An Equilibrium Model of Production and Expansion

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Outline

1. Single Producer
2. Market Clearing; Equilibrium
3. Ramifications

Outline

Price-taking Producer

Decisions

1. quantity $Q_t \mathrm{d}t$ to produce.
2. investment $I_t \mathrm{d}t$ in expanding capacity (better, capability).

Instantaneous P/L

$$\left(P_t Q_t - \texttt{prod.cost}(Q_t, C_t) - I_t\right) \mathrm{d}t.$$

with prices P and capacity C. Leads to *myopic* optimal Q^*.

Evolution of capacity C

$$\mathrm{d}C_t = \sqrt{k I_t} \mathrm{d}t,$$

where $k > 0$. Present decisions affect future value.

Optimal Production

Cost: decreasing returns to scale

$$\texttt{prod.cost}(q, c) = \frac{1}{4}\left(\frac{q}{c}\right)^2.$$

Optimally produced quantity

$$Q^* \ := \ 2C^2P.$$

Optimal instantaneous P/L

$$\left(C_t^2 P_t^2 - I_t\right) \mathrm{d}t.$$

Control Problem

Producer's dynamic value function

$$u_s(c) = \sup_{I_t;\, t \geq s} \mathbb{E}_s \left[\int_s^\infty \exp(-r(t-s)) \left(C_t^2 P_t^2 - I_t \right) \mathrm{d}t \right]$$

with capacity evolution

$$C_t = c + \int_s^t \sqrt{k I_u} \mathrm{d}u; \quad t \geq s; \quad c > 0.$$

We wish to compute u and the optimal (I^*, C^*).

How to characterise u?

- Assume randomness driven by Brownian motion W.
- Martingale method, with Itô-Wentzell, provides a "BSPDE".
- Inspect dynamics, and conjecture $u_s(c) = c^2 Y_s$ for a process Y.

Verification

Lemma

Let (Y, Z) satisfy the "BSDE"

$$\mathrm{d}Y_t = -\left(P_t^2 - rY_t + kY_t^2\right)\mathrm{d}t + Z_t \mathrm{d}W_t.$$

Assume Y is *positive*, and some transversality(+) conditions. Define

$$I^* \ := \ C_0^2 k \exp\left(2k \int_0^{\cdot} Y_t \mathrm{d}t\right) Y^2,$$

which leads to capacity

$$C^* \ := \ C_0 \exp\left(k \int_0^{\cdot} Y_t \mathrm{d}t\right).$$

Assuming that I^* is admissible, it is optimal, and $u(c) = c^2 Y$.

Outline

Producers; Demand

Continuum of producers

- C_0: random variable of initial producer capacities...
- ...independent of W, which will drive demand.
- Competitive framework: the law of C_0 has no atoms.

Exogenous demand function

$$D_t(p) = D_t p^{-\delta},$$

where $\delta > 0$, and $D \equiv D(1)$ is such that

$$\log D_t = \log D_0 + \rho t + \sigma W_t,$$

with $D_0 > 0$, $\rho > 0$.

Market Clearing

Demand should equal supply

$$D_t P_t^{-\delta} = \mathbb{E}\left[2 C_t^2 P_t \mid \mathcal{W}_t\right]$$
$$= 2\mathbb{E}[C_0^2] P_t \exp\left(2k \int_0^{\cdot} Y_t \mathrm{d}t\right).$$

Solving for equilibrium prices,

$$X := \frac{1+\delta}{2k} \log P = x_0 + \int_0^{\cdot} (\mu - Y_t)\, \mathrm{d}t + \nu W;$$

$$x_0 = \frac{1}{2k} \log \frac{D_0}{2\mathbb{E}[C_0^2]}; \quad \mu := \frac{\rho}{2k} > 0; \quad \nu := \frac{\sigma}{2k}.$$

Coupled Forward-Backward SDE system

The FBSDE system

Recalling the dynamics for Y in terms of $P = p(X)$, we obtain

$$\mathrm{d}X_t = (\mu - Y_t)\,\mathrm{d}t + \nu \mathrm{d}W_t; \quad X_0 = x_0;$$
$$\mathrm{d}Y_t = -\left(\exp(\beta X_t) - rY_t + kY_t^2\right)\mathrm{d}t + Z_t \mathrm{d}W_t;$$

where

$$\beta := \frac{4k}{1+\delta} > 0.$$

Associated ODE

Conjecture solution satisfying $Y = g(X)$, use Itô's formula, compare coefficients: $Z = \nu g'(X)$, and, for $x \in \mathbb{R}$,

$$\frac{\nu^2}{2} g''(x) + (\mu - g(x))\, g'(x) - rg(x) + kg(x)^2 + \exp(\beta x) = 0.$$

Solution

Theorem

Assume that $\beta > 0$, $k > 0$, $\mu > 0$, as well as

$$r > k\mu + \frac{1}{2}(k\nu)^2.$$

Then, there exists a (unique) strictly positive, strictly increasing C^2 function g such that the previous ODE holds. Furthermore:

- There is a unique (strong) solution to the SDE

$$\mathrm{d}X_t = (\mu - g(X_t))\,\mathrm{d}t + \nu \mathrm{d}W_t; \quad X_0 = x_0;$$

- The triple $(X,\ Y \equiv g(X),\ Z \equiv \nu g'(X))$ solves the FBSDE system.
- The previous also satisfy all technical conditions for optimality verification; therefore, they lead to equilibrium.

Outline

Ramifications

On capacity

- Linear returns: $\mathrm{d}C_t = k\mathrm{d}I_t$.
- Depreciation: $\mathrm{d}C_t = \ldots - \psi C_t \mathrm{d}t$.
- Reversible: $\mathrm{d}C_t = (\mathrm{d}C_t)^+ - (\mathrm{d}C_t)^-$.

On heterogeneity

- Producer coefficients; different abilities.

On market structure

- Monopoly: single agent; price impact.
- Oligopoly: finite number of agents.
- General: arbitrary law of C_0.

The End

Thank you!