Predicting the Last Zero of a Spectrally Negative Lévy process.

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Research Showcase LSE

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[1] joint work with José Pedraza

Insurance

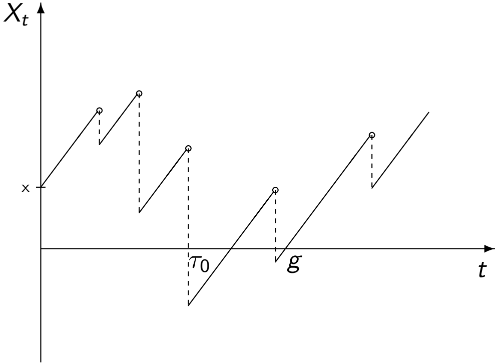
Crámer–Lundberg Process

$$X_t = x + ct - \sum_{j=1}^{N_t} Y_j,$$

where $x, c > 0$, N_t is a Poisson process with intensity $\lambda > 0$ and $\{Y_j\}_{j\geq 1}$ is a sequence of positive i.i.d random variables independent of N_t.

Two quantities of interest are the moment of ruin and the last zero of the process

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}$$
$$g = \sup\{t \geq 0 : X_t \leq 0\}$$


X_t
x
τ_0
g
t

Degradation models

We can model the ageing of a device with $D = (D_t, t \geq 0)$ where

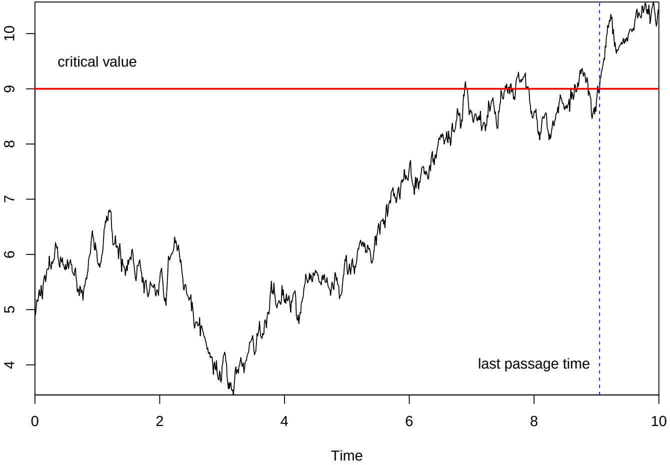
$$D_t = G_t + \sigma B_t$$

where $\sigma \geq 0$, $(G_t, t \geq 0)$ is a subordinator and $(B_t, t \geq 0)$ is an standard Brownian motion. Then, D is an spectrally positive Lévy process.

The failure time of the device can be defined as

$$g^* = \sup\{t > 0 : X_t \geq f_*\}$$

where f_* is a critical value.


critical value
last passage time
Time
0
2
4
6
8
10
4
5
6
7
8
9
10

Last passage times

Last passage times are random times which are not stopping times. For example, if

$$g = \sup\{t > 0 : X_t \leq 0\}$$

then we have that

$$\{g < t\} = \{X_s > 0 \text{ for all } s > t\} \in \mathcal{F}.$$

On the other hand, stopping times are random times such that the decision whether to stop or not depends only on the past and present information.

Some further motivation, Shiryaev 2002

In a 2002 paper by Shiryaev: *We will consider below the case where the process X is standard linear Brownian motion B. From the viewpoint of the modern mathematical finance this model due to Bachelier is too idealized. However we will further see that even in this relatively simple case the solution to the corresponding optimization problem is rather nontrivial. On the other hand the solution of the case of a Brownian motion gives a way to solve this problem of more general cases*

Albert Shiryaev at LSE

Guanajuato/CIMAT

José would soon be showing me the way

Optimal prediction problems

Consider X a stochastic process and let be g a last passage time. An optimal prediction problem is

$$V_* = \inf_{\tau \in \mathcal{T}} \mathbb{E}(|g - \tau|)$$

where $\mathcal{T}$ is the set of all stopping times of X.

- [Du Toit et al., 2008] predicted the last zero of a Brownian Motion with drift.
- [du Toit and Peskir, 2008] predicted the time of the ultimate maximum for Brownian motion with drift.

Optimal prediction problems

- [Glover et al., 2013] predicted the time in which a transient difussion attains its ultimate minimum.
- [Glover and Hulley, 2014] predicted the last passage time of a level $z > 0$ for an arbitrary nonnegative time-homogeneous transient diffusion.
- [Baurdoux and Van Schaik, 2014] predicted the time at which a Lévy process attains its ultimate supremum.
- [Baurdoux et al., 2016] predicted when a positive self-similar Markov process attain its pathwise global supremum or infimum before hitting zero for the first time.

and more...

A process $X = (X_t, t \geq 0)$ is said to be a Lévy process if

- The paths of X are $\mathbb{P}$-a.s. càdlàg
- X has independent increments.
- X has stationary increments.
- $X_0 = 0$ a.s.

Examples

- Brownian motion.
- Compound Poisson process.
- Gamma process.
- Stable processes.

The law of a Lévy process is characterised by the characteristic exponent,

$$\Psi(\theta) = -\log\left(\mathbb{E}(e^{i\theta X_1})\right).$$

Lévy–Khintchine Formula for Lévy processes

Exist $\sigma \geq 0$, $\mu \in \mathbb{R}$ and measure Π (Lévy measure) concentrated on $\mathbb{R} \setminus \{0\}$, with $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$, such that

$$\Psi(\theta) = i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - e^{i\theta x} + i\theta x \mathbb{I}_{\{|x|<1\}})\Pi(dx)$$

for all $\theta \in \mathbb{R}$.

Lévy–Itô decomposition

$$X_t = \sigma B_t - \mu t + \int_0^t \int_{\{|x|\geq 1\}} xN(ds, dx) + \int_0^t \int_{\{|x|<1\}} x(N(ds, dx) - ds\Pi(dx))$$

Infinite horizon problem

Let X a spectrally negative Lévy process such that $\lim_{t\to\infty} X_t = \infty$. Consider g the last that time that X is below the level zero,

$$g = \sup\{t \geq 0 : X_t \leq 0\}.$$

We solve the optimal prediction problem

$$V_* = \inf_{\tau \in \mathcal{T}} \mathbb{E}(|g - \tau|) \tag{1}$$

where $\mathcal{T}$ is the set of all stopping times.

Lemma

Assume that $\int_{(-\infty,1)} x^2 \Pi(dx) < \infty$. The optimal prediction problem (1) is equivalent to the standard optimal stopping problem

$$V(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left(\int_0^{\tau} G(X_s) ds \right), \tag{2}$$

where the function G is given by

$$G(x) = 2F(x) - 1$$

and $F(x) = \mathbb{P}(-\inf_{t \geq 0} X_t \leq x) = \psi'(0+) W(x)$. The function V_* is given by $V_* = V(0) + \mathbb{E}(g)$.

Function G

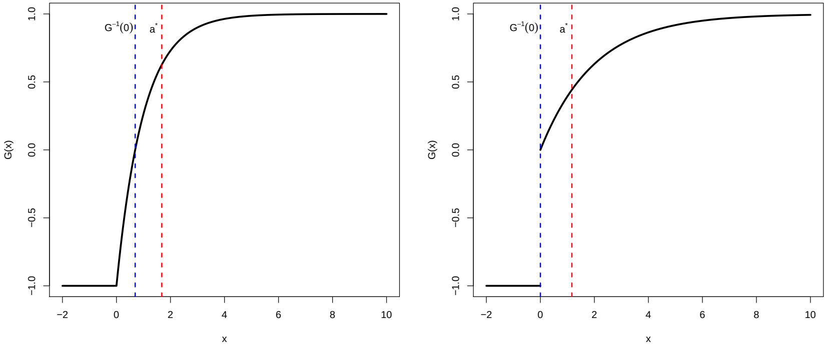


Figure: Left side: $\Pi(dx) = e^{2x}(e^x - 1)^{-3}dx$, $x > 0$ without Gaussian component. Right side: Crámer–Lundberg process with $c = 2$, $\lambda = 1$, $\xi \sim \exp(1)$.

Theorem

Suppose that X is a spectrally negative Lévy process drifting to infinity with Lévy measure Π satisfying

$$\int_{(-\infty,-1)} x^2 \Pi(dx) < \infty.$$

Let

$$a^* = \inf\left\{ x \geq 0 : \int_{[0,x]} F(x-y)dF(y) \geq 1/2 \right\}$$

Then the optimal stopping time is given by

$$\tau^* = \inf\{t \geq 0 : X_t \geq a^*\}.$$

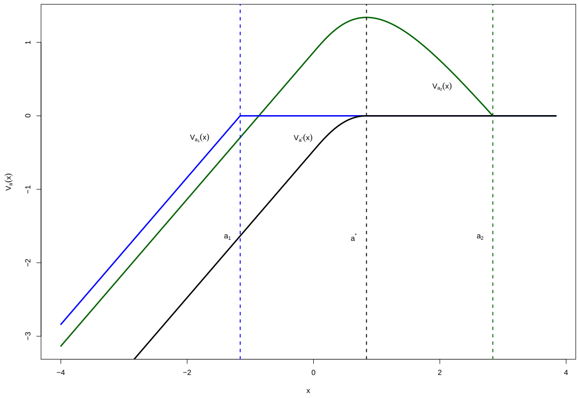


Figure: Brownian motion with drift. Function $x \mapsto V_a$ for different values of a. Blue: $a < a^*$; green: $a > a^*$; black: $a = a^*$.

The last zero process

Optimal prediction in $L^p, p > 1$ sense is much trickier.

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Let X be a spectrally negative Lévy process drifting to infinity. Let $t \geq 0$ and $x \in \mathbb{R}$, we define as $g_t^{(x)}$ as the last time that the process is below x before time t, i.e.,

$$g_t^{(x)} = \sup\{0 \leq s \leq t : X_s \leq x\},$$

with the convention $\sup \emptyset = 0$. We simply denote $g_t := g_t^{(0)}$ for all $t \geq 0$. We define

$$U_t := t - g_t$$

the time of the current excursion before time t above zero.

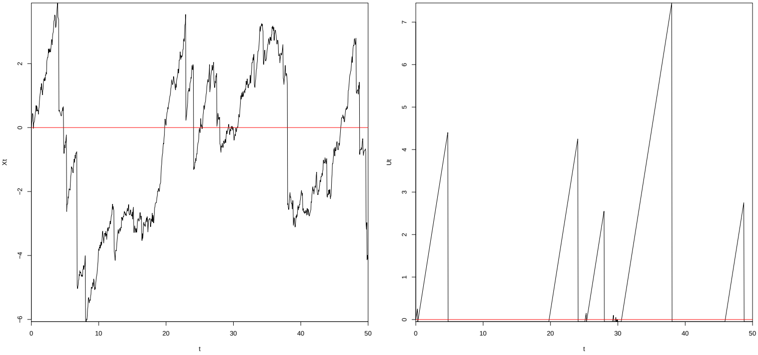

Figure: Sample path of X on the left hand side. Sample path of U_t on the right hand side.

Let X be a spectrally negative Lévy process drifting to infinity with $\int_{(-\infty,-1)} |x|^{p+1} < \infty$. Define the optimal prediction problem

$$V_* = \inf_{\tau \in \mathcal{T}} \mathbb{E}(|\tau - g|^p) \tag{3}$$

where $\mathcal{T}$ is the set of all stopping times and $p > 1$.

Lemma

Problem (3) is equivalent to the optimal stopping problem

$$V(u,x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^\tau G(U_s, X_s) ds \right), \tag{4}$$

where the function G is given by

$$G(u,x) = u^{p-1}\psi'(0+)W(x) - \mathbb{E}_x(g^{p-1}),$$

Properties of V that can be shown

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3. V is a continuous function in E.

We define for any $u > 0$

$$b(u) := \inf\{x > 0 : V(u,x) = 0\}.$$

Then $\tau_D = \inf\{t > 0 : X_t > b(U_t)\}$.

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6. b can be characterised, and approximated numerically

Preprints

- Main paper: L^p optimal prediction of the last zero of a spectrally negative Lévy process https://arxiv.org/abs/2003.06869

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- Auxiliary results: On the last zero process with applications in corporate bankruptcy https://arxiv.org/abs/2003.06871

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