ON MEAN FIELD GAMES AND APPLICATIONS

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WHAT? WHY?

- The previous video was shown to the inventors of MFGs by some engineers working for Huawei that they were subsidizing their research; “The next mobile telecommunication networks are going to look much more like an intersection like that”.
- **Decentralized / Centralized** intelligence; if you think about the future mobile communication networks, you expect very soon to have about two millions connected objects per square kilometer, and there is no way you can have some kind of centralized intelligence that organize all the traffic.
- Mathematical models for situations involving many agents/players.

INTRODUCTION

- New class of models for the average (**Mean Field**) behavior of "small" agents (**Games**) started in the early 2000's by J.M. Lasry and P.L. Lions; "small" means that one single agent is not going to change the outcome of the game.
- Project ("Finance and Sustainable Development") sponsored by two major companies in France, EDF (French Electricity Company) and Credit Agricole. Three characteristics: **long-term planning, a lot of externalities, rational decisions.**
- Requires new **mathematical theories**.
- Numerous applications: economics, finance, social networks, crowd motions, telecommunications, Meaningful Data and Machine Learning.
- Independent introduction of a particular class of MFG models by M. Huang, P.E. Caines and R.P. Malhamé in 2006.
- Previous related works in Economics: anonymous games (discrete-time games), Krusell-Smith .

- Combination of Mean Field theories (classical in Physics and Mechanics) and the notion of Nash equilibria in Games theory.

- Nash equilibria for continua of “small” players: a single heterogeneous group of players (adaptions to several groups …).

- Each generic player is “rational”, i.e. she/he tries to optimize (control) a criterion that depends on the others (the whole group) and the optimal decision affects the behavior of the group (however, this interpretation is limited to some particular situations …).

- Nota: Herd behavior is rational because this means that I am trying to have the same type of behavior than the others.

- Huge class of models: agents —> particles, no dep. on the group are two extreme particular cases.

A REALLY SIMPLE EXAMPLE

- Simple example, not new but gives an idea of the general class of models: **Where do we put our towels on the beach?**

- E metric space, N players ($1 \leq i \leq N$) choose a position $x_i \in E$ according to a criterion $F_i(X)$ where $X = (x_1, \ldots, x_N) \in E^N$.

- Nash equilibrium: $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_N)$ if for all $1 \leq i \leq N$ we have that $\bar{x}_i$ min over E of $F_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_N)$; i.e., you freeze all the others except the i's one.

- Usual difficulties with the notion.

- $N \to \infty$? simpler?

- Indistinguishable players; for the player *i* all the others look the same.

$F_i(X) = F(x_i, \{x_j\}_{j \neq i})$, F sym in $(x_j)_{j \neq i}$

- Part of the mathematical theories is about $N \to \infty$: You have a common framework that does not depend on N.

$$F_i = F(x, m) \quad x \in E, \quad m \in \mathscr{P}(E)$$

where

$$x = x_i, \quad m = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$$

- **"Theorem":** Nash equilibria converges, as $N \to \infty$, to solutions of

$$\text{(MFG)} \quad \forall x \in \operatorname{Supp} m, F(x, m) = \inf_{y \in E} F(y, m) .$$

- Facts:
  1. General existence and stability results.
  2. Uniqueness if $(m \to F(\cdot, m))$ monotone.
  3. If $F = \Phi^{'}(m)$, then $(\min_{\mathscr{P}(E)} \Phi)$ yields one solution of (MFG).

A SECOND SIMPLY EXAMPLE

$$E = \mathbb{R}^d, F_i(X) = f(x_i) + g\left(\frac{\#\{j / |x_i - x_j| < \varepsilon\}}{(N-1)|B_\varepsilon|}\right)$$

- $g \uparrow$ aversion crowds, $g \downarrow$ like crowds

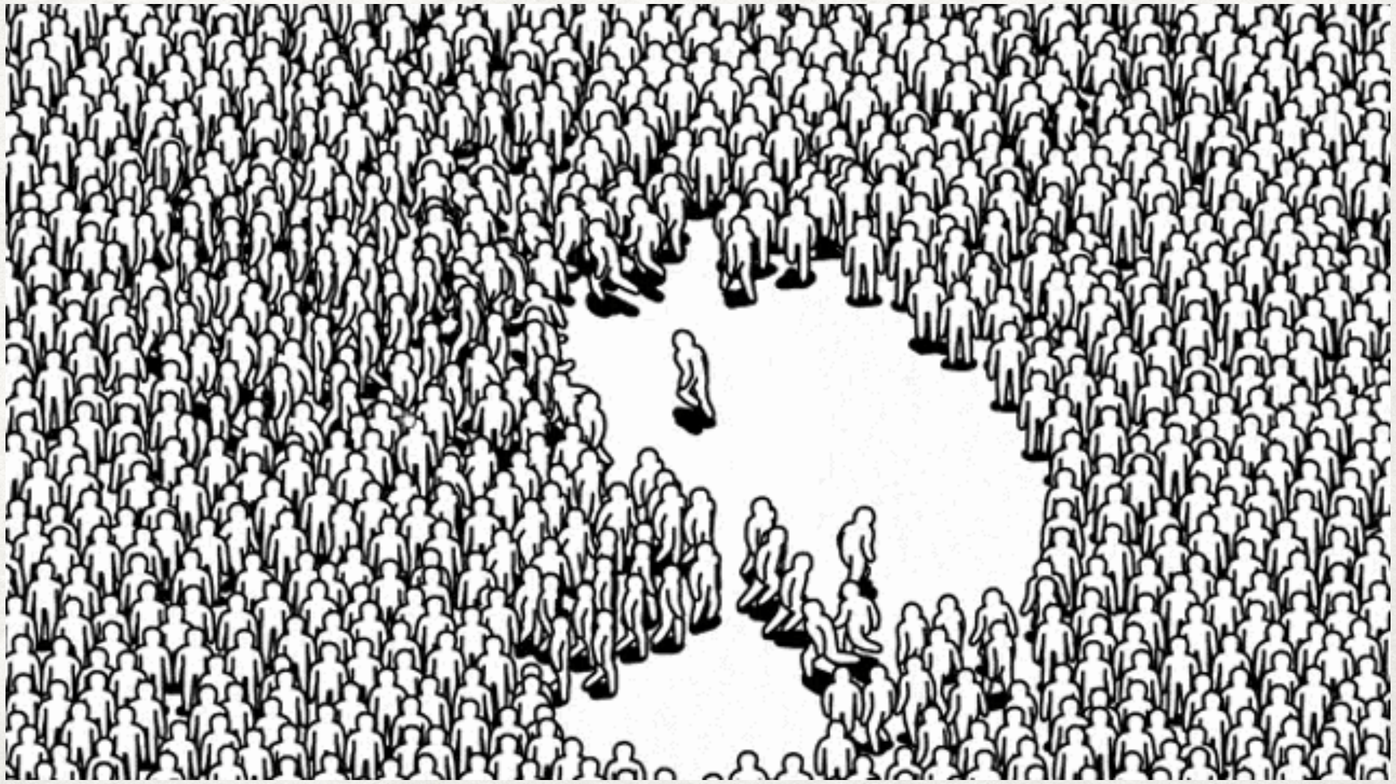
$$F(x,m) = f(x) + g(m + 1_{B_\varepsilon}(x)(|B_\varepsilon|^{-1})$$

$$\varepsilon \to 0 \quad F(x,m) = f(x) + g(m(x))$$

(MFG) supp $m \subset$ Arg min $\left(f(x) + g(m(x))\right)$

- $g \uparrow$ uniqueness, $g \downarrow$ non uniqueness,

- $g \downarrow$ Everyones will flock to one place or to another.



$$\min\left\{\int fm + \int G(m) \,|\, m \in \mathscr{P}(E)\right\}$$

$$G = \int_0^z f(s)\,ds$$

- Explicit solution if $g \uparrow$.

GENERAL STRUCTURE (in a very particular example)

- Dynamical problem, horizon T, continuous time and space, Brownian noises (both independent and common), no inter temporal preference rate, control on drifts (Hamiltonian H), criterion dep. only on m.

- $U(x,m,t)$, $x \in \mathbb{R}^d, m \in \mathscr{P}(\mathbb{R}^d)$ or $\mathscr{M}_+(\mathbb{R}^d), t \in [0,T]$ and $H(x,p,m)$ (convex in $p \in \mathbb{R}^d$)

- MFG master equation is an infinite dimensional object

$$
\begin{cases}
\frac{\partial U}{\partial t} - (\nu + \alpha)\nabla_x U + H(x, \nabla_x U, m) \\
+\langle \frac{\partial U}{\partial m}, -(\nu+\alpha)\Delta m + \mathrm{div}\left(\frac{\partial H}{\partial p} m\right)\rangle + \\
-\alpha \frac{\partial U}{\partial m^2}\langle \nabla m, \nabla m\rangle + 2\alpha \langle \frac{\partial}{\partial m}\nabla_x U, \nabla m\rangle = 0
\end{cases}
$$

and $U|_{t=0} = U_0(x,m)$ (final cost) .

- ν amount of ind. rand., α amount of common rand.

TWO PARTICULAR CASES

- The previous infinite dimensional problem reduces to a finite dimensional one in two cases:
- **(1) Independent noises** ($\alpha = 0$), no common noise; int. along caract. in m yields:

$$(\text{MFG}) \begin{cases} \frac{\partial U}{\partial t} - \nu \nabla_x U + H(x, \nabla_x U, m) = 0, \\ u\,|_{t=0} = U_0(x, m(0)),\ m\,|_{t=T} = \bar{m}, \\ \frac{\partial m}{\partial t} + \nu \Delta m + \text{div}\left(\frac{\partial H}{\partial p} m\right) = 0 \end{cases}$$

where $\bar{m}$ is given. We have a **FORWARD BACKWARD SYSTEM.**

- **(2) Finite state MFG** which refers to mean field games on graphs/networks. Work in progress: "A mean field game of shipping: modeling and the use of real data" co-authored with Michele Bergami, Simone Moawad, Barath Raaj Suria Narayanan (PG students, LSE), Evan Chien Yi Chow (ADIA); Charles-Albert Lehalle.

A LAST S. EXAMPLE: the Mexican Wave

- In a stadium we have about 100 rows: let us put them into a column and say that the population on the column is huge.
- We have a density on that column which is being parametrized by the angle (let us think that the stadium is a circle).
- The variable of each agent in that column is the position $z \in [0,1]$; $z = 0$ means sitting, $z = 1$ means standing.
- The cost function has three term: (1) clearly moving (kinetic energy) is part of the cost; (2) second, a position between 0 and 1 is uncomfortable (concave cost function); (3) finally, each person wants to be in the same state than his/her neighbor (mimicking).
- One can prove the following statement: “If the mimicking coefficient is not too small you have up to a change of orientation a unique stable periodic wave”.

THANK YOU