}} \int_{\tau \leq t} x_{\tau}^{\rho_{\tau}} (\nu S_{\tau}) L_{e,\tau} d\tau \quad (64)$$

$$L_{p,t} + L_{e,t} \leq 1 \quad (65)$$

$$S_t = \int_{\tau \leq t} (1 - e^{-\lambda_{\tau}(t-\tau)})^{1-\rho_{\tau}} (\nu S_{\tau}) L_{e,\tau} d\tau \quad (66)$$

$$\tilde{v}_t^{(e)}(x) = \mathbb{E}_t \left[\int_t^{\infty} e^{-(r+\gamma(x))(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right] \quad (67)$$

$$\lambda_t = \gamma(x_t) \quad (68)$$

The aggregate stock of intangibles, N_t , and aggregate value added, Y_t , are given by:

$$N_t = \int_{\tau(i) \leq t} N_{i,t} di = \int_{\tau \leq t} (\nu S_{\tau}) L_{e,\tau} d\tau \quad (69)$$

$$Y_t = \int_{\tau(i) \leq t} Y_{i,t} di = \frac{\Lambda_t}{1 - \zeta} \int_{\tau \leq t} x_{\tau}^{\rho_{\tau}} (\nu S_{\tau}) L_{e,\tau} d\tau \quad (70)$$

A.1.4 Evolution of project-level variables

In order to distinguish project-level variables for entrepreneurs from those of the imitator fringe, we index the former using $^{(e)}$, and the latter using $^{(m)}$. Project-wide variables are without superscript.

$$x_{i,t}^{(e)} = e^{-\lambda_i(t-\tau(i))} x_{\tau(i)}$$

$$N_{i,t}^{(e)} = e^{-\lambda_i(1-\rho_i)(t-\tau(i))} (\nu S_{\tau(i)})$$

$$Y_{i,t}^{(e)} = \frac{\Lambda_t}{1 - \zeta} e^{-\lambda_i(t-\tau(i))} x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)})$$

$$\begin{aligned}
L_{p,i,t}^{(e)} &= \left(\frac{\Lambda_t}{1-\zeta} \right)^{\frac{1}{\zeta}} e^{-\lambda_i(t-\tau(i))} x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
\Pi_{i,t}^{(e)} &= \Lambda_t e^{-\lambda_i(t-\tau(i))} x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
V_{i,t}^{(e)} &= \Lambda_t x_t^{\rho_t} (\nu S_t) v_t^{(e)} \\
x_{i,t}^{(m)} &= \left(1 - e^{-\lambda_i(t-\tau(i))} \right) x_{\tau(i)} \\
N_{i,t}^{(m)} &= \left(1 - e^{-\lambda_i(t-\tau(i))} \right)^{1-\rho_i} (\nu S_{\tau(i)}) \\
Y_{i,t}^{(m)} &= \frac{\Lambda_t}{1-\zeta} \left(1 - e^{-\lambda_i(t-\tau(i))} \right) x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
L_{p,i,t}^{(m)} &= \left(\frac{\Lambda_t}{1-\zeta} \right)^{\frac{1}{\zeta}} \left(1 - e^{-\lambda_i(t-\tau(i))} \right) x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
\Pi_{i,t}^{(m)} &= \Lambda_t \left(1 - e^{-\lambda_i(t-\tau(i))} \right) x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
V_{i,t}^{(m)} &= \Lambda_t x_t^{\rho_t} (\nu S_t) (v_t - v_t^{(e)}) \\
x_{i,t} &= x_{\tau(i)} \\
N_{i,t} &= \nu S_{\tau(i)} \\
Y_{i,t} &= \frac{\Lambda_t}{1-\zeta} x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
L_{p,i,t} &= \left(\frac{\Lambda_t}{1-\zeta} \right)^{\frac{1}{\zeta}} x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
\Pi_{i,t} &= \Lambda_t x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) \\
V_{i,t} &= \Lambda_t x_{\tau(i)}^{\rho_i} (\nu S_{\tau(i)}) v_t \\
v_t &= \mathbb{E}_t \left[\int_t^\infty e^{-(r+\delta)(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right]
\end{aligned}$$

A.1.5 Obsolescence risk

Next, we describe a version of the model with obsolescence risk, and derive the equilibrium conditions that obtain in this case.

Model description

We represent obsolescence as a Poisson process that affects the *entire* project, as opposed to production streams individually. Upon obsolescence, intangibles invested in the project lose their productive value. Prior to the realization of this shock, the value of the project is unchanged, so the exposition of the static allocation decision, of expropriation risk, and of spillovers are unchanged.

The value of a new project to an entrepreneur is now given by:

$$V_{i,t}^{(e)}(x, N) = \mathbb{E}_t \left[\int_t^{+\infty} e^{-(r+\delta)(s-t)} \Pi_{i,s} dt \right] = \Lambda_t x^{\rho_t} N \tilde{v}_t^{(e)}(x) \quad (71)$$

where

$$\tilde{v}_t^{(e)}(x) \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-(r+\delta+\gamma(x))(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right]. \quad (72)$$

The entrepreneur still chooses the initial span of the project to solve:

$$x_t = \arg \max_x x^{\rho_t} \tilde{v}_t^{(e)}(x). \quad (73)$$

The total value of the project at its creation is now:

$$V_{i,t} = V_{i,t}^{(c)} + V_{i,t}^{(e)} = \Lambda_t x_t^{\rho_t} \nu S_t v_t \quad (74)$$

where

$$v_t \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-(r+\delta)(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right]. \quad (75)$$

Finally, the law of motion for the stock of intangibles is given by:

$$S_t = \int_{\tau \leq t} (1 - e^{-\lambda_\tau(t-\tau)})^{1-\rho_\tau} (\nu S_\tau) e^{-\delta(t-\tau)} L_{e,\tau} d\tau, \quad (76)$$

reflecting the fact that the intangible capital of projects ceases to have productive value (and therefore to contribute to the stock of spillovers) once the project has become obsolete.

Equilibrium conditions

An equilibrium of this model is a set of processes for:

$$\left\{ \Lambda_t, W_t, x_t, v_t^{(e)}, L_{e,t}, S_t, L_{p,t}, \lambda_t \right\}_{t \geq 0}$$

that satisfy the following conditions:

$$\Lambda_t = (1 - \zeta) \left(\frac{\zeta}{W_t} \right)^{\frac{\zeta}{1-\zeta}} \quad (77)$$

$$x_t = \arg \max_{x \geq 1} x^{\rho_t} \tilde{v}_t^{(e)}(x) \quad (78)$$

$$v_t^{(e)} = \tilde{v}_t^{(e)}(x_t) \quad (79)$$

$$0 = L_{e,t} \left(W_t - \Lambda_t x_t^{\rho_t} (\nu S_t) v_t^{(e)} \right) \quad (80)$$

$$0 \leq L_{e,t} \quad (81)$$

$$0 \leq W_t - \Lambda_t x_t^{\rho_t} (\nu S_t) v_t^{(e)} \quad (82)$$

$$L_{p,t} = \left(\frac{\Lambda_t}{1 - \zeta} \right)^{\frac{1}{\zeta}} \int_{\tau \leq t} x_\tau^{\rho_\tau} (\nu S_\tau) e^{-\delta(t-\tau)} L_{e,\tau} d\tau \quad (83)$$

$$L_{p,t} + L_{e,t} = 1 \quad (84)$$

$$S_t = \int_{\tau \leq t} (1 - e^{-\lambda_\tau(t-\tau)})^{1-\rho_\tau} (\nu S_\tau) e^{-\delta(t-\tau)} L_{e,\tau} d\tau \quad (85)$$

$$\tilde{v}_t^{(e)}(x) = \mathbb{E}_t \left[\int_t^\infty e^{-(r+\delta+\gamma(x))(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \right] \quad (86)$$

$$\lambda_t = \gamma(x_t) \quad (87)$$

Additional results

Concentration between projects is given by:

$$H_t = \int_{\tau(i) \leq t} \left(\frac{Y_{i,t}}{Y_t} \right)^2 di \quad (88)$$

$$= \frac{\int_{\tau \leq t} x_\tau^{2\rho_\tau} S_\tau^2 e^{-\delta(t-\tau)} L_{e,\tau} d\tau}{\left(\int_{\tau \leq t} x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \right)^2} \quad (89)$$

Along the balanced growth path, this expression simplifies to:

$$H = \frac{1}{L_e} \frac{\int_{\tau \leq t} e^{-(2g+\delta)(t-\tau)} d\tau}{\left(\int_{\tau \leq t} e^{-(g+\delta)(t-\tau)} d\tau \right)^2} \quad (90)$$

$$= \frac{2(g+\delta)n}{2g+\delta} \frac{n}{2}, \quad (91)$$

where we used the fact that $nL_e = g + \delta$ along the BGP.

Concentration between entrepreneurs is given by:

$$H_t^{(e)} = \int_{\tau(i) \leq t} \left(\frac{Y_{i,t}^{(e)}}{Y_t^{(e)}} \right)^2 di \quad (92)$$

$$= \frac{\int_{\tau \leq t} e^{-2\lambda_\tau(t-\tau)} x_\tau^{2\rho_\tau} S_\tau^2 e^{-\delta(t-\tau)} L_{e,\tau} d\tau}{\left(\int_{\tau \leq t} e^{-\lambda_\tau(t-\tau)} x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \right)^2} \quad (93)$$

Along the balanced growth path, this expression simplifies to:

$$H = \frac{1}{L_e} \frac{\int_{\tau \leq t} e^{-(2(g+\lambda)+\delta)(t-\tau)} d\tau}{\left(\int_{\tau \leq t} e^{-(\lambda+g+\delta)(t-\tau)} d\tau \right)^2} \quad (94)$$

$$= \frac{2(\lambda+g+\delta)^2}{(2(\lambda+g)+\delta)(g+\delta)} \frac{n}{2}, \quad (95)$$

where we used the fact that $nL_e = g + \delta$ along the BGP.

A.2 Appendix to Section 2

A.2.1 The law of motion for aggregate spillovers

We prove the following Lemma, which holds for the general model with obsolescence risk.

Lemma 4 (Evolution of spillovers). *Assume that $\rho_t < 1$ for all t . Then the change in the stock of spillover intangibles is given by*

$$dS_t = (dt) \int_{\tau \leq t} \left(1 - e^{-\lambda_\tau(t-\tau)} \right)^{-\rho_\tau} \left[((1 - \rho_\tau)\lambda_\tau + \delta) e^{-\lambda_\tau(t-\tau)} - \delta \right] (\nu S_\tau) e^{-\delta(t-\tau)} L_{e,\tau} d\tau. \quad (96)$$

Proof of Lemma 4. Assume that $\rho_t < 1$ for all t . Then:

$$S_{t+dt} = \underbrace{0}_{\text{spillovers from projects started at } t+dt} + \underbrace{\int_{\tau(i) \leq t} \left(1 - e^{-\lambda_i(t+dt-\tau(i))}\right)^{1-\rho_i} (\nu S_{\tau(i)}) a_{i,t} di}_{\text{spillovers from existing projects}}$$

where $a_{i,t}$ is an indicator for whether the intangibles associated with project i have become obsolete by time t . To see why there are no immediate spillovers from new projects, recall that, absent obsolescence, spillovers from an individual project are given by:

$$S_{i,t} = \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} (\nu S_{\tau(i)}),$$

which is zero when $t = \tau(i)$, i.e. when the project is just created, except if $\rho_i = 1$. Using the expression above, we get:

$$\begin{aligned} dS_t &= \nu \int_{\tau(i) \leq t} S_{\tau(i)} \left[\left(1 - e^{-\lambda_i(t+dt-\tau(i))}\right)^{1-\rho_i} a_{i,t+dt} - \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} a_{i,t} \right] di \\ &= \nu \int_{\tau(i) \leq t} S_{\tau(i)} \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} \left[\left(\frac{1 - e^{-\lambda_i(t+dt-\tau(i))}}{1 - e^{-\lambda_i(t-\tau(i))}}\right)^{1-\rho_i} a_{i,t+dt} - a_{i,t} \right] di \\ &= \nu \int_{\tau(i) \leq t} S_{\tau(i)} \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{1-\rho_i} \left[\left(1 + \frac{e^{-\lambda_i(t-\tau(i))}}{1 - e^{-\lambda_i(t-\tau(i))}} \lambda_i dt\right)^{1-\rho_i} a_{i,t+dt} - a_{i,t} \right] di \\ &= \nu \int_{\tau(i) \leq t} S_{\tau(i)} \left(1 - e^{-\lambda_i(t-\tau(i))}\right)^{-\rho_i} \left[\left(\left(1 - e^{-\lambda_i(t-\tau(i))}\right) + e^{-\lambda_i(t-\tau(i))} (1 - \rho_i) \lambda_i dt\right) a_{i,t+dt} \right. \\ &\quad \left. - \left(1 - e^{-\lambda_i(t-\tau(i))}\right) a_{i,t} \right] di \\ &= \nu \int_{\tau \leq t} L_{e,\tau} S_{\tau} \left(1 - e^{-\lambda_{\tau}(t-\tau)}\right)^{-\rho_{\tau}} \left[\left(\left(1 - e^{-\lambda_{\tau}(t-\tau)}\right) + e^{-\lambda_{\tau}(t-\tau)} (1 - \rho_{\tau}) \lambda_{\tau} dt\right) e^{-\delta(t+dt-\tau)} \right. \\ &\quad \left. - \left(1 - e^{-\lambda_{\tau}(t-\tau)}\right) e^{-\delta(t-\tau)} \right] d\tau \\ &= (dt) \int_{\tau \leq t} \left(1 - e^{-\lambda_{\tau}(t-\tau)}\right)^{-\rho_{\tau}} \left[\left((1 - \rho_{\tau}) \lambda_{\tau} + \delta\right) e^{-\lambda_{\tau}(t-\tau)} - \delta \right] (\nu S_{\tau}) e^{-\delta(t-\tau)} L_{e,\tau} d\tau. \end{aligned}$$

In the penultimate line, we used the fact that $L_{e,\tau}$ projects get created in each cohort τ . We also used the fact that all project starting at the same date choose the same span x_t , and are therefore

exposed to the same degree of expropriation risk $\lambda_t = \gamma(x_t)$. Finally, we used the fact that the obsolescence shock is identically and independently over time and across projects. This establishes the result. ■

A.2.2 Existence and unicity of BGP in baseline model

We make the following technical assumptions about the function $\gamma(\cdot)$ governing expropriation risk:

$$\gamma(1) \geq 0, \quad (97)$$

$$\gamma'(1) = 0, \quad (98)$$

$$\forall x \geq 1, \quad \gamma'(x) > 0, \quad (99)$$

$$\forall x \geq 1, \quad \gamma''(x) > 0, \quad (100)$$

$$\lim_{x \rightarrow +\infty} \gamma(x) - x\gamma'(x) < 0. \quad (101)$$

In the proof below, we will make use of the following lemmas.

Lemma 5. *Define:*

$$R(x) \equiv x\gamma'(x) - \rho\gamma(x).$$

R is a strictly increasing mapping from $[1, +\infty)$ to $[-\rho\gamma(1), +\infty)$. Let R^{-1} be its inverse, and let:

$$\underline{x}(\rho) \equiv R^{-1}(\rho r), \quad \underline{\lambda}(\rho) \equiv \gamma(\underline{x}(\rho)). \quad (102)$$

Then both $\underline{x}(\rho)$ and $\underline{\lambda}(\rho)$ are strictly increasing functions of ρ , with $\underline{x}(0) = 1$ and $\underline{\lambda}(\rho) = \gamma(1)$.

Proof of Lemma 5. By Assumptions (97) - (103), the function R satisfies:

$$R(1) = -\rho\gamma(1) \leq 0$$

$$\lim_{x \rightarrow +\infty} R(x) = +\infty$$

$$\forall x \geq 1, \quad R'(x) = \gamma'(x) + x\gamma''(x) - \rho\gamma'(x) > 0$$

Therefore it is a strictly increasing mapping of from $[1, +\infty)$ to $[-\rho\gamma(1), +\infty)$. Its inverse is well-defined and strictly increasing. The monotonicity of $\underline{x}(\rho)$ follows. The fact that γ is strictly increasing then implies is also strictly increasing $\underline{\lambda}(\rho)$. ■

Finally, we assume that the parameter ν satisfies:

$$\nu \geq \frac{\zeta}{1-\zeta} (r + \lambda(1)), \quad (103)$$

This assumption is necessary and sufficient for the existence and unicity of the balanced growth path for all values of ρ . Intuitively, it states when spillovers are strong, the size of new projects is large enough for entry to be attractive to entrepreneurs, relative to working in production.

Existence and unicity of balanced growth path in baseline model. Assume that $\rho_t = \rho > 0$. We guess and verify that there exists a unique balanced growth equilibrium in which W_t is growing at rate $(1 - \zeta)g$, and entrepreneurial labor is $L_{e,t} = L_e$, for some $g > 0$ and L_e to be determined. Assume that W_t is growing at rate $(1 - \zeta)g$. Equation (67) implies that:

$$\tilde{v}_t^{(e)}(x) = \frac{1}{r + \gamma(x) + \zeta g}. \quad (104)$$

Next, denote by $\omega(x) = x\tilde{v}_t^{(e)}(x)$ the objective function in Equation (59). We have:

$$\begin{aligned} \omega(1) &= \frac{1}{r + \gamma(1) + \zeta g} > 0, \\ \forall x \geq 1, \quad \omega'(x) &= \frac{x^{\rho-1}}{(r + \gamma(x) + \zeta g)^2} (\rho(r + \gamma(x) + \zeta g) - x\gamma'(x)), \\ \omega'(1) &= \frac{\rho}{r + \gamma(1) + \zeta g} > 0, \\ \lim_{x \rightarrow +\infty} \omega'(x) &< 0, \end{aligned}$$

where the latter two inequalities follow from Assumptions (98) and (101). Moreover, we see that:

$$\omega'(x) \geq 0 \quad \iff \quad \rho(r + \gamma(x) + \zeta g) \geq x\gamma'(x). \quad (105)$$

By Lemma 5, ω has exactly one interior global maximum, which is given by:

$$x = R^{-1}(\rho(r + \zeta g)). \quad (106)$$

Under our guess, x_t must therefore be constant, $x_t = x$. In particular, we also have $\lambda_t = \lambda = \gamma(x)$ and $v_t^{(e)} = \tilde{v}^{(e)}(x) \equiv v^{(e)}$. Additionally, from Lemma 5, we have that:

$$x > \underline{x}(\rho) > 1, \quad (107)$$

and so $\lambda = \gamma(x) > \underline{\lambda}(\rho) > \gamma(1) \geq 0$. Therefore there must be positive expropriation risk.

Next, assume that $L_{e,t} = L_e > 0$. Then the free-entry conditions imply that:

$$W_t = \Lambda_t x^\rho (\nu S_t) v^{(e)}. \quad (108)$$

Combining this with Equations (58) and (65), we see that S_t must grow at rate $g > 0$. Moreover, we obtain:

$$\begin{aligned} L_{p,t} &= \frac{\zeta}{1-\zeta} \frac{1}{x^\rho (\nu S_t) v^{(e)}} \int_{\tau \leq t} x_\tau^\rho (\nu S_\tau) L_e d\tau \\ &= \frac{\zeta}{1-\zeta} \frac{r + \gamma(x) + \zeta g}{g} L_e, \end{aligned}$$

where we used the expression for $v^{(e)}$ in Equation (104) in the last line. Therefore production labor demand is constant. Using the labor market clearing condition, we can write:

$$g = \frac{g}{L_e} - \frac{\zeta}{1-\zeta} (r + \gamma(x) + \zeta g). \quad (109)$$

Finally, using our guess and the fact that $\lambda = \gamma(x) > 0$, we can rewrite Equation (68) as:

$$\begin{aligned} 1 &= \nu L_e \int_{\tau \leq t} (1 - e^{-\gamma(x)(t-\tau)})^{1-\rho} e^{-g(t-\tau)} d\tau \\ &= \frac{\nu L_e}{\gamma(x)} \int_0^1 (1-u)^{1-\rho} u^{\frac{g}{\gamma(x)}-1} du \end{aligned}$$

where we used the change of variable $u = e^{-\gamma(x)(t-\tau)}$ to go from the first to the second line. We can rewrite this relationship as:

$$\frac{g}{L_e} = n\left(\frac{g}{\gamma(x)}, \rho\right)$$

where:

$$n(y, z) \equiv \nu y \mathcal{B}(y, 2-z), \quad (110)$$

where \mathcal{B} is the Beta function. We note that the function n satisfies:

$$\forall y > 0, \quad n(y, 1) = \nu \quad (111)$$

$$\forall y > 0, z \in (0, 1], \quad \frac{\partial n}{\partial z}(y, z) = \nu (\psi(y+2-z) - \psi(2-z)) y \mathcal{B}(y, 2-z) > 0 \quad (112)$$

$$\forall y > 0, z \in (0, 1], \quad \frac{\partial n}{\partial y}(y, z) = \nu(1 + y\psi(y) - y\psi(y + 2 - z)) \mathcal{B}(y, 2 - z) \leq 0 \quad (113)$$

Here ψ denotes the Digamma function.

Combining Equation (110) with Equation (109), we see that the following relationship must hold:

$$g = n\left(\frac{g}{\gamma(x)}, \rho\right) - \frac{\zeta}{1 - \zeta} (r + \gamma(x) + \zeta g). \quad (114)$$

To conclude, we need to show that the system of equations:

$$\rho(r + \gamma(x) + \zeta g) = x\gamma'(x) \quad (115)$$

$$g = n\left(\frac{g}{\gamma(x)}, \rho\right) - \frac{\zeta}{1 - \zeta} (r + \gamma(x) + \zeta g) \quad (116)$$

has a unique solution (g, x) with $g > 0$ and $x \geq 1$. Recall that Equation (115) can be written as:

$$x = R^{-1}(\rho(r + \zeta g)),$$

Define $\lambda(g) = \gamma(x(g))$, and:

$$\Delta(g) \equiv n\left(\frac{g}{\lambda(g)}, \rho\right) - \frac{\zeta}{1 - \zeta} (r + \lambda(g) + \zeta g) - g.$$

Establishing that the system (115)-(116) has a unique solution is equivalent to showing that the equation $\Delta(g) = 0$ has a unique, strictly positive solution.

To start, note that $\lambda(g)$ is strictly increasing, with $\lambda(0) = \underline{\lambda}(\rho) > 0$, and $\lim_{x \rightarrow +\infty} \lambda(x) = +\infty$. At $y = 0$, for any $z > 0$, we have the following Taylor expansion:

$$\mathcal{B}(y, z) = \frac{1}{y} + O(1).$$

Therefore:

$$n(y, z) = \nu + O(y),$$

so that:

$$\lim_{y \rightarrow 0} n(y, z) = \nu,$$

and therefore:

$$\lim_{g \rightarrow 0} \Delta(g) = \nu - \frac{\zeta}{1 - \zeta} (r + \underline{\lambda}(\rho)) > 0, \quad (117)$$

To see why this limit must be positive, recall that from Lemma 5, $\underline{\lambda}(\rho)$ is strictly increasing with respect to ρ ; so the sign of the limit in Equation (117) then follows from assumption (103).

Additionally, from Equations (111) and (112), it follows that $n(y, z) \leq \nu$, and therefore:

$$\Delta(g) \leq \nu - \frac{\zeta}{1-\zeta} (r + \lambda(g) + \zeta g), \quad (118)$$

so that

$$\lim_{g \rightarrow +\infty} \Delta(g) = -\infty. \quad (119)$$

Taken together, the two limits (117) and (119) show that there is at least one strictly positive solution to the equation $\Delta(g) = 0$. ■

Regions of non-monotonicity. Throughout we fix the values of $\xi > 0$, $\zeta \in]0, 1[$, and $r > 0$. First, assume that $\nu > 0$ and $\eta > 0$. From the proof of Lemma 1, we know that when $\rho = 1$, $g/\gamma(x)$ is finite. We now differentiate the system of Equations (115)-(116) with respect to ρ , and evaluate the differentials at $\rho = 1$. We obtain:

$$\begin{aligned} \frac{\partial g}{\partial \rho} &= \frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta} \left(r + \zeta \frac{\partial g}{\partial \rho} + \gamma'(x) \frac{\partial x}{\partial \rho} \right), \\ x\gamma''(x) \frac{\partial x}{\partial \rho} &= x\gamma'(x) + \zeta \frac{\partial g}{\partial \rho}. \end{aligned}$$

Here, we have used, in particular, the fact that $\frac{\partial n}{\partial y}(y, 1) = 0$ to simplify the first equation. We have also guessed that $\frac{\partial x}{\partial \rho}$ and $\frac{\partial g}{\partial \rho}$ are both finite at $\rho = 1$, but we will verify this guess below. We can rewrite the system as:

$$\begin{aligned} \frac{1-\zeta+\zeta^2}{1-\zeta} \frac{\partial g}{\partial \rho} + \frac{\zeta}{1-\zeta} \gamma'(x) \frac{\partial x}{\partial \rho} &= \frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta} r \\ -\zeta \frac{\partial g}{\partial \rho} + x\gamma''(x) \frac{\partial x}{\partial \rho} &= x\gamma'(x). \end{aligned}$$

The discriminant is:

$$\Delta(x) = \frac{1-\zeta+\zeta^2}{1-\zeta} x\gamma''(x) + \frac{\zeta^2}{1-\zeta} \gamma'(x) > 0$$

where the latter inequality is because the optimal span is $x > 1$, and γ is convex. The solution is:

$$\frac{\partial g}{\partial \rho} = \frac{1}{\Delta(x)} \left(x\gamma''(x) \left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta} r \right) - \frac{\zeta}{1-\zeta} x(\gamma'(x))^2 \right)$$

$$\frac{\partial x}{\partial \rho} = \frac{1}{\Delta(x)} \left(\zeta \left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta} r \right) + \frac{1-\zeta+\zeta^2}{1-\zeta} x \gamma'(x) \right)$$

We rewrite the solution for $\frac{\partial g}{\partial \rho}$ as:

$$\begin{aligned} \frac{\partial g}{\partial \rho} &= \frac{x \gamma''(x)}{\Delta(x)} \left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta} \left(r + \frac{(\gamma'(x))^2}{\gamma''(x)} \right) \right) \\ &= \frac{1-\zeta}{1-\zeta+\zeta^2+\zeta^2 \frac{\gamma'(x)}{x \gamma''(x)}} \left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta} \left(r + \frac{(\gamma'(x))^2}{\gamma''(x)} \right) \right). \end{aligned}$$

We finally substitute the functional form for $\gamma(x)$ and note that $\frac{1}{\eta}(x-1)^{1+\xi} = (1+\xi)\gamma(x)$ to get:

$$\frac{\partial g}{\partial \rho} = \frac{1-\zeta}{1-\zeta+\zeta^2+\frac{\zeta^2}{\xi} \frac{x-1}{x}} \left(\frac{\partial n}{\partial \rho} - \frac{\zeta}{1-\zeta} \left(r + \frac{1+\xi}{\xi} \gamma(x) \right) \right).$$

This expression shows that:

$$\frac{\partial g}{\partial \rho} < 0 \quad \iff \quad \frac{\partial n}{\partial \rho} < \frac{\zeta}{1-\zeta} \left(r + \frac{1+\xi}{\xi} \gamma(x) \right).$$

Figure 5 then reports a partition of the space of values for (ν, η) such that this inequality holds. ■

A.2.3 Proofs for model extensions

Preliminaries For the case with obsolescence risk, we make the same assumptions on the function governing expropriation risk, $\gamma(x)$, as in Equations (97)-(100). Additionally, we state the analog to Lemma (5) without proof.

Lemma 6. *Define R as in Lemma 5, and let:*

$$\underline{x}(\rho) \equiv R^{-1}(\rho(r+\delta)), \quad \underline{\lambda}(\rho) \equiv \gamma(\underline{x}(\rho)). \quad (120)$$

Then both $\underline{x}(\rho)$ and $\underline{\lambda}(\rho)$ are strictly increasing functions of ρ , with $\underline{x}(0) = 1$ and $\underline{\lambda}(\rho) = \gamma(1)$.

Finally, we define the following function of ρ :

$$H(\rho) \equiv n \left(\frac{\delta}{\underline{\lambda}(\rho)}, \rho \right) - \frac{\zeta}{1-\zeta} (r + \delta + \underline{\lambda}(\rho)) - \delta, \quad (121)$$

where recall that the function n is defined as:

$$n(y, z) \equiv \nu \mathcal{B}(y, 2-z), \quad (122)$$

where \mathcal{B} is the Beta function. The Lemma shows that there exists a unique balanced growth path with strictly positive growth in the model with obsolescence risk, if and only if:

$$H(\rho) > 0. \quad (123)$$

Existence and unicity of balanced growth path in the model with obsolescence. Assume that $\rho_t = \rho > 0$. As in the baseline model, we guess and verify that there exists a unique balanced growth equilibrium in which W_t is growing at rate $(1 - \zeta)g$, and entrepreneurial labor is $L_{e,t} = L_e$, for some $g > 0$ and L_e to be determined.

Assume that W_t is growing at rate $(1 - \zeta)g$. Equation (86) implies that:

$$\tilde{v}_t^{(e)}(x) = \frac{1}{r + \delta + \gamma(x) + \zeta g}. \quad (124)$$

Next, denote by $\omega(x) = x\tilde{v}_t^{(e)}(x)$ the objective function in Equation (78). Following the same steps as in the proof of Lemma 1, we see that $\omega(x)$ has exactly one interior global maximum, given by:

$$x = R^{-1}(\rho(r + \delta + \zeta g)) \quad (125)$$

Under our guess, x_t must therefore be constant, $x_t = x$. Additionally, $\lambda_t = \lambda = \gamma(x)$ and $v_t^{(e)} = \tilde{v}^{(e)}(x) \equiv v^{(e)}$. Finally, it must be that $x > \underline{x}(\rho) > 1$, and so $\lambda = \gamma(x) > \underline{\lambda}(\rho) > 0$.

Next, assume that $L_{e,t} = L_e > 0$. Then the free-entry conditions imply that:

$$W_t = \Lambda_t x^\rho (\nu S_t) v^{(e)}. \quad (126)$$

Combining this with Equations (77) and (84), we see that S_t must grow at rate $g > 0$. Moreover, we obtain:

$$\begin{aligned} L_{p,t} &= \frac{\zeta}{1 - \zeta} \frac{1}{x^\rho (\nu S_t) v^{(e)}} \int_{\tau \leq t} x_\tau^\rho (\nu S_\tau) e^{-\delta(t-\tau)} L_e d\tau \\ &= \frac{\zeta}{1 - \zeta} \frac{r + \delta + \gamma(x) + \zeta g}{g + \delta} L_e, \end{aligned}$$

where we used the expression for $v^{(e)}$ in Equation (124) in the last line. Therefore production labor demand is constant. Using the labor market clearing condition, we then have:

$$g + \delta = \frac{g + \delta}{L_e} - \frac{\zeta}{1 - \zeta} (r + \delta + \gamma(x) + \zeta g). \quad (127)$$

Finally, using our guess and the fact that $\lambda = \gamma(x) > 0$, we can rewrite Equation (87) as:

$$\begin{aligned} 1 &= \nu L_e \int_{\tau \leq t} (1 - e^{-\gamma(x)(t-\tau)})^{1-\rho} e^{-(g+\delta)(t-\tau)} d\tau \\ &= \frac{\nu L_e}{\gamma(x)} \int_0^1 (1-u)^{1-\rho} u^{\frac{g+\delta}{\gamma(x)}-1} du \end{aligned}$$

where we used the change of variable $u = e^{-\gamma(x)(t-\tau)}$ to go from the first to the second line. We can rewrite this relationship as:

$$\frac{g+\delta}{L_e} = n\left(\frac{g+\delta}{\gamma(x)}, \rho\right)$$

where $n(y, z)$ is defined as in the proof of Lemma 1.

To conclude, we need to show that the system of equations:

$$\rho(r + \delta + \gamma(x) + \zeta g) = x\gamma'(x) \quad (128)$$

$$g + \delta = n\left(\frac{g+\delta}{\gamma(x)}, \rho\right) - \frac{\zeta}{1-\zeta}(r + \delta + \gamma(x) + \zeta g) \quad (129)$$

has a unique solution (g, x) with $g > 0$ and $x \geq 1$. Recall that Equation (128) can be written as:

$$R(x) = r + \delta + \zeta g \iff x = x(g),$$

where $x(g) = R^{-1}(r + \delta + \zeta g)$. Similar to the proof of Lemma 1, define $\lambda(g) = \gamma(x(g))$, and:

$$\Delta(g) \equiv n\left(\frac{g+\delta}{\lambda(g)}, \rho\right) - \frac{\zeta}{1-\zeta}(r + \delta + \lambda(g) + \zeta g) - (g + \delta).$$

We need to prove that the equation $\Delta(g) = 0$ has a unique, strictly positive solution.

To start, note that $\lambda(g)$ is strictly increasing, with $\lambda(0) = \underline{\lambda}(\rho) > 0$. Therefore:

$$\lim_{g \rightarrow 0} \Delta(g) = n\left(\frac{\delta}{\underline{\lambda}(\rho)}, \rho\right) - \frac{\zeta}{1-\zeta}(r + \delta + \underline{\lambda}(\rho)) - \delta = H(\rho). \quad (130)$$

Additionally, from Equation (112), it follows that $n(y, z) \leq \nu$, and therefore:

$$\Delta(g) \leq \nu - \frac{\zeta}{1-\zeta}(r + \delta + \lambda(g) + \zeta g) - \delta, \quad (131)$$

so that

$$\lim_{g \rightarrow +\infty} \Delta(g) = -\infty. \quad (132)$$

Taken together, the two limits (130) and (132) show that, for any ρ such that $H(\rho) > 0$, there is at least one strictly positive solution to the equation $\Delta(g) = 0$. \blacksquare

A.3 Transitional dynamics

This appendix discusses the computation of the transitional dynamics of the model after a change in ρ . We compute these transitional dynamics in the context of the model with obsolescence risk, which is described in Section 1.3 and Appendix A.1.5.

A.3.1 Equations characterizing the transition path

After eliminating wages W_t from the conditions characterizing equilibrium in Appendix A.1.5, we can describe an equilibrium in which the free-entry always binds as a set of processes $\{S_t, \Lambda_t, x_t, v_t^{(e)}, L_{e,t}\}$ that satisfy:

$$S_t = \int_{\tau \leq t} \left(1 - e^{-\gamma(x_\tau)(t-\tau)}\right)^{1-\rho_\tau} (\nu S_\tau) e^{-\delta(t-\tau)} L_{e,\tau} d\tau \quad (133)$$

$$L_{e,t} = 1 - \frac{\zeta}{1-\zeta} \frac{1}{x_t^{\rho_t} v_t^{(e)} S_t} \int_{\tau \leq t} x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \quad (134)$$

$$\Lambda_t = (1-\zeta)^{1-\zeta} \zeta^\zeta \left(\nu x_t^{\rho_t} v_t^{(e)} S_t\right)^{-\zeta} \quad (135)$$

$$x_t = \arg \max_x x^{\rho_t} \int_t^{+\infty} e^{-(r+\delta+\gamma(x))(s-t)} \Lambda_s ds \quad (136)$$

$$v_t^{(e)} = \int_t^{+\infty} e^{-(r+\delta+\gamma(x_t))(s-t)} \frac{\Lambda_s}{\Lambda_t} ds \quad (137)$$

Let $\{\rho_t\}$ be a process that satisfies:

$$\forall t < 0, \quad \rho_t = \underline{\rho} > 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \rho_t = \bar{\rho}, \quad (138)$$

where both $\underline{\rho}$ and $\bar{\rho}$ satisfy $H(\underline{\rho}) > 0$ and $H(\bar{\rho}) > 0$, where H is the function described in Lemma 3 in the main text. Definition (2) states that a transition path is a set of processes $\{S_t, \Lambda_t, x_t, v_t^{(e)}, L_{e,t}\}$, such that, given $\{\rho_t\}$:

1. $\forall t < 0$, $\{S_t, \Lambda_t, x_t, v_t^{(e)}, L_{e,t}\}$ are given by the balanced growth path associated with $\underline{\rho}$;
2. $\forall t \geq 0$, $\{S_t, \Lambda_t, x_t, v_t^{(e)}, L_{e,t}\}$ satisfy equations (133)-(137);

3. $S_0 = 1$.

Note that the balanced growth path associated with $\underline{\rho}$ must be a solution to equations (133)-(137) where $\rho_t = \underline{\rho}$ for all t . This solution is unique up to a multiplicative constant for S_t , which we fix by the normalization $S_0 = 1$.

A.3.2 Numerical computation

Main steps

Equations (133) and (134) are backward-looking, while Equations (136) and (137) are forward-looking. We start from initial guesses for the paths of $\{x_t, v_t^{(e)}, S_t\}_{t \geq 0}$, and then proceed as follows:

Step 1: Update the corresponding path for $\{\Lambda_t\}_{t \geq 0}$ from equation (135).

Step 2: Given the path for $\{\Lambda_t\}_{t \geq 0}$, solve Equations (136) and (137) by backward iteration to obtain a new path for $\{x_t, v_t^{(e)}\}_{t \geq 0}$.

Step 3: Given the paths for $\{x_t, v_t^{(e)}, S_t\}_{t \geq 0}$, solve Equation (134) through forward iteration to obtain a new path for $\{L_{e,t}\}_{t \geq 0}$.

Step 4: Given the paths for $\{x_t, v_t^{(e)}, L_{e,t}\}_{t \geq 0}$, solve Equation (133) through forward iteration to obtain a new path for $\{S_t\}_{t \geq 0}$.

We then iterate on steps 1-4 until convergence on $\{L_{e,t}\}_{t \geq 0}$.

As initial guess for $\{x_t, v_t^{(e)}, S_t\}_{t \geq 0}$, we use the values of x and v on the BGP associated with the long-run value of ρ_t , $\bar{\rho}$. For S_t , note that, when $\rho_t < 1$, we established in Appendix A.1.5 that S_t is not a jump variable. So we use the initial guess $S_t = e^{gt}$, where g is the growth rate associated with the BGP for $\rho = \underline{\rho}$. $L_{e,t}$, however, may be a jump variable, since it depends on $v_t^{(e)}$.

For Step 4, we describe two possible approaches: one that uses the law of motion for S_t in levels, and another than uses the expression for dS_t obtained in Lemma 2. For the results presented in Section 4, we use the latter approach.

Grid

In what follows we approximate all integrals on a time grid with fixed step size Δt . We compute the transition path at the dates:

$$t_n = (n - 1)\Delta t, \quad n = 1, \dots, N + 1, \quad (139)$$

where N is a large number, and we denote by $T = N\Delta t$ the maximum horizon at which we compute the transition path.

Step 1

Let $\mathbf{\Lambda}$ be an $(N + 1) \times 1$ column vector with entries $\mathbf{\Lambda}_n = \Lambda_{t_n}$, and let \mathbf{x} , $\mathbf{v}^{(e)}$ and \mathbf{S} be column vectors for $(x_t, v_t^{(e)}, S_t, \rho_t)$ defined similarly. We update $\mathbf{\Lambda}$ pointwise using:

$$\mathbf{\Lambda}_n = (1 - \zeta)^{1-\zeta} \zeta^\zeta \left(\nu \mathbf{x}_n^{\rho_n} \mathbf{v}_n^{(e)} \mathbf{S}_n \right)^{-\zeta} \quad (140)$$

Step 2

For this step, we need to compute integrals of the form:

$$O_t(x) \equiv \int_{s \geq t} e^{-(r+\delta+\gamma(x))(s-t)} \Lambda_s ds. \quad (141)$$

For $1 \leq n \leq N$, we use the following approximation:

$$\begin{aligned} O_{t_n}(x) &= \int_{t_n}^T e^{-(r+\delta+\gamma(x))(s-t_n)} \Lambda_s ds + \int_{s \geq T} e^{-(r+\delta+\gamma(x))(s-t_n)} \Lambda_s ds \\ &\approx \sum_{k=n}^N \int_{(k-1)\Delta t}^{k\Delta t} e^{-(r+\delta+\gamma(x))(s-t_n)} \Lambda_s ds \\ &\quad + e^{-(r+\delta+\gamma(x))(T-t_n)} \Lambda_T \sum_{k=N+1}^{+\infty} \int_{(k-1)\Delta t}^{k\Delta t} e^{-(r+\delta+\gamma(x))(s-T)} e^{-\zeta \bar{g}(s-T)} ds \\ &\approx (\Delta t) \left(\sum_{k=n}^N e^{-(r+\delta+\gamma(x))((k-n)\Delta t)} \Lambda_{(k-1)\Delta t} + e^{-(r+\delta+\gamma(x))(T-t_n)} \Lambda_T \sum_{k=N+1}^{+\infty} e^{-(r+\delta+\gamma(x)+\zeta \bar{g})(k-N-1)\Delta t} \right) \\ &= (\Delta t) e^{(r+\delta+\gamma(x))(n-1)\Delta t} \left(\sum_{k=n}^N e^{-(r+\delta+\gamma(x))(k-1)\Delta t} \Lambda_{(k-1)\Delta t} + \frac{e^{-(r+\delta+\gamma(x))T} \Lambda_T}{1 - e^{-(r+\delta+\gamma(x)+\zeta \bar{g})\Delta t}} \right). \end{aligned}$$

For $n = N + 1$, we use the approximation:

$$O_{t_{N+1}}(x) = (\Delta t) \frac{\Lambda_T}{1 - e^{-(r+\delta+\gamma(x)+\zeta \bar{g})\Delta t}}.$$

We can write this in matrix form as:

$$\mathbf{O}_n(x) = (\Delta t) e^{(r+\delta+\gamma(x))(n-1)\Delta t} \mathbf{P}_{O,n}(x) \mathbf{\Lambda}, \quad (142)$$

where the $(1 \times (N + 1))$ row vector $\mathbf{P}_{O,n}(x)$ has entries:

$$\mathbf{P}_{O,n}^{(k)}(x) = \begin{cases} 0 & \text{if } k < n \\ e^{-(r+\delta+\gamma(x))(k-1)\Delta t} & \text{if } n \leq k \leq N \\ \frac{e^{-(r+\delta+\gamma(x))T}}{1 - e^{-(r+\delta+\gamma(x)+\zeta\bar{g})\Delta t}} & \text{if } k = N + 1 \end{cases} \quad (143)$$

Given this approximation, for each value of n , we compute numerically:

$$\mathbf{x}_n = \max_x \mathbf{O}_n(x),$$

to update \mathbf{x} , and then update \mathbf{v} using:

$$\mathbf{v}_n^{(e)} = \frac{\mathbf{O}_n(\mathbf{x}_n)}{\mathbf{\Lambda}_n \mathbf{x}_n^{\rho_n}}.$$

Step 3

Let $0 \leq t \leq T$. We first write Equation (134) as:

$$\begin{aligned} L_{e,t} &= 1 - \frac{\zeta}{1 - \zeta} \frac{1}{x_t^{\rho_t} v_t^{(e)} S_t} \left(T_{L,t}^{(-)} + T_{L,t}^{(+)} \right) \\ T_{L,t}^{(-)} &= \int_{\tau \leq 0} x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \\ T_{L,t}^{(+)} &= \int_{\tau=0}^{\tau=t} x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \end{aligned}$$

For the term $T_{L,t}^{(-)}$, we use the following approximation. We first write:

$$T_{L,t}^{(-)} = \underbrace{\int_{\tau \leq -(T-t)} x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau}_{T_{1,L,t}^{(-)}} + \underbrace{\int_{\tau=-(T-t)}^0 x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau}_{T_{2,L,t}^{(-)}}$$

We approximate each term separately, so that the overall approximation is done over a window of size T . For $T_{1,L,t}^{(-)}$ and $t = t_n = (n - 1)\Delta t$, we write:

$$T_{1,L,t}^{(-)} = \underline{x}^{\rho} \underline{L}_e \int_{\tau \leq -(T-t)} e^{-\delta(t-\tau)} e^{g\tau} d\tau$$

$$\begin{aligned}
&= \underline{x}^\rho \underline{L}_e e^{-\delta t} \int_{\tau \leq -(T-t)} e^{-(\underline{g}+\delta)\tau} d\tau \\
&= \underline{x}^\rho \underline{L}_e e^{-\delta t} \int_{u \leq 0} e^{(\underline{g}+\delta)(u-(T-t))} du \\
&\approx \frac{\Delta t}{1 - e^{-(\underline{g}+\delta)\Delta t}} e^{\underline{g}t} e^{-(\underline{g}+\delta)T} \underline{x}^\rho \underline{L}_e
\end{aligned}$$

For $T_{2,L,t}^{(-)}$ and $t = t_n = (n-1)\Delta t$, we write:

$$\begin{aligned}
T_{2,L,t}^{(-)} &= \underline{x}^\rho \underline{L}_e \int_{\tau = -(T-t)}^0 e^{-\delta(t-\tau)} S_\tau d\tau \\
&= \underline{x}^\rho \underline{L}_e e^{\underline{g}t} \int_{\tau = -(T-t)}^0 e^{-(\underline{g}+\delta)(t-\tau)} d\tau \\
&= \underline{x}^\rho \underline{L}_e e^{\underline{g}t} \int_{u=t}^T e^{-(\underline{g}+\delta)u} d\tau \\
&\approx \underline{x}^\rho \underline{L}_e (\Delta t) e^{\underline{g}t} \sum_{k=n}^N \left(e^{-(\underline{g}+\delta)\Delta t} \right)^k
\end{aligned}$$

Then we can write the approximation to $T_{L,t}^{(-)}$ in matrix form as:

$$\mathbf{T}_L^{(-)} = (\Delta t) \mathbf{P}_L^{(-)} \mathbf{J}$$

where \mathbf{J} is an $(N+1) \times 1$ matrix with all entries equal to 1, and $\mathbf{P}_L^{(-)}$ is an $(N+1) \times (N+1)$ matrix with entries:

$$\mathbf{P}_L^{(-),(n,k)} = \underline{x}^\rho \underline{L}_e \times \begin{cases} e^{\underline{g}(n-1)\Delta t} e^{-(\underline{g}+\delta)k\Delta t} & \text{if } n \leq k \leq N \\ e^{\underline{g}(n-1)\Delta t} \frac{e^{-(\underline{g}+\delta)T}}{1 - e^{-(\underline{g}+\delta)\Delta t}} & \text{if } k = N+1 \end{cases},$$

and zeros elsewhere. For the term $T_{L,t}^{(+)}$, we use $T_{L,t_1}^{(+)} = 0$, and for $n \geq 2$, we make the approximation:

$$\begin{aligned}
T_{L,t_n}^{(+)} &= \int_0^{t_n} x_\tau^{\rho\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau = \sum_{k=1}^{n-1} \int_{(k-1)\Delta t}^{k\Delta t} x_\tau^{\rho\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \\
&\approx (\Delta t) e^{-\delta(n-1)\Delta t} \sum_{k=1}^{n-1} x_{(k-1)\Delta t}^{\rho(k-1)\Delta t} S_{(k-1)\Delta t} e^{\delta(k-1)\Delta t} L_{e,(k-1)\Delta t}
\end{aligned}$$

This approximation can be written in matrix form as:

$$\mathbf{T}_L^{(+)} = (\Delta t)\mathbf{P}_L^{(+)}\mathbf{L}_e$$

where $\mathbf{P}_L^{(+)}$ is an $(N+1) \times (N+1)$ matrix with entries:

$$\mathbf{P}_L^{(+),(n,k)} = \underline{x}^\rho \underline{L}_e \times \begin{cases} 0 & \text{if } k \geq n \\ e^{-\delta(n-1)\Delta t} \mathbf{x}_k^\rho e^{\delta(k-1)\Delta t} \mathbf{S}_k & \text{if } 1 \leq k \leq n-1 \end{cases}$$

Finally, let \mathbf{D} be the $(N+1) \times (N+1)$ matrix given by:

$$\mathbf{D} = \text{diag} \left(\left\{ (\mathbf{x}_n^\rho \mathbf{v}_n^{(e)} \mathbf{S}_n)^{-1} \right\}_{n=1}^{N+1} \right)$$

Thus $L_{e,t}$ must satisfy:

$$\mathbf{L}_e = \mathbf{J} - \frac{\zeta}{1-\zeta} \mathbf{D} \left((\Delta t)\mathbf{P}_L^{(-)}\mathbf{J} + (\Delta t)\mathbf{P}_L^{(+)}\mathbf{L}_e \right)$$

or equivalently:

$$\mathbf{L}_e = \left(\mathbf{I} + (\Delta t) \frac{\zeta}{1-\zeta} \mathbf{D} \mathbf{P}_L^{(+)} \right)^{-1} \left(\mathbf{I} - (\Delta t) \frac{\zeta}{1-\zeta} \mathbf{D} \mathbf{P}_L^{(-)} \right) \mathbf{J}$$

where \mathbf{I} is the identity matrix of size $(N+1) \times (N+1)$.

Step 4, using the law of motion for S_t in levels

Let $0 \leq t \leq T$. We write Equation (133) as:

$$\begin{aligned} S_t &= \nu \left(T_{S,t}^{(-)} + T_{S,t}^{(+)} \right) \\ T_{S,t}^{(-)} &= \int_{\tau \leq 0} (1 - e^{-\lambda(t-\tau)})^{1-\rho} S_\tau e^{-\delta(t-\tau)} \underline{L}_e d\tau \\ T_{S,t}^{(+)} &= \int_{\tau=0}^{\tau=t} (1 - e^{-\lambda_\tau(t-\tau)})^{1-\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \end{aligned}$$

We proceed analogously as for L . For the term $T_{S,t}^{(-)}$, we use the following approximation:

$$T_{S,t}^{(-)} = \underbrace{\int_{\tau \leq -(T-t)} (1 - e^{-\lambda(t-\tau)})^{1-\rho} S_\tau e^{-\delta(t-\tau)} \underline{L}_e d\tau}_{T_{1,S,t}^{(-)}} + \underbrace{\int_{\tau=-(T-t)}^0 (1 - e^{-\lambda(t-\tau)})^{1-\rho} S_\tau e^{-\delta(t-\tau)} \underline{L}_e d\tau}_{T_{2,S,t}^{(-)}}$$

For $T_{1,S,t}^{(-)}$ and $t = t_n = (n-1)\Delta t$, we write:

$$\begin{aligned} T_{1,S,t}^{(-)} &= \underline{L}_e \int_{\tau \leq -(T-t)} (1 - e^{-\lambda(t-\tau)})^{1-\rho} e^{\underline{g}\tau} e^{-\delta(t-\tau)} d\tau \\ &\approx \underline{L}_e (1 - e^{-\lambda T})^{1-\rho} e^{-\delta t} \int_{\tau \leq -(T-t)} e^{(\underline{g}+\delta)\tau} d\tau \\ &\approx \frac{e^{\underline{g}t}}{1 - e^{-(\underline{g}+\delta)\Delta t}} e^{-(\underline{g}+\delta)T} (1 - e^{-\lambda T})^{1-\rho} \underline{L}_e \end{aligned}$$

For $T_{2,S,t}^{(-)}$ and $t = t_n = (n-1)\Delta t$, we write:

$$\begin{aligned} T_{2,S,t}^{(-)} &= \underline{L}_e \int_{\tau = -(T-t)}^0 (1 - e^{-\lambda(t-\tau)})^{1-\rho} e^{-\delta(t-\tau)} S_\tau d\tau \\ &= \underline{L}_e e^{\underline{g}t} \int_{\tau = -(T-t)}^0 (1 - e^{-\lambda(t-\tau)})^{1-\rho} e^{-(\underline{g}+\delta)(t-\tau)} d\tau \\ &= \underline{L}_e e^{\underline{g}t} \int_{u=t}^T e^{-(\underline{g}+\delta)u} du \\ &\approx (\Delta t) \underline{L}_e e^{\underline{g}t} \sum_{k=n}^N \left(e^{-k(\underline{g}+\delta)\Delta t} \right) \left(1 - e^{-\lambda k \Delta t} \right)^{1-\rho} \end{aligned}$$

Then we can write the approximation to $T_{S,t}^{(-)}$ in matrix form as:

$$\mathbf{T}_S^{(-)} = (\Delta t) \mathbf{P}_S^{(-)} \mathbf{J}$$

where $\mathbf{P}_S^{(-)}$ is an $(N+1) \times (N+1)$ matrix with entries:

$$\mathbf{P}_S^{(-),(n,k)} = \underline{L}_e \times \begin{cases} e^{\underline{g}(n-1)\Delta t} e^{-(\underline{g}+\delta)k\Delta t} (1 - e^{-\lambda k \Delta t})^{1-\rho} & \text{if } n \leq k \leq N \\ e^{\underline{g}(n-1)\Delta t} \frac{e^{-(\underline{g}+\delta)T}}{1 - e^{-(\underline{g}+\delta)\Delta t}} (1 - e^{-\lambda T})^{1-\rho} & \text{if } k = N+1 \end{cases},$$

and zeros elsewhere. For the term $T_{S,t}^{(+)}$, we use $T_{S,t_1}^{(+)} = 0$, and for $n \geq 2$, we make the approximation:

$$\begin{aligned} T_{S,t_n}^{(+)} &= \int_0^{t_n} (1 - e^{-\lambda_\tau(t-\tau)})^{1-\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \\ &= \sum_{k=1}^{n-1} \int_{(k-1)\Delta t}^{k\Delta t} (1 - e^{-\lambda_\tau(t-\tau)})^{1-\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \end{aligned}$$

$$\approx (\Delta t)e^{-\delta(n-1)\Delta t} \sum_{k=1}^{n-1} (1 - e^{-\lambda_{(k-1)\Delta t}(n-k)\Delta t})^{1-\rho_{(k-1)\Delta t}} S_{(k-1)\Delta t} e^{\delta(k-1)\Delta t} L_{e,(k-1)\Delta t}$$

This approximation can be written in matrix form as:

$$\mathbf{T}_S^{(+)} = (\Delta t)\mathbf{P}_S^{(+)}\mathbf{S}$$

where $\mathbf{P}_S^{(+)}$ is an $(N+1) \times (N+1)$ matrix with entries:

$$\mathbf{P}_S^{(+),(n,k)} = \begin{cases} 0 & \text{if } k \geq n \\ e^{-\delta(n-1)\Delta t} (1 - e^{-\lambda_k(n-k)\Delta t})^{1-\rho_k} e^{\delta(k-1)\Delta t} \mathbf{L}_{e,k} & \text{if } 1 \leq k \leq n-1 \end{cases}$$

We can then write Equation (133) in approximate matrix form as:

$$\mathbf{S} = \nu(\Delta t) \left(\mathbf{P}_S^{(-)}\mathbf{J} + \mathbf{P}_S^{(+)}\mathbf{S} \right),$$

or equivalently:

$$\mathbf{S} = \nu(\Delta t) \left(\mathbf{I} - \nu(\Delta t)\mathbf{P}_S^{(+)} \right)^{-1} \mathbf{P}_S^{(-)}\mathbf{J}.$$

Step 4, using the expression for dS_t from Lemma 2

The rate of change in S_t is given by:

$$\dot{S}_t = \int_{\tau \leq t} \left(1 - e^{-\lambda_\tau(t-\tau)} \right)^{-\rho_\tau} \left[((1 - \rho_\tau)\lambda_\tau + \delta) e^{-\lambda_\tau(t-\tau)} - \delta \right] (\nu S_\tau) e^{-\delta(t-\tau)} L_{e,\tau} d\tau \quad (144)$$

We can write this as:

$$\dot{S}_t = \nu [((1 - \rho_\tau)\lambda_\tau + \delta) y_t - \delta z_t] \quad (145)$$

with:

$$\begin{aligned} y_t &= \int_{\tau \leq t} \left(1 - e^{-\lambda_\tau(t-\tau)} \right)^{-\rho_\tau} S_\tau e^{-(\lambda_\tau + \delta)(t-\tau)} L_{e,\tau} d\tau \\ z_t &= \int_{\tau \leq t} \left(1 - e^{-\lambda_\tau(t-\tau)} \right)^{-\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \end{aligned} \quad (146)$$

Each of these integrals can then be approximated using the same steps as for the approximation to S . Namely, we write:

$$\mathbf{y} = \mathbf{P}_y^{(-)}\mathbf{J} + \mathbf{P}_y^{(+)}\mathbf{S} \quad (147)$$

where:

$$\begin{aligned}
\mathbf{P}_y^{(-),(n,k)} &= \underline{L}_e \times \begin{cases} e^{\underline{g}(n-1)\Delta t} e^{-(\underline{g}+\lambda+\delta)k\Delta t} (1 - e^{-\lambda k\Delta t})^{-\rho} & \text{if } n \leq k \leq N \\ e^{\underline{g}(n-1)\Delta t} \frac{e^{-(\underline{g}+\lambda+\delta)T}}{1 - e^{-(\underline{g}+\lambda+\delta)\Delta t}} (1 - e^{-\lambda T})^{-\rho} & \text{if } k = N + 1 \end{cases}, \\
\mathbf{P}_y^{(+),(n,k)} &= \begin{cases} 0 & \text{if } k \geq n \\ e^{-\delta(n-1)\Delta t} (1 - e^{-\lambda_k(n-k)\Delta t})^{-\rho_k} e^{\delta(k-1)\Delta t} e^{-\lambda_k(n-k)\Delta t} \mathbf{L}_{e,k} & \text{if } 1 \leq k \leq n-1 \end{cases} \\
\mathbf{S} &= \begin{cases} 1 & \text{if } n = 1 \\ \mathbf{S}^{(n-1)} e^{\frac{\dot{\mathbf{S}}^{(n-1)}}{\mathbf{S}^{(n-1)}} \Delta t} & \text{if } n > 1 \end{cases}
\end{aligned} \tag{148}$$

(Here, $\mathbf{S}^{(k)}$ refers to the k -th entry of the vector $\tilde{\mathbf{S}}$, and $\dot{\mathbf{S}}^{(k)}$ refers to the k -th entry of the vector $\dot{\mathbf{S}}^{(k)}$.) For z_t , we write

$$\mathbf{z} = \mathbf{P}_z^{(-)} \mathbf{J} + \mathbf{P}_z^{(+)} \mathbf{S} \tag{149}$$

where:

$$\begin{aligned}
\mathbf{P}_z^{(-),(n,k)} &= \underline{L}_e \times \begin{cases} e^{\underline{g}(n-1)\Delta t} e^{-(\underline{g}+\delta)k\Delta t} (1 - e^{-\lambda k\Delta t})^{-\rho} & \text{if } n \leq k \leq N \\ e^{\underline{g}(n-1)\Delta t} \frac{e^{-(\underline{g}+\delta)T}}{1 - e^{-(\underline{g}+\delta)\Delta t}} (1 - e^{-\lambda T})^{-\rho} & \text{if } k = N + 1 \end{cases}, \\
\mathbf{P}_z^{(+),(n,k)} &= \begin{cases} 0 & \text{if } k \geq n \\ e^{-\delta(n-1)\Delta t} (1 - e^{-\lambda_k(n-k)\Delta t})^{-\rho_k} e^{\delta(k-1)\Delta t} \mathbf{L}_{e,k} & \text{if } 1 \leq k \leq n-1 \end{cases}
\end{aligned} \tag{150}$$

Define:

$$\mathbf{D}_y \equiv \text{diag}((1 - \rho)\lambda + \delta), \quad \mathbf{D}_z = \delta \mathbf{I}. \tag{151}$$

The path for the rate of change $\dot{\mathbf{S}}$ must then be a solution to:

$$\dot{\mathbf{S}} = \nu(\Delta t) \left[\left(\mathbf{D}_y \mathbf{P}_y^{(-)} - \mathbf{D}_z \mathbf{P}_z^{(-)} \right) \mathbf{J} + \left(\mathbf{D}_y \mathbf{P}_y^{(+)} - \mathbf{D}_z \mathbf{P}_z^{(+)} \right) \mathbf{S} \right]$$

Note that this equation is not linear in $\dot{\mathbf{S}}$, since \mathbf{S} is not a linear function of $\dot{\mathbf{S}}$. However, it can be solved recursively, starting from the first entry of $\dot{\mathbf{S}}$, which only depends on pre-shock variables and on the current (time-0) value of $L_{e,t}$.

Computation of other variables

The aggregate stock of intangible capital is given by:

$$N_t = \int_{\tau(i) \leq t} \nu S_{i,\tau(i)} a_{i,t} di = \int_{\tau \leq t} \nu S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \quad (152)$$

The corresponding approximation can be written as:

$$\mathbf{N} = \nu(\Delta t) \left(\mathbf{P}^{(-)} \mathbf{J} \underline{L}_e + \mathbf{P}^{(+)} (\mathbf{S} \cdot \mathbf{L}_e) \right), \quad (153)$$

where \cdot indicates element-wise operations, and where the matrices $\mathbf{P}_N^{(-)}$ and $\mathbf{P}_N^{(+)}$ are given by:

$$\mathbf{P}^{(-),(n,k)} = \begin{cases} e^{\underline{g}(n-1)\Delta t} e^{-(\underline{g}+\delta)k\Delta t} & \text{if } n \leq k \leq N \\ e^{\underline{g}(n-1)\Delta t} \frac{e^{-(\underline{g}+\delta)T}}{1 - e^{-(\underline{g}+\delta)\Delta t}} & \text{if } k = N + 1 \end{cases},$$

$$\mathbf{P}^{(+),(n,k)} = \begin{cases} 0 & \text{if } k \geq n \\ e^{-\delta(n-1)\Delta t} e^{\delta(k-1)\Delta t} & \text{if } 1 \leq k \leq n - 1 \end{cases}$$

The aggregate value of existing projects is given by:

$$V_t = \Lambda_t v_t \int_{\tau(i) \leq t} x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{i,\tau(i)} a_{i,t} di = \Lambda_t v_t \int_{\tau \leq t} \nu x_\tau^{\rho_\tau} S_\tau e^{-\delta(t-\tau)} L_{e,\tau} d\tau \quad (154)$$

The corresponding approximation can be written as:

$$\mathbf{V} = \nu(\Delta t) \left(\mathbf{P}^{(-)} \mathbf{J} \underline{L}_e \underline{x}^\rho + \mathbf{P}^{(+)} (\mathbf{S} \cdot \mathbf{L}_e \cdot \mathbf{x}^\rho) \right) \cdot (\mathbf{\Lambda} \cdot \mathbf{v}), \quad (155)$$

where \mathbf{v} , the normalized value of new projects (including their value to imitators) is computed using the same approach as described for $\mathbf{v}^{(e)}$ (the normalized value of new projects to entrepreneurs), but setting $\gamma(x) = 0$. The approximation to the price of intangibles in output units is given by:

$$\mathbf{p}_N = \mathbf{W} \cdot / (\nu \mathbf{S}), \quad (156)$$

where \mathbf{W} is the wage rate. Aggregate Q is then given by $\mathbf{Q} = \mathbf{V} \cdot / (\mathbf{p}_N \cdot \mathbf{N})$.

Aggregate operating revenue is given by:

$$\Pi_t = \Lambda_t \int_{\tau(i) \leq t} x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{i,\tau(i)} a_{i,t} di = \frac{V_t}{v_t}, \quad (157)$$

so that it can be approximated as $\mathbf{\Pi} = \mathbf{V} \cdot / \mathbf{v}$. Aggregate output is given by:

$$Y_t = \frac{\Lambda_t}{1 - \zeta} \int_{\tau(i) \leq t} x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{i,\tau(i)}) a_{i,t} di = \frac{1}{1 - \zeta} \mathbf{\Pi}_t, \quad (158)$$

so that it can be approximated as $\mathbf{Y} = \frac{\mathbf{\Pi}}{1 - \zeta}$. The Herfindhal index of sales among existing projects is given by:

$$c_t = \left(\frac{V_t}{\Lambda_t v_t} \right) \int_{\tau(i) \leq t} \left(x_{\tau(i)}^{\rho_{\tau(i)}} (\nu S_{i,\tau(i)}) a_{i,t} \right)^2 di = \left(\frac{V_t}{\Lambda_t v_t} \right) \underbrace{\int_{\tau \leq t} x_{\tau}^{2\rho_{\tau}} \nu^2 S_{\tau}^2 L_{e,\tau} e^{-\delta(t-\tau)} d\tau}_{\equiv u_t}. \quad (159)$$

The integral u_t can be approximated by:

$$\mathbf{u} = \nu^2 (\Delta t) \left(\mathbf{P}_u^{(-)} \mathbf{J} \underline{x}^{2\rho} \underline{L}_e + \mathbf{P}^{(+)} (\mathbf{S} \cdot \mathbf{S} \cdot \mathbf{x}^{2\rho} \cdot \mathbf{L}_e) \right), \quad (160)$$

where the matrix $\mathbf{P}_u^{(-)}$ is given by:

$$\mathbf{P}_u^{(-),(n,k)} = \begin{cases} e^{2g(n-1)\Delta t} e^{-(2g+\delta)k\Delta t} & \text{if } n \leq k \leq N \\ e^{2g(n-1)\Delta t} \frac{e^{-(2g+\delta)T}}{1 - e^{-(2g+\delta)\Delta t}} & \text{if } k = N + 1 \end{cases}.$$

Consistency with balanced growth path

In order to make approximation errors consistent, we compute the values used for the BGP at time $t < 0$ and $t > T$ in the method above using the same numerical schemes as for the transitional dynamics. Specifically, we compute the BGP values of (x, L) by iterating on the following steps:

Step 1: Given a value for g , say $g = g_-$, define $\mathbf{\Lambda} = (e^{\zeta g_-(n-1)\Delta t})_{n=1}^{N+1}$, and update the corresponding values of $(x, v^{(e)})$ using:

$$x = \max_{\tilde{x}} \mathbf{O}^{(1)}(\tilde{x}), \quad v^{(e)} = \frac{\mathbf{O}^{(1)}(x)}{\mathbf{\Lambda}^{(1)} x^{\rho}}.$$

Step 2: Given the values of $g, x, v^{(e)}$, update the value of L_e through:

$$L_e = \frac{1}{1 + \frac{\zeta}{1 - \zeta} \frac{\Delta t}{x^{\rho} v^{(e)}} \mathbf{A}'_L \mathbf{J}} \quad (161)$$

where \mathbf{A}_L is a vector defined analogously to the first row of $\mathbf{P}_L^{(-)}$:

$$\mathbf{A}_L^{(n,k)} = x^\rho \times \begin{cases} e^{-(g_+ + \delta)k\Delta t} & \text{if } 1 \leq k \leq N \\ \frac{e^{-(g_+ + \delta)T}}{1 - e^{-(g_+ + \delta)\Delta t}} & \text{if } k = N + 1 \end{cases}$$

Step 3: Given the values of $x, v^{(e)}, L^e$, update the value of g through:

$$g = (\nu\Delta t) (\mathbf{A} - \mathbf{B}) \mathbf{J}$$

and where \mathbf{A} and \mathbf{B} , are, respectively, the first rows of the matrices $\mathbf{D}_y \mathbf{P}_y^{(-)}$ and $\mathbf{D}_z \mathbf{P}_z^{(-)}$:

$$\mathbf{A}^{(n)} = ((1 - \rho)\lambda + \delta)L_e \times \begin{cases} e^{-(g_+ + \lambda + \delta)n\Delta t} (1 - e^{-\lambda n\Delta t})^{-\rho} & \text{if } 1 \leq n \leq N \\ \frac{e^{-(g_+ + \lambda + \delta)T}}{1 - e^{-(g_+ + \lambda + \delta)\Delta t}} (1 - e^{-\lambda T})^{-\rho} & \text{if } n = N + 1 \end{cases},$$

$$\mathbf{B}^{(n)} = \delta L_e \times \begin{cases} e^{-(g_+ + \delta)n\Delta t} (1 - e^{-\lambda n\Delta t})^{-\rho} & \text{if } 1 \leq n \leq N \\ \frac{e^{-(g_+ + \delta)T}}{1 - e^{-(g_+ + \delta)\Delta t}} (1 - e^{-\lambda T})^{-\rho} & \text{if } n = N + 1 \end{cases}.$$

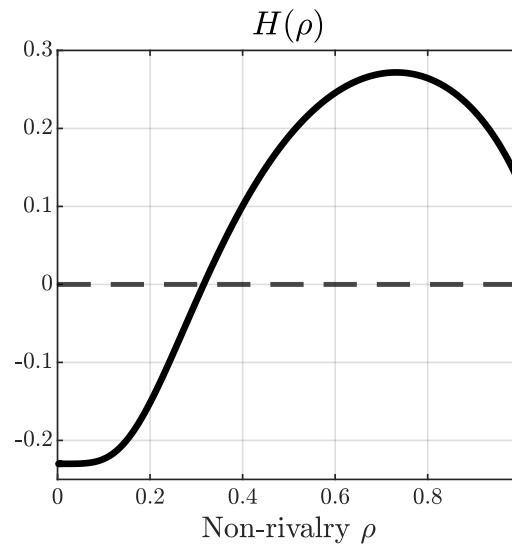


Figure A1: The function $H(\rho)$ in the model with obsolescence risk. As described in Lemma 3, a balanced growth path with strictly positive growth only exists when $H(\rho) > 0$. Parameter values used for this plot are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\eta = 1/0.03$, $\xi = 0.3$, and $\delta = 0.02$.

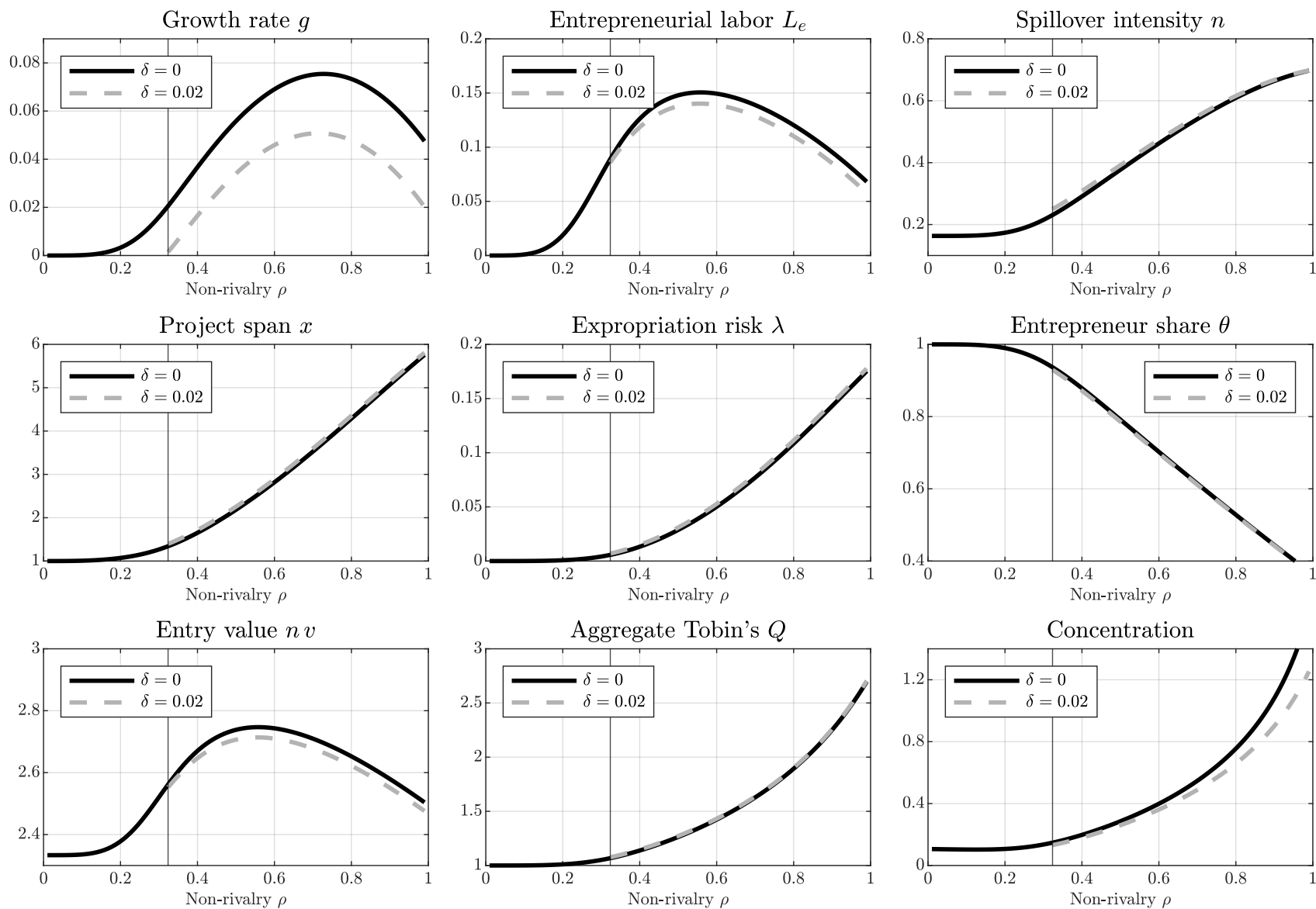


Figure A2: Comparative statics of the balanced growth path with respect to the degree of non-rivalry, ρ , in the baseline model (black line) and in the model with obsolescence risk (dashed gray line). The expropriation risk function is assumed to be $\gamma(x) = \frac{(x-1)^{1+\xi}}{\eta(1+\xi)}$. Parameter values used are $r = 0.07$, $\zeta = 0.7$, $\nu = 0.7$, $\eta = 1/0.03$, and $\xi = 0.3$. In the model with obsolescence risk, we use an annual rate of obsolescence of $\delta = 0.02$. The measure of concentration used is the Herfindahl index of sales among projects.