Kernelized Cumulants: Beyond Mean Embeddings

Zoltán Szabó

Joint work with:

- Patric Bonnier, Harald Oberhauser
- @ Mathematical Institute, University of Oxford.


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Moments and cumulants on $\mathbb{R} \ni X \sim \gamma$

- Moments $\mu(\gamma) := \left(\mu^{(i)}(\gamma)\right)_{i\in\mathbb{N}}$:

$$\mu^{(i)}(\gamma) := \mathbb{E}\left(X^i\right) \in \mathbb{R}, \qquad \mu^{(0)}(\gamma) := 1.$$

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- Cumulants $\kappa(\gamma) = \left(\kappa^{(i)}(\gamma)\right)_{i\in\mathbb{N}}$: from the moment-generating function

$$\sum_{i\in\mathbb{N}} \kappa^{(i)}(\gamma)\frac{\theta^i}{i!} = \log\left(\sum_{i\in\mathbb{N}} \mu^{(i)}(\gamma)\frac{\theta^i}{i!}\right).$$

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$\kappa^{(1)}(\gamma) = \mathbb{E}(X)$	mean
$\kappa^{(2)}(\gamma) = \mathbb{E}(X - \mathbb{E}X)^2$	variance
$\kappa^{(3)}(\gamma) = \mathbb{E}(X - \mathbb{E}X)^3$	3rd central moment
$\kappa^{(4)}(\gamma) = \mathbb{E}(X - \mathbb{E}X)^4 - 3\left[\mathbb{E}(X - \mathbb{E}X)^2\right]^2$	
$\kappa^{(5)}(\gamma) = \mathbb{E}(X - \mathbb{E}X)^5 - 10\mathbb{E}(X - \mathbb{E}X)^3\mathbb{E}(X - \mathbb{E}X)^2$	

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$$\kappa^{(3)}(\gamma) = \mathbb{E}\left(X^3\right) - \overbrace{3\mathbb{E}\left(X^2\right)\mathbb{E}(X)}^{\mathbb{E}(XXX')+\mathbb{E}(XX'X)+\mathbb{E}(X'XX)} + 2\mathbb{E}^3(X), \qquad \{\{1,2,3\}\}, \{\{1,2\},\{3\}\}, \{\{1,3\},\{2\}\}, \{\{2,3\},\{1\}\}, \{\{1\},\{2\},\{3\}\},$$

$\ldots$

where $X, X' \sim \gamma$, independent.

Question

What are the weights in front of the moments?

Unzipping cumulants on $\mathbb{R}$: the weights

m	elements of $\pi \in P(m)$	$\lvert\pi\rvert$	c_π
1	$\{1\}$	1	1
2	$\{1,2\}$	1	1
	$\{1\},\{2\}$	2	-1
3	$\{1,2,3\}$	1	1
	$\{1,2\}, \{3\}$	2	-1
	$\{1,3\}, \{2\}$	2	-1
	$\{2,3\}, \{1\}$	2	-1
	$\{1\}, \{2\}, \{3\}$	3	2

with $P(m) :=$ all partitions of $[m]$, $c_\pi = (-1)^{|\pi|-1}(|\pi|-1)!$

Motivation, i.e. one reason why one likes cumulants

Moment and cumulants on $\mathbb{R}^d$

Change $\mathbb{E}\left(X^i\right) \in \mathbb{R}$ to $\mathbb{E}\left[X_1^{i_1} \cdots X_d^{i_d}\right] \in \mathbb{R}$ ($\mathbf{i} \in \mathbb{N}^d$). log, $P(m)$: ✓

Known theorem [Billingsley, 2012]

Let γ be a probability measure on a bounded subset of $\mathbb{R}^d$ with cumulants $\kappa(\gamma)$ and let $(X_1, \ldots, X_d) \sim \gamma$. Then

1. $\gamma \mapsto \kappa(\gamma)$ is injective.
2. $X_1, \ldots, X_d$ are independent $\Leftrightarrow \kappa^{\mathbf{i}}(\gamma) = 0$ for all $\mathbf{i} \in \mathbb{N}_+^d$.

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Motivation

1. Various data types, nonlinear features: kernels.
2. Linear: not even characteristic (see MMD and HSIC).
3. Computable estimators.

Idea

Lifting

$(X_1, \ldots, X_d) \in \times_{j=1}^{d} \mathcal{X}_j \to (\Phi_1(X_1), \ldots, \Phi_d(X_d)) \in \times_{j=1}^{d} \mathcal{H}_{k_j}.$

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Ingredients:

1. Moments: swap out $\mathbb{E}\left[X_1^{i_1} \cdots X_d^{i_d}\right] \in \mathbb{R}$ to

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2. From moments to cumulants:
   - log on tensor algebras, or
   - combinatorial description of cumulants ($\leftarrow$ a bit simpler, but $\Leftrightarrow$).

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2. From moments to cumulants:
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3. Computation: by the 'expected kernel trick' (V-statistics).

Kernel (generalization of $\mathbf{a}^\top \mathbf{b}$), RKHS

- Def-1 (feature space):

$$k(a,b) = \langle \Phi(a), \Phi(b) \rangle_{\mathcal{H}}.$$

- Def-2 (reproducing kernel):

$$k(\cdot, b) \in \mathcal{H}, \qquad f(b) = \langle f, k(\cdot, b) \rangle_{\mathcal{H}}.$$

- Def-3 (Gram matrix): $\mathbf{G} = [k(x_i, x_j)]_{i,j=1}^n \in \mathbb{R}^{n \times n} \succeq 0$.

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Notes

- $k \overset{1:1}{\leftrightarrow} \mathcal{H}_k = \overline{\operatorname{Span}}(k(\cdot, x) : x \in \mathcal{X})$: Fourier analysis, polynomials, splines, ...
- Examples: $k_p(\mathbf{x}, \mathbf{y}) = (\langle \mathbf{x}, \mathbf{y} \rangle + \gamma)^p$, $k_G(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x}-\mathbf{y}\|_2^2}$.
- Kernels exist on various domains!

Mean embedding

- Mean embedding (Bochner integral):

$$\mu_k(\mathbb{P}) := \int_{\mathcal{X}} \underbrace{k(\cdot, x)}_{\Phi(x) \in \mathcal{H}_k} \mathrm{d}\mathbb{P}(x) \in \mathcal{H}_k.$$

Mean embedding, MMD

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$$\mathrm{MMD}_k(\mathbb{P}, \mathbb{Q}) := \|\mu_k(\mathbb{P}) - \mu_k(\mathbb{Q})\|_{\mathcal{H}_k}.$$

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$$= \Big\| \underbrace{\mu_{\otimes_{j=1}^{d} k_j}(\mathbb{P}) - \otimes_{j=1}^{d}\mu_{k_j}(\mathbb{P}_j)}_{\text{cross-covariance operator}} \Big\|_{\mathcal{H}_k}.$$

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Clarification of what $\otimes_{j=1}^d k_j$ and $\otimes_{j=1}^d \mu_{k_j}(\mathbb{P}_j)$ are follows.

Tensor product: $\bigotimes_{j=1}^{d} a_j$

- If $\mathbf{a} \in \mathbb{R}^{n_1}$, $\mathbf{b} \in \mathbb{R}^{n_2}$:

$$\mathbb{R} \ni \mathbf{v}^\top \left(\mathbf{a}\mathbf{b}^\top\right) \mathbf{w} = \left(\mathbf{v}^\top \mathbf{a}\right) \left(\mathbf{b}^\top \mathbf{w}\right) = \langle \mathbf{a}, \mathbf{v} \rangle_{\mathbb{R}^{n_1}} \langle \mathbf{b}, \mathbf{w} \rangle_{\mathbb{R}^{n_2}},$$

  $\mathbf{a} \otimes \mathbf{b} := \mathbf{a}\mathbf{b}^\top$ is an $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ bilinear form.

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- For $a \in \mathcal{H}_1$, $b \in \mathcal{H}_2$ Hilbert spaces, i.e. for $d = 2$:

$$(a \otimes b)(v, w) := \langle a, v \rangle_{\mathcal{H}_1} \langle b, w \rangle_{\mathcal{H}_2}.$$

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- For $d \geq 2$ and $a_j \in \mathcal{H}_j$,

$$\left(\otimes_{j=1}^{d} a_j\right) (b_1, \ldots, b_d) := \prod_{j=1}^{d} \langle a_j, b_j \rangle_{\mathcal{H}_j}.$$

Tensor product: $\otimes_{j=1}^{d}\mathcal{H}_j$

$$\otimes_{j=1}^{d}\mathcal{H}_j := \overline{\mathrm{Span}}(\otimes_{j=1}^{d} a_j \,:\, a_j \in \mathcal{H}_j).$$

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$\xrightarrow{\text{spec.}}$ The tensor product of RKHSs is an RKHS

$$\mathcal{H}_k = \otimes_{j=1}^d \mathcal{H}_{k_j},$$

$$k(x, x') := (\otimes_{j=1}^d k_j)(x, x') := \prod_{j=1}^d \underbrace{k_j(x_j, x_j')}_{\text{coordinate-wise similarity}}.$$

Validness of MMD & HSIC, their estimation

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Properties:

1. Injectivity of μ_k on probability / finite signed measures, so

   universal $\Rightarrow$ characteristic.

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1. Injectivity of μ_k on probability / finite signed measures, so

   universal $\Rightarrow$ characteristic.

2. Easy-to-estimate: expected kernel trick

$$\langle \mu_k(\mathbb{P}), \mu_k(\mathbb{Q})\rangle_{\mathcal{H}_k} = \int_{\mathcal{X}}\int_{\mathcal{X}} k(x,y)\mathrm{d}\mathbb{P}(x)\mathrm{d}\mathbb{Q}(y).$$

Kernelized moments – towards kernelized cumulants

- From now:
  - $X = (X_j)_{j=1}^d \in \times_{j=1}^d \mathcal{X}_j$, $X \sim \gamma$,
  - kernels $k_j : \mathcal{X}_j \times \mathcal{X}_j \to \mathbb{R}$, $j \in [d]$,
  - lifting $\Phi(X) = (\Phi_j(X_j))_{j=1}^d$ with $\Phi_j(x_j) := k_j(\cdot, x_j)$,
  - RKHS $\mathcal{H}^{\otimes \mathbf{i}} := \mathcal{H}_{k_1}^{\otimes i_1} \otimes \cdots \otimes \mathcal{H}_{k_d}^{\otimes i_d}$ with kernel $k^{\otimes \mathbf{i}} := k_1^{\otimes i_1} \otimes \cdots \otimes k_d^{\otimes i_d}$, and feature

$$\Phi^{\otimes \mathbf{i}}(X) := [\Phi_1(X_1)]^{\otimes i_1} \otimes \cdots \otimes [\Phi_d(X_d)]^{\otimes i_d}.$$

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- Moment sequence:

$$\mu(\gamma) = \left(\mu^{\mathbf{i}}(\gamma)\right)_{\mathbf{i} \in \mathbb{N}^d}, \qquad \mu^{\mathbf{i}}(\gamma) := \mathbb{E}\left[\Phi^{\otimes \mathbf{i}}(X)\right] \in \mathcal{H}^{\otimes \mathbf{i}}.$$

Kernelized cumulants: examples first, analogous to $\mathbb{R}$

- $d = 1$, $m \in [3]$: $X \sim \gamma$,

$$\kappa_k^{(1)}(\gamma) = \mathbb{E}\left[\Phi(X)\right]$$

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$$\begin{aligned}\kappa_k^{(3)}(\gamma) = {} & \mathbb{E}\Big[\Phi^{\otimes 3}(X)\ \Big] - \mathbb{E}\big[\Phi(X) \otimes \Phi(X) \otimes \Phi(X')\big] \\ & - \mathbb{E}\big[\Phi(X) \otimes \Phi(X') \otimes \Phi(X)\big] - \mathbb{E}\big[\Phi(X') \otimes \Phi(X) \otimes \Phi(X)\big] \\ & + 2\mathbb{E}^{\otimes 3}\big[\Phi(X)\big].\end{aligned}$$

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- $d = 2$, $m = 2$: $(X_1, X_2) \sim \gamma$,

$$\kappa_{k_1,k_2}^{(2,0)}(\gamma) = \mathbb{E}\left[\Phi_1^{\otimes 2}(X_1)\right] - \mathbb{E}^{\otimes 2}\left[\Phi_1(X_1)\right],$$

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\kappa_k^{(3)}(\gamma) &= \mathbb{E}\Big[\Phi^{\otimes 3}(X)\ \Big] - \mathbb{E}\big[\Phi(X) \otimes \Phi(X) \otimes \Phi(X')\big]\\
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\kappa_{k_1,k_2}^{(1,1)}(\gamma) &= \mathbb{E}\left[\Phi_1(X_1) \otimes \Phi_2(X_2)\right] - \mathbb{E}\left[\Phi_1(X_1)\right] \otimes \mathbb{E}\left[\Phi_2(X_2)\right]
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Kernelized cumulants: examples first, analogous to $\mathbb{R}$

- $d = 1$, $m \in [3]$: $X, X' \sim \gamma$, independent,

$$\kappa_k^{(1)}(\gamma) = \mathbb{E}[\Phi(X)],$$
$$\kappa_k^{(2)}(\gamma) = \mathbb{E}[\Phi(X) \otimes \Phi(X)] - \mathbb{E}[\Phi(X)] \otimes \mathbb{E}[\Phi(X)],$$
$$\begin{aligned}\kappa_k^{(3)}(\gamma) = {} & \mathbb{E}\left[\Phi^{\otimes 3}(X)\right] - \mathbb{E}[\Phi(X) \otimes \Phi(X) \otimes \Phi(X')] \\ & - \mathbb{E}[\Phi(X) \otimes \Phi(X') \otimes \Phi(X)] - \mathbb{E}[\Phi(X') \otimes \Phi(X) \otimes \Phi(X)] \\ & + 2\mathbb{E}^{\otimes 3}[\Phi(X)].\end{aligned}$$

- $d = 2$, $m = 2$: $(X_1, X_2) \sim \gamma$,

$$\kappa_{k_1,k_2}^{(2,0)}(\gamma) = \mathbb{E}\left[\Phi_1^{\otimes 2}(X_1)\right] - \mathbb{E}^{\otimes 2}\left[\Phi_1(X_1)\right],$$
$$\kappa_{k_1,k_2}^{(1,1)}(\gamma) = \mathbb{E}\left[\Phi_1(X_1) \otimes \Phi_2(X_2)\right] - \mathbb{E}\left[\Phi_1(X_1)\right] \otimes \mathbb{E}\left[\Phi_2(X_2)\right],$$
$$\kappa_{k_1,k_2}^{(0,2)}(\gamma) = \mathbb{E}\left[\Phi_2^{\otimes 2}(X_2)\right] - \mathbb{E}^{\otimes 2}\left[\Phi_2(X_2)\right].$$

Wanted: repetition and partitioning. Weights: as before (c_π).

- Repetition (diagonal measure): $\mathbf{i} \in \mathbb{N}^d$,

$$\gamma^{\mathbf{i}} := \mathrm{Law}(\underbrace{X_1,\ldots,X_1}_{i_1 \text{ times}},\underbrace{X_2,\ldots,X_2}_{i_2 \text{ times}},\ldots,\underbrace{X_d,\ldots,X_d}_{i_d \text{ times}}).$$

Kernelized cumulants: $X \sim \gamma$ prob. measure on $\times_{j=1}^{d} \mathcal{X}_j$

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- Partitioning (partition measure): $\pi \in P(d)$, $b = |\pi|$,

$$\gamma_\pi := \gamma|_{\mathcal{X}_{\pi_1}} \otimes \cdots \otimes \gamma|_{\mathcal{X}_{\pi_b}}.$$

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- Kernelized cumulants: $m = \deg(\mathbf{i}) := \sum_{j=1}^d i_j \xrightarrow{\text{OK}} \gamma_\pi^{\mathbf{i}} = (\gamma^{\mathbf{i}})_\pi$,

$$\kappa_{k_1,\ldots,k_d}(\gamma) := \left(\kappa^{\mathbf{i}}_{k_1,\ldots,k_d}(\gamma)\right)_{\mathbf{i} \in \mathbb{N}^d},$$
$$\kappa^{\mathbf{i}}_{k_1,\ldots,k_d}(\gamma) := \sum_{\pi \in P(m)} c_\pi \mathbb{E}_{\gamma_\pi^{\mathbf{i}}} k^{\otimes \mathbf{i}}(\cdot, (X_1, \ldots, X_m)).$$

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$\Rightarrow$ expected kernel trick is applicable

Cumulants characterize distributions

Point-separating k := injectivity of $\Phi \Leftarrow$ characteristic $k \Leftarrow$ universal k.

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Theorem

- Assume:
  - γ, η: probability measures on $\times_{j=1}^{d}\mathcal{X}_j$,
  - $(\mathcal{X}_j)_{j=1}^{d}$ are Polish spaces,
  - k_j: bounded, continuous, point-separating kernel ($j \in [d]$).

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- Then, $\gamma = \eta \Leftrightarrow \kappa_{k_1,\dots,k_d}(\gamma) = \kappa_{k_1,\dots,k_d}(\eta)$, and

$$
\begin{aligned}
d^{\mathbf{i}}(\gamma,\eta) &:= \|\kappa^{\mathbf{i}}_{k_1,\dots,k_d}(\gamma) - \kappa^{\mathbf{i}}_{k_1,\dots,k_d}(\eta)\|^2_{\mathcal{H}^{\otimes \mathbf{i}}} \\
&= \sum_{\pi,\tau \in P(m)} c_\pi c_\tau \Big[\mathbb{E}_{\gamma^{\mathbf{i}}_\pi \otimes \gamma^{\mathbf{i}}_\tau} k^{\otimes \mathbf{i}}((X_1,\dots,X_m),(Y_1,\dots,Y_m)) \\
&\qquad + \mathbb{E}_{\eta^{\mathbf{i}}_\pi \otimes \eta^{\mathbf{i}}_\tau} k^{\otimes \mathbf{i}}((X_1,\dots,X_m),(Y_1,\dots,Y_m)) \\
&\qquad - 2\mathbb{E}_{\gamma^{\mathbf{i}}_\pi \otimes \eta^{\mathbf{i}}_\tau} k^{\otimes \mathbf{i}}((X_1,\dots,X_m),(Y_1,\dots,Y_m))\Big].
\end{aligned}
$$

Cumulants characterize independence

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- Then, $\gamma = \gamma|_{\mathcal{X}_1} \otimes \cdots \otimes \gamma|_{\mathcal{X}_d} \Leftrightarrow \kappa^{\mathbf{i}}_{k_1,\ldots,k_d}(\gamma) = 0$ for every $\mathbf{i} \in \mathbb{N}_+^d$

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$$\|\kappa^{\mathbf{i}}_{k_1,\ldots,k_d}(\gamma)\|^2_{\mathcal{H}^{\otimes \mathbf{i}}} = \sum_{\pi,\tau \in P(m)} c_\pi c_\tau \mathbb{E}_{\gamma^{\mathbf{i}}_\pi \otimes \gamma^{\mathbf{i}}_\tau} k^{\otimes \mathbf{i}}((X_j)_{j=1}^m, (Y_j)_{j=1}^m),$$

where $m = \deg(\mathbf{i})$.

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  where $m = \deg(\mathbf{i})$.

Estimation in both cases

$\mathbb{E}k^{\otimes \mathbf{i}}((X_1,\ldots,X_m),(Y_1,\ldots,Y_m)) \Rightarrow$ V-statistics ✓

Distance between kernel variance embeddings

- By our theorem: if $\gamma = \eta$, then $d^{(2)}(\gamma, \eta) = 0$.
- V-statistic estimator of $d^{(2)}(\gamma, \eta)$:

$$\frac{1}{N^2}\mathrm{Tr}\left[(\mathbf{K}_x\mathbf{J}_N)^2\right] + \frac{1}{M^2}\mathrm{Tr}\left[(\mathbf{K}_y\mathbf{J}_M)^2\right] - \frac{2}{NM}\mathrm{Tr}\left[\mathbf{K}_{xy}\mathbf{J}_M\mathbf{K}_{xy}^\top\mathbf{J}_N\right],$$

with $(x_n)_{n=1}^N \overset{\text{i.i.d.}}{\sim} \gamma$, $(y_m)_{m=1}^M \overset{\text{i.i.d.}}{\sim} \eta$, $\mathbf{K}_x = [k(x_i, x_j)]_{i,j=1}^N$, $\mathbf{K}_y = [k(y_i, y_j)]_{i,j=1}^M$, $\mathbf{K}_{x,y} = [k(x_i, y_j)]_{i,j=1}^{N,M}$, $\mathbf{J}_n = \mathbf{I}_n - \mathbf{H}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$.

Distance between kernel variance/skewness embeddings

- By our theorem: if $\gamma = \eta$, then $d^{(2)}(\gamma, \eta) = 0$.
- V-statistic estimator of $d^{(2)}(\gamma, \eta)$:

$$\frac{1}{N^2}\mathrm{Tr}\left[(\mathbf{K}_x\mathbf{J}_N)^2\right] + \frac{1}{M^2}\mathrm{Tr}\left[(\mathbf{K}_y\mathbf{J}_M)^2\right] - \frac{2}{NM}\mathrm{Tr}\left[\mathbf{K}_{xy}\mathbf{J}_M\mathbf{K}_{xy}^\top\mathbf{J}_N\right],$$

with $(x_n)_{n=1}^N \overset{\text{i.i.d.}}{\sim} \gamma$, $(y_m)_{m=1}^M \overset{\text{i.i.d.}}{\sim} \eta$, $\mathbf{K}_x = [k(x_i, x_j)]_{i,j=1}^N$, $\mathbf{K}_y = [k(y_i, y_j)]_{i,j=1}^M$, $\mathbf{K}_{x,y} = [k(x_i, y_j)]_{i,j=1}^{N,M}$, $\mathbf{J}_n = \mathbf{I}_n - \mathbf{H}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$.

Time complexity

Quadratic as MMD.

- $d^{(3)}(\gamma, \eta)$: similarly; quadratic time.

Cross-skewness independence criterion (CSIC)

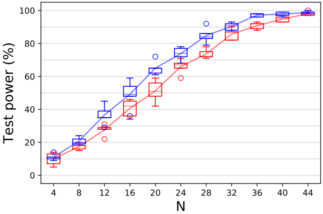
- By our theorem: if $\gamma = \gamma|_{\mathcal{X}_1} \otimes \gamma|_{\mathcal{X}_2}$, then $\kappa^{(2,1)}_{k,\ell}(\gamma) = 0$ and $\kappa^{(1,2)}_{k,\ell}(\gamma) = 0$.
- V-statistic estimator of $\|\kappa^{(1,2)}_{k,\ell}(\gamma)\|^2_{\mathcal{H}_k^{\otimes 1} \otimes \mathcal{H}_\ell^{\otimes 2}}$:

$$\frac{1}{N^2}\Bigg\langle \mathbf{K} \circ \mathbf{K} \circ \mathbf{L} - 4\mathbf{K} \circ \mathbf{KH} \circ \mathbf{L} - 2\mathbf{K} \circ \mathbf{K} \circ \mathbf{LH} + 4\mathbf{KH} \circ \mathbf{K} \circ \mathbf{LH}$$
$$+2\mathbf{K} \circ \mathbf{L}\left\langle \frac{\mathbf{K}}{N^2}\right\rangle + 2\mathbf{KH} \circ \mathbf{HK} \circ \mathbf{L} + 4\mathbf{K} \circ \mathbf{HK} \circ \mathbf{LH} + \mathbf{K} \circ \mathbf{K}\left\langle \frac{\mathbf{L}}{N^2}\right\rangle$$
$$-8\mathbf{K} \circ \mathbf{LH}\left\langle \frac{\mathbf{K}}{N^2}\right\rangle - 4\mathbf{K} \circ \mathbf{HK}\left\langle \frac{\mathbf{L}}{N^2}\right\rangle + 4\left\langle \frac{\mathbf{K}}{N^2}\right\rangle^2 \mathbf{L}\Bigg\rangle,$$

  with kernels $k : \mathcal{X}_1^2 \to \mathbb{R}$, $\ell : \mathcal{X}_2^2 \to \mathbb{R}$, $\mathbf{K} := \mathbf{K}_x, \mathbf{L} := \mathbf{L}_y$, $\langle \mathbf{A} \rangle := \sum_{i,j} A_{i,j}$.
- Time complexity: quadratic.

Numerical illustration: improved power

- Seoul bicycle rental data:
  - two-sample test (MMD, $d^{(2)}$): winter vs fall, $d = 11$,



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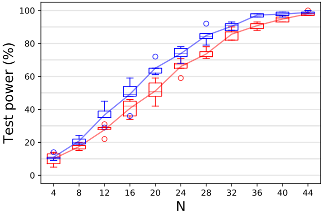
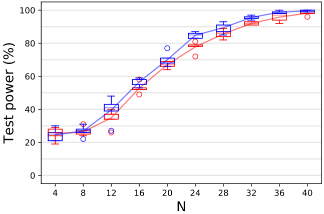


- Brazilian traffic data:
  - independence test (HSIC, CSIC); (blockage, fire, ...) vs slowness of traffic; $d_1 = 16$, $d_2 = 1$; l.h.s.



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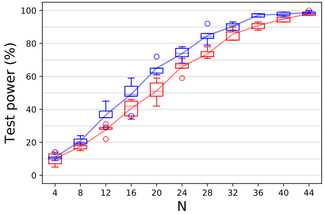

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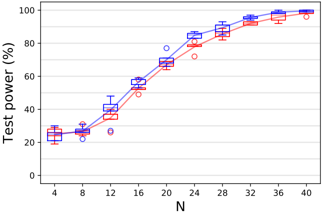
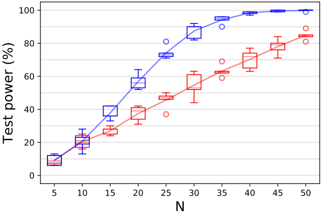
  - two-sample test (MMD, $d^{(3)}$): slow vs fast moving traffic, $d = 16$; r.h.s.





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- k-Lancaster interaction $\xleftarrow{d=3}$ k-Streitberg interaction $\xleftarrow{\mathbf{i}=\mathbf{1}_d}$ k-cumulants.

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Appendix

- Bell numbers.
- Characteristic kernels.
- Universal kernels.
- Moments and cumulants on $\mathbb{R}^d$.
- Estimator for $d^{(3)}(\gamma, \eta)$.

Bell numbers

- $B(m) :=$ number of elements in $P(m)$.
- $B_0 = B_1 = 1,\ B_2 = 2,\ B_3 = 5,\ B_4 = 15,\ B_5 = 52,\ B_6 = 203,$ $B_7 = 877,\ B_8 = 4140,\ \ldots$

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- Recursion:

$$B_{m+1} = |P(m+1)| = \sum_{k=0}^{m} \binom{m}{k} B_k.$$

Bell numbers – continued

- Easy computation by the Bell triangle

$$
\begin{array}{lllll}
1 & & & & \\
1 & 2 & & & \\
2 & 3 & 5 & & \\
5 & 7 & 10 & 15 & \\
15 & 20 & 27 & 37 & 52 \\
52 & \dots & & &
\end{array}
$$

Bell numbers – continued

- Easy computation by the Bell triangle

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\end{array}
$$

- Asymptotics:

$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + \mathcal{O}\left(\frac{\ln\ln n}{\ln^2 n}\right)$$

as $n \to \infty$.

Contents

Description of characteristic kernels on $\mathbb{R}^d$

For continuous bounded shift-invariant kernels on $\mathbb{R}^d$:

$$k(\mathbf{x}, \mathbf{x}') = k_0(\mathbf{x} - \mathbf{x}') \overset{(*)}{=} \int_{\mathbb{R}^d} e^{-i\langle \mathbf{x}-\mathbf{x}', \boldsymbol{\omega}\rangle} \mathrm{d}\Lambda(\boldsymbol{\omega})$$

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$$\|\mu_k(\mathbb{P}) - \mu_k(\mathbb{Q})\|_{\mathcal{H}_k} = \|c_{\mathbb{P}} - c_{\mathbb{Q}}\|_{L^2(\Lambda)}.$$

$(*)$: Bochner's theorem, $c_{\mathbb{P}}$: characteristic function of $\mathbb{P}$.

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Theorem ([Sriperumbudur et al., 2010])

k is characteristic iff. $supp(\Lambda) = \mathbb{R}^d$.

Examples on $\mathbb{R}$; similarly $\mathbb{R}^d$ [Sriperumbudur et al., 2010]

For Poisson kernel: $\sigma \in (0,1)$.

kernel name	k_0	$\widehat{k_0}(\omega)$	$supp\left(\widehat{k_0}\right)$
Gaussian	$e^{-\frac{x^2}{2\sigma^2}}$	$\sigma e^{-\frac{\sigma^2\omega^2}{2}}$	$\mathbb{R}$
Laplacian	$e^{-\sigma\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{\sigma}{\sigma^2+\omega^2}$	$\mathbb{R}$
B_{2n+1}-spline	$*^{2n+2}\chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x)$	$\frac{4^{n+1}}{\sqrt{2\pi}}\frac{\sin^{2n+2}\left(\frac{\omega}{2}\right)}{\omega^{2n+2}}$	$\mathbb{R}$
Sinc	$\frac{\sin(\sigma x)}{x}$	$\sqrt{\frac{\pi}{2}}\chi_{[-\sigma,\sigma]}(\omega)$	$[-\sigma,\sigma]$
Poisson	$\frac{1-\sigma^2}{\sigma^2-2\sigma\cos(x)+1}$	$\sqrt{2\pi}\sum_{j=-\infty}^{\infty}\sigma^{\lvert j\rvert}\delta(\omega-j)$	$\mathbb{Z}$
Dirichlet	$\frac{\sin\left(\frac{(2n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$	$\sqrt{2\pi}\sum_{j=-\infty}^{\infty}\delta(\omega-j)$	$\{0,\pm1,\pm2,\ldots,\pm n\}$
Fejér	$\frac{1}{n+1}\frac{\sin^2\frac{(n+1)x}{2}}{\sin^2\left(\frac{x}{2}\right)}$	$\sqrt{2\pi}\sum_{j=-n}^{n}\left(1-\frac{\lvert j\rvert}{n+1}\right)\delta(\omega-j)$	$\{0,\pm1,\pm2,\ldots,\pm n\}$
Cosine	$\cos(\sigma x)$	$\sqrt{\frac{\pi}{2}}\left[\delta(\omega-\sigma)+\delta(\omega+\sigma)\right]$	$\{-\sigma,\sigma\}$

Examples on $\mathbb{R}$; similarly $\mathbb{R}^d$ [Sriperumbudur et al., 2010]

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Cosine	$\cos(\sigma x)$	$\sqrt{\frac{\pi}{2}}\left[\delta(\omega-\sigma)+\delta(\omega+\sigma)\right]$	$\{-\sigma,\sigma\}$

For $x \in \mathbb{R}^d$: $k_0(x)=\prod_{j=1}^d k_0(x_j)$, $\widehat{k_0}(\omega)=\prod_{j=1}^d \widehat{k_0}(\omega_j)$.

Properties of universal kernels [Steinwart, 2001, Steinwart and Christmann, 2008]

If k is universal, then

- $k(x, x) > 0$ for all $x \in \mathcal{X}$.

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- The normalized kernel (like corr)

  $$\tilde{k}(x,y) := \frac{k(x,y)}{\sqrt{k(x,x)k(y,y)}}$$

  is universal.

Universal Taylor kernels [Steinwart, 2001, Steinwart and Christmann, 2008]

- For an $C^\infty \ni f : (-r, r) \to \mathbb{R}$

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad t \in (-r, r),\ r \in (0, \infty].$$

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- If $a_n > 0\ \forall n$, then

$$k(\mathbf{x}, \mathbf{y}) = f(\langle \mathbf{x}, \mathbf{y} \rangle)$$

is universal on $\mathcal{X} := \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 \leq \sqrt{r} \right\}$.

Universal kernels on compact subsets of $\mathbb{R}^d$, $\alpha > 0$

- $k(\mathbf{x}, \mathbf{y}) = e^{\alpha \langle \mathbf{x}, \mathbf{y} \rangle}$: previous result with $f(t) = e^{\alpha t} \Rightarrow a_n = \frac{\alpha^n}{n!}$.

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- $k(\mathbf{x}, \mathbf{y}) = e^{-\alpha\|\mathbf{x}-\mathbf{y}\|_2^2}$: exp. kernel & normalization.

Universal kernels on compact subsets of $\mathbb{R}^d$, $\alpha > 0$

- $k(\mathbf{x}, \mathbf{y}) = (1 - \langle \mathbf{x}, \mathbf{y} \rangle)^{-\alpha}$ binomial kernel
  - on $\mathcal{X}$ compact $\subset \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 < 1\}$.
  - $f(t) = (1 - t)^{-\alpha} = \sum_{n=0}^{\infty} \underbrace{\binom{-\alpha}{n} (-1)^n}_{>0} t^n \quad (|t| < 1)$,

where $\binom{b}{n} = \sum_{i=1}^{n} \frac{b-i+1}{i}$.

Contents

Moments and cumulants on $\mathbb{R}^d \ni X \sim \gamma$, $\mathbf{i} \in \mathbb{N}^d$

	$d = 1$	$d \geq 1$
moment sequence	$\mu(\gamma) := \left(\mu^{(i)}(\gamma)\right)_{i \in \mathbb{N}}$	$\mu(\gamma) := \left(\mu^{\mathbf{i}}(\gamma)\right)_{\mathbf{i} \in \mathbb{N}^d}$
moments	$\mu^{(i)}(\gamma) := \mathbb{E}\left(X^i\right) \in \mathbb{R}$	$\mu^{\mathbf{i}}(\gamma) := \mathbb{E}\left[X_1^{i_1} \cdots X_d^{i_d}\right] \in \mathbb{R}$

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m-th moment	$\mu^{(m)}(\gamma)$	$\mu^m(\gamma) := \left(\mu^{\mathbf{i}}(\gamma)\right)_{\deg(\mathbf{i})=m}$

where $\deg(\mathbf{i}) := i_1 + \cdots + i_d$, $\mu^0(\gamma) = 1$

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and cumulants $\kappa(\gamma) = (\kappa^{\mathbf{i}}(\gamma))_{\mathbf{i}\in\mathbb{N}^d}$

$$\sum_{\mathbf{i}\in\mathbb{N}^d} \kappa^{\mathbf{i}}(\gamma)\frac{\boldsymbol{\theta}^{\mathbf{i}}}{\mathbf{i}!} = \log\left(\sum_{\mathbf{i}\in\mathbb{N}^d} \mu^{\mathbf{i}}(\gamma)\frac{\boldsymbol{\theta}^{\mathbf{i}}}{\mathbf{i}!}\right), \quad \boldsymbol{\theta} \in \mathbb{R}^d,$$

where $\deg(\mathbf{i}) := i_1 + \cdots + i_d$, $\mu^0(\gamma) = 1$, $\mathbf{i}! = i_1!\cdots i_d!$, $\boldsymbol{\theta}^{\mathbf{i}} = \theta_1^{i_1}\cdots\theta_d^{i_d}$.

Contents, moments and cumulants on $\mathbb{R}$, motivation of cumulants

Estimator for $d^{(3)}(\gamma, \eta) = \|\kappa_k^{(3)}(\gamma) - \kappa_k^{(3)}(\eta)\|^2_{\mathcal{H}_k^{\otimes 3}}$, $N = M$

$$d^{(3)}(\gamma, \eta) = \|\kappa_k^{(3)}(\gamma)\|^2_{\mathcal{H}_k^{\otimes 3}} + \|\kappa_k^{(3)}(\eta)\|^2_{\mathcal{H}_k^{\otimes 3}} - 2\langle \kappa_k^{(3)}(\gamma), \kappa_k^{(3)}(\eta)\rangle_{\mathcal{H}_k^{\otimes 3}}$$

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$$\begin{aligned}\langle \kappa_k^{(3)}(\gamma), \kappa_k^{(3)}(\eta)\rangle_{\mathcal{H}_k^{\otimes 3}} \approx \frac{1}{N^2}\Bigg\langle & \mathbf{K}_{xy} \circ \mathbf{K}_{xy} \circ \mathbf{K}_{xy} - 3\mathbf{K}_{xy} \circ \mathbf{K}_{xy} \circ \mathbf{H}\mathbf{K}_{xy} \\ & - 3\mathbf{K}_{xy} \circ \mathbf{K}_{xy} \circ \mathbf{K}_{xy}\mathbf{H} + 6\mathbf{K}_{xy} \circ \mathbf{K}_{xy}\mathbf{H} \circ \mathbf{H}\mathbf{K}_{xy} \\ & + 3\mathbf{K}_{xy} \circ \mathbf{K}_{xy}\left\langle \frac{\mathbf{K}_{xy}}{N^2}\right\rangle + 2\mathbf{K}_{xy} \circ \mathbf{H}\mathbf{K}_{xy} \circ \mathbf{H}\mathbf{K}_{xy} \\ & + 2\mathbf{K}_{xy} \circ \mathbf{K}_{xy}\mathbf{H} \circ \mathbf{K}_{xy}\mathbf{H} - 6\mathbf{K}_{xy} \circ \mathbf{K}_{xy}\mathbf{H}\left\langle \frac{\mathbf{K}_{xy}}{N^2}\right\rangle \\ & - 6\mathbf{K}_{xy} \circ \mathbf{H}\mathbf{K}_{xy}\left\langle \frac{\mathbf{K}_{xy}}{N^2}\right\rangle + 4\left\langle \frac{\mathbf{K}}{N^2}\right\rangle^2 \mathbf{K}_{xy}\Bigg\rangle.\end{aligned}$$

Note: Matrix multiplication takes precedence over the Hadamard one.

Estimator for $d^{(3)}(\gamma,\eta)$ – continued

$$\|\kappa_k^{(3)}(\gamma)\|^2_{\mathcal{H}_k^{\otimes 3}} \approx \frac{1}{N^2}\Big\langle \mathbf{K}_x \circ \mathbf{K}_x \circ \mathbf{K}_x - 6\mathbf{K}_x \circ \mathbf{K}_x\mathbf{H} \circ \mathbf{K}_x$$
$$+ 4\mathbf{K}_x\mathbf{H} \circ \mathbf{K}_x \circ \mathbf{K}_x\mathbf{H} + 3\mathbf{K}_x \circ \mathbf{K}_x \left\langle \frac{\mathbf{K}_x}{N^2} \right\rangle$$
$$+ 6\mathbf{K}_x\mathbf{H} \circ \mathbf{H}\mathbf{K}_x \circ \mathbf{K}_x - 12\mathbf{K}_x \circ \mathbf{H}\mathbf{K}_x \left\langle \frac{\mathbf{K}_x}{N^2} \right\rangle$$
$$+ 4\left\langle \frac{\mathbf{K}_x}{N^2} \right\rangle^2 \mathbf{K}_x \Big\rangle.$$

$\|\kappa_k^{(3)}(\eta)\|^2_{\mathcal{H}_k^{\otimes 3}}$: similarly (change $\mathbf{K}_x$ to $\mathbf{K}_y$).

Contents, $d^2(\gamma,\eta)$ estimation

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